Class functions and G-homomorphisms

1) Every function on G

$$f : G \to \mathbb{C}$$

determines a linear map on any representation V ($\rho : G \to GL(V)$)

by $T_f : V \to V$

$$T_f = \sum_{g\in G} f(g)\rho(g),$$

i.e. T_f acts on $v \in V$ by

$$T_f v = \sum_{g\in G} f(g)\cdot(g.v).$$

2) <u>Claim:</u> for a function $f : G \to \mathbb{C}$,

(f is a class function) $\Longleftrightarrow$ ($T_f : V \to V$ is G-linear for every G-rep. V)

<u>Pf.</u> If f is a class function and $h \in G$, we need to show

$$T_f \circ \rho(h) = \rho(h) \circ T_f.$$

Indeed, $T_f \circ \rho(h) = \left(\sum_{g\in G} f(g)\rho(g)\right) \circ \rho(h)$

$$= \sum_{g\in G} f(g)\rho(g)\rho(h) = \sum_{g\in G} f(g)\rho(gh)$$

change the summation variable $g \rightsquigarrow hgh^{-1}$,

and since f is constant on conj. classes

$$\sum_{g\in G} f(hgh^{-1})\rho(hgh^{-1}\cdot h) = \sum_{g\in G} f(g)\rho(hg)$$

$$= \rho(h)\sum_{g\in G} f(g)\rho(g) = \underline{\underline{\rho(h)\circ T_f}}$$

and this concludes the ($\Longrightarrow$) direction.

Conversely, suppose T_f is a G-linear map for every G-rep.

Take $V = \mathbb{C}[G]$ - the regular rep.

Let $h \in G$ be any element.

Let T_f act on $e_h = h.(e_1)$:

On the one hand

$$T_f e_h = \sum_{g\in G} f(g)\, g.(e_h) = \sum_{g\in G} f(g) e_{gh}.$$

while on the other hand,

$$T_f e_h = T_f(h.e_1) = h.(T_f e_1)$$

$$= h\sum_{g\in G} f(g)(g.e_1) = \sum_{g\in G} f(g)[(hg).e_1]$$

$$= \sum_{g\in G} f(g) e_{hg}.$$

Change the summation variable - $g \rightsquigarrow h^{-1}gh$

$$= \sum_{g\in G} f(h^{-1}gh)e_{h(h^{-1}gh)} = \sum_{g\in G} f(h^{-1}gh)e_{gh}$$

But $\{e_g : g\in G\}$ is a basis for V

and thus $\sum_g f(g)e_{gh} = \sum_g f(h^{-1}gh)e_{gh}$

implies $f(g) = f(h^{-1}gh) \quad \forall g \in G$.

h was arbitrary $\Rightarrow$ f is constant on conj. classes. ∎

3) If V_i is an irrep. of G and $f : G \to \mathbb{C}$ is a class func. then $T_f : V_i \to V_i$ is G-linear

$\Longrightarrow$ it's scalar!

Suppose $T_f v = \lambda v \quad \forall v \in V_i$.

Then $\lambda \cdot \dim V_i = \mathrm{Tr}(T_f)$.

On the other hand, by linearity of the trace:

$$\mathrm{Tr}(T_f) = \sum_{g\in G} f(g)\,\mathrm{Tr}(\rho(g)) = \sum_{g\in G} f(g)\chi_{V_i}(g)$$

$$= \underline{\underline{|G|}} < \chi_{V_i}, \bar{f} >$$

$$\Longrightarrow \underline{\lambda = \frac{|G|}{\dim V_i} < \chi_{V_i}, \bar{f} >}$$

So we get an explicit formula for T_f on any irrep, and consequently ∀rep.

Example / Application:

⊛ The projection on the $V_i^{n_i}$ part of $V_1^{n_1} \oplus \dots \oplus V_k^{n_k}$:

Take $f: G \to \mathbb{C}$ to be the class function $f = \left(\frac{\dim V_i}{|G|}\right) \overline{\chi_{V_i}}$.

Then the action of T_f on V_j is multiplication by $\lambda = \frac{\dim V_i}{\dim V_j} \langle \chi_{V_j}, \chi_{V_i} \rangle$

i.e. $T_f : V_j \to V_j$ is multiplication by 0 if $j \neq i$ and by 1 if $j = i$.

$\Longrightarrow$ T_f is the projection $P_i : V_1^{n_1} \oplus \dots \oplus V_k^{n_k} \to V_i^{n_i}$.

⊛ We can use this to almost find copies of irreps. inside the regular rep. $\mathbb{C}[G]$.

Suppose $\chi = \chi_{V_i}$ is an irreducible character.

$$P_i = \frac{\dim V_i}{|G|} \sum_{g \in G} \overline{\chi(g)} \cdot g$$

is the projection $\mathbb{C}[G] \twoheadrightarrow V_i^{\dim V_i}$.

- Apply this projection to the vector $e_1 \in \mathbb{C}[G]$:

$$P_i(e_1) = \frac{\dim V_i}{|G|} \sum_{g \in G} \overline{\chi(g)} e_g \in V_i^{\dim V_i}$$

This is some non-zero element of the part $(V_i^{\dim V_i})$.

Note: This is not enough for finding a copy of V_i inside $\mathbb{C}[G]$.

Example: $G = S_3$ $V = V_{std}$ - the 2-dim irrep.

We can find a vector $v \in V \oplus V$ that is not contained in any single copy of V!

e.g. $v = (e_1 - e_2, e_2 - e_3) \in V^2$.

Suppose $v \in W \leq V^2$ where $W \cong V$ a copy of V inside V^2.

