Basic operations on G-reps
(and their effect on characters)

1) Direct sum: Suppose V and W are G-reps.

We define $V \oplus W$ to be the G-rep with the G-action
$$g.(v \oplus \omega) = (g.v \oplus g.\omega)$$
i.e. $\rho_{V\oplus W}(g)(v, \omega) = (\rho_V(v), \rho_W(\omega))$.

⊛ If $v_1, \dots, v_n \in V$ and $\omega_1, \dots, \omega_m \in W$ are bases, then $(v_1, 0), \dots, (v_n, 0), (0, \omega_1), \dots, (0, \omega_m) \in V \oplus W$ is a basis with the property that $\rho_{V\oplus W}(g)$ has the block form
$$\left(\begin{array}{c|c} \rho_V(g) & 0 \\ \hline 0 & \rho_W(g) \end{array}\right)$$
in this basis.

⊛ Computing the trace of this block matrix, we find
$$\chi_{V\oplus W}(g) = \mathrm{Tr}(\rho_{V\oplus W}(g)) = \mathrm{Tr}(\rho_V(g)) + \mathrm{Tr}(\rho_W(g))$$
$$= \underline{\chi_V(g) + \chi_W(g)}$$
$$\left(\begin{array}{c|c} \cancel{A} & B \\ \hline C & \cancel{D} \end{array}\right)$$

So characters are additive w.r.t. direct sums.

2) Dual space: Suppose V is a G-rep. We wish to define a natural G-action on the dual $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$.

There is a natural way in which a linear map $T : V \to W$ induces a linear map on the dual spaces -
$$V^* \leftarrow W^* \ : T^*$$
(going in the opposite direction)

namely - if $\varphi \in W^*$ is a linear functional on W, define $T^*\varphi \in V^*$ to be the linear functional $\varphi \circ T : V \to \mathbb{C}$.

T^* is called the <u>dual map</u>.

⊛ How does $(\cdot)^*$ react to composition? If
$$\begin{tikzcd} V \arrow[r, "T"] \arrow[rr, bend right, dashed, "S\circ T"'] & W \arrow[r, "S"] & U \end{tikzcd}$$
is dualized, we get
$$\begin{tikzcd} V^* & W^* \arrow[l, "T^*"'] & U^* \arrow[l, "S^*"'] \arrow[ll, bend left, dashed, "T^*\circ S^*"] \end{tikzcd}$$

i.e. $(S \circ T)^* = T^* \circ S^*$ - reverses the direction of composition!

⊛ Given a G-rep. ρ_V on the space V we can try defining
$$V^* \leftarrow V^* \ : \rho(g)^*$$
for all $g \in G$, but because of the arrow reversal we would get $\rho(gh)^* = (\rho(g)\rho(h))^* = \rho(h)^*\rho(g)^*$ and this does <u>not</u> satisfy the composition rule.

$$\begin{tikzcd} V \arrow[d, "\rho(h)"'] \arrow[dr, "\rho(gh)"] & \\ V \arrow[r, "\rho(g)"'] \arrow[ur, phantom, "\checkmark"] & V \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} V^* & \\ V^* \arrow[u, "\rho(h)^*"] \arrow[ur, phantom, "\times"] & V^* \arrow[l, "\rho(g)^*"] \arrow[ul, "\rho(gh)^*"'] \end{tikzcd}$$

To get the arrows back in the right direction, we replace g by g^{-1}:

$$\begin{tikzcd} V & \\ V \arrow[u, "\rho(h^{-1})"] & V \arrow[l, "\rho(g^{-1})"] \arrow[ul, "\rho((gh)^{-1})"'] \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} V^* \arrow[d, "\rho(h^{-1})^*"'] \arrow[dr, "\rho((gh)^{-1})^*"] & \\ V^* \arrow[r, "\rho(g^{-1})^*"'] \arrow[ur, phantom, "\checkmark"] & V^* \end{tikzcd}$$

and this diagram has the form of a G-rep!

⊛ We therefore <u>define</u>
$$\rho_{V^*}(g) := \rho_V(g^{-1})^*$$
the <u>dual rep.</u> to V.

⊛ Let's compute the character of the dual rep.

Fix some $g \in G$. Since $g^n = 1$ we have $\rho_V(g)$ is a diagonalizable transformation, and all it's eigenvalues are n-th roots of unity. In particular they all lie on the unit circle $\subseteq \mathbb{C}$.

Let $v_1, \dots, v_n \in V$ be a diagonalizing basis for $\rho_V(g)$, and

$$\rho_V(g) v_i = \lambda_i v_i . \quad (\lambda_i^n = 1 \ \forall i)$$

In matrix form $\rho_V(g) = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

and $\chi_V(g) = \mathrm{Tr}(\rho_V(g)) = \lambda_1 + \dots + \lambda_n$.

$$\rho_V(g^{-1}) = \rho_V(g)^{-1} = \begin{pmatrix} \lambda_1^{-1} & & 0 \\ & \ddots & \\ 0 & & \lambda_n^{-1} \end{pmatrix} \underset{\uparrow}{=} \begin{pmatrix} \bar{\lambda}_1 & & 0 \\ & \ddots & \\ 0 & & \bar{\lambda}_n \end{pmatrix}$$

$$[\lambda \bar{\lambda} = |\lambda|^2 = 1]$$

so $\chi_V(g^{-1}) = \bar{\lambda}_1 + \dots + \bar{\lambda}_n = \overline{\chi_V(g)}$!