Then $v \in W \Rightarrow \sigma . v \in W \quad \forall \sigma \in S_3$

$\Rightarrow v + \sigma . v \in W \quad \forall \sigma \in S_3$.

But $(12) v = (e_2 - e_1, e_1 - e_3)$

$\Longrightarrow v + (12).v = (0, \underbrace{e_1 + e_2 - 2e_3}_{\omega})$

i.e. $(0, \omega) \in W$.

$S_3 . \omega$ is a non-zero subrep. of $V \Longrightarrow S_3 . \omega = V$

$\Longrightarrow 0 \oplus V = S_3 . (0, \omega) \leq W$

and similarly $V \oplus 0 \leq W \Longrightarrow \underline{W = V^2}$.

Bilinear forms and V^*

1) There is a correspondence:

$$\left\{ \begin{array}{c} \text{bilinear forms} \\ B: V \times V \to \mathbb{C} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{linear maps} \\ T_B : V \to V^* \end{array} \right\}$$

and this is given by -

$$B \rightsquigarrow \left[T_B(v) = B(\cdot, v) \in V^* \right]$$

(a linear functional on V)

$$\left[B_T(v, w) = (Tw)(v) \right] \leftsquigarrow T$$

(a linear functional on V)

Ex. Prove that $T_B : V \to V^*$ is a linear map, and $B_T : V \times V \to \mathbb{C}$ is a bilinear map.

Moreover prove that

$B \rightsquigarrow T_B \rightsquigarrow B_{(T_B)} = B$

$T \rightsquigarrow B_T \rightsquigarrow T_{(B_T)} = T$

So this is indeed a natural bijection.

2) If V is a G-rep. then V^* is also a G-rep.

Claim: B is G-invariant iff T_B is G-linear.

Pf. ($\Longrightarrow$) Suppose B is G-inv. and $g \in G$, $v, w \in V$.

$$T_B(gv)(w) = B(w, gv) = B(g^{-1}w, g^{-1}gv) = B(g^{-1}w, v) = T_B(v)(g^{-1}w)$$

and this is our def. of the G-action on V^*:

$$[g.(T_B v)](w) = T_B v(g^{-1}w),$$

So we found $T(gv) = g(Tv)$ as linear functionals.

($\Longleftarrow$) Suppose T is G-linear, i.e.

$$T(gv) = g(Tv) = (g^{-1})^*(Tv)$$

and thus

$$B(gw, gv) = T(gv)(gw) = (g^{-1})^*(Tv)(gw)$$

$$= Tv(g^{-1}gw) = Tv(w) = B(w, v)$$

$\blacksquare$

Notes on dual spaces:
(Things you need to know!)

① If we choose a basis for V $\{v_1, \dots, v_n\} \in V$ then every vector $v \in V$ has a unique presentation -

$$v = \sum_{i=1}^{n} \lambda_i(v) v_i .$$

We can make these coefficients into linear functionals

$$v \longmapsto \lambda_i(v)$$

and these are in fact the elements of the dual basis to $\{v_i\}$

i.e. $\lambda_i(v_j) = \delta_{ij}$.

Cor. The coefficients of vectors w.r.t. a basis are elements of V^*.

② Now suppose V is equipped with a G-action.

G acts on the basis

$$\{v_1, \dots, v_n\} \longmapsto \{gv_1, \dots, gv_n\}$$

another basis!

So if we expand

$$v = \sum_{i=1}^{n} \lambda_i(v) v_i, \text{ then}$$

$$gv = \sum_{i=1}^{n} \lambda_i(v) gv_i .$$

But this demonstrates that if $\{\delta_i\}$ are the coefficients associated to the basis $\{gv_i\}$

i.e. $w = \sum_{i=1}^{n} \delta_i(w)(gv_i)$,

then we find:

$$\sum \lambda_i(v) gv_i = \sum \delta_i(gv) gv_i$$

$$\Downarrow$$

$$\lambda_i(v) = \delta_i(gv) = (g^*\delta_i)(v) .$$

Equivalently,

$$(g.\lambda_i)(v) = \lambda_i(g^{-1}v) = \delta_i(v) \quad \forall v \in V$$

i.e. $\underline{g.\lambda_i = \delta_i}$

Cor. When expressing v w.r.t. some basis $\{v_i\} \leftrightarrow \{\lambda_i\}$, the G-action on V can be interpreted in two equivalent ways

① G acts on the basis vectors $\{v_i\} \rightsquigarrow \{gv_i\}$ while keeping the coefficients fixed.

② G acts on the coefficients $\{\lambda_i\} \rightsquigarrow \{g\lambda_i\}$ while keeping the basis vectors fixed.

Example: If $\mathbb{C}\{e_g : g \in G\}$ is the regular representation, the functions $f: G \to \mathbb{C}$ are it's dual rep!

A vector is given by

$$v = \sum_{g \in G} a_g e_g \quad \text{for } a_g \in \mathbb{C}.$$

(coefficient)

(basis vector)

The coefficient a_g is just a function $a : G \to \mathbb{C}$.

The G-action on V_{reg} may be equivalently interpreted as

① $h(\sum a_g e_g) = \sum a_g e_{hg}$

② $h(\sum a_g e_g) = \sum a_{h^{-1}g} e_g$

[I hope this saves you some brain processing time in figuring out when G acts by g^{-1} or by g ...]

And remember!

$$\mathbb{C}[G] \neq \{f : G \to \mathbb{C}\}$$

They are actually the <u>duals</u> of each other, and happen to be isomorphic as G-reps (notice that the character is a real function $\Rightarrow V \cong V^*$)