! [The character of g^{-1} is the complex conjugate of that of g ! ".]

<u>Ex:</u> If $T : V \to V$ has matrix form (a_{ij}) w.r.t. a basis $(v_i)_i$ and $(f_i)_i \subseteq V^*$ is the dual basis (i.e. $f_i(v_j) = \delta_{ij}$) then $T^* : V^* \to V^*$ has matrix form $(a_{ji}) = a^t$ in the dual basis!

– In particular $\mathrm{Tr}(T) = \mathrm{Tr}(a) = \mathrm{Tr}(a^t)$
$= \mathrm{Tr}(T^*)$

<u>Cor.</u> $\chi_{V^*}(g) = \mathrm{Tr}(\rho_V(g^{-1})^*) = \chi_V(g^{-1})$

and we find:

$$\underline{\chi_{V^*} = \overline{\chi_V}} \quad \text{complex conjugates!}$$

3) Tensor products:

If V and W are G-reps. then $V \otimes W$ is a G-rep via the induced action $\rho_{V \otimes W}(g)(v \otimes w) = (\rho_V(g) v) \otimes (\rho_W(g) w)$.

⊛ If $v_1, \dots, v_n \in V$ and $w_1, \dots, w_m \in W$ are bases, then $(v_i \otimes w_j)_{i,j}$ is a basis for $V \otimes W$,

we compute the character of $V \otimes W$.

If $\rho_V(g) v_j = \sum_i a_{ij} v_i$
$\rho_W(g) w_l = \sum_k b_{kl} w_k$

then $\rho_{V \otimes W}(g)(v_j \otimes w_l) = \sum_{i,k} a_{ij} b_{kl} v_i \otimes w_k$.

The diagonal coefficient - the (j,l)-th coefficient - is $a_{jj} \cdot b_{ll}$.

$$\chi_{V \otimes W}(g) = \sum_{j,l} a_{jj} b_{ll} = \left(\sum_j a_{jj}\right)\left(\sum_l b_{ll}\right)$$
$$= \chi_V(g) \cdot \chi_W(g)$$

$$\Longrightarrow \underline{\chi_{V \otimes W} = \chi_V \cdot \chi_W} \quad \text{the product of characters.}$$

4) Homomorphism space: Let V and W be G-reps.

By similar considerations to the one made for V^*, there is a natural G-action on the space

$$\mathrm{Hom}_{\mathbb{C}}(V,W) = \{T : V \to W : T \text{ linear}\}$$

given by $g \bullet (T) = g T g^{-1}$ i.e.

$$\rho_{\mathrm{Hom}}(g)(T) = \rho_W(g) \circ T \circ \rho_V(g^{-1}) \in \mathrm{Hom}_{\mathbb{C}}(V,W).$$

⊛ We described an isomorphism of vector spaces $V^* \otimes W \cong \mathrm{Hom}(V,W)$.

<u>Ex.</u> Prove that this is an iso. of G-representations!

⊛ Computing the character,

$$\chi_{\mathrm{Hom}(V,W)} = \chi_{V^* \otimes W} = \chi_{V^*} \cdot \chi_W = \underline{\overline{\chi_V} \cdot \chi_W} .$$

<u>Rem:</u> G-homomorphisms $\mathrm{Hom}_G(V,W)$ are precisely the maps fixed by all $g \in G$.

Properties of the character table

Rem 1: By the Fundamental thm. of Character theory,

V is irreducible $\iff \frac{1}{|G|}\sum_{g\in G}|\chi_V(g)|^2 = 1$.

Since $|\chi_V(g)| = |\overline{\chi_V(g)}| = |\chi_{V^*}(g)|$, we find that V^* is irreducible $\iff$ V is, and $V \cong V^* \iff \chi_V$ is real.

$\Longrightarrow$ For every irrep. we find; if χ_V is not a real function, then V^* is a distinct irrep. of G!

Rem 2: For every vec. space V,

$Tr(id_V) = \dim V$.

- Therefore if χ is the character of some G-rep. W, then

$\chi(1) = \dim W$.

- $\forall g\in G$, $\chi(g) = \sum_{i=1}^{n}\lambda_i$ - the eigenvalues of $\rho_V(g)$.

$\Longrightarrow |\chi(g)| \leq \sum_{i=1}^{n}|\lambda_i| = \sum_{i=1}^{n} 1 = \dim W$

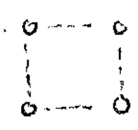
and equality holds $\iff \lambda_1 = \lambda_2 = \ldots = \lambda_n$

i.e. $\iff \rho(g) = \lambda\cdot(id_W)$

So the character tells us some direct information about the rep.

Example: Character table for D_4.

The Dihedral group of order 8 is the symmetry group of a square



it's generated by:

- a rotation by $90°$ - τ $(\tau^4 = 1)$
- a reflection along the horisontal axis - σ $(\sigma^2 = 1)$

and these are subject to the relation $\sigma\tau\sigma = \tau^{-1}$ (rotation in the opposite direction)

⊛ The conjugacy classes are

$\{1\}$, $\{\sigma, \tau^2\sigma\}$, $\{\tau\sigma, \tau^3\sigma\}$

$\{\tau, \tau^3\}$, $\{\tau^2\}$

so we will have 5 irreps!

⊛ We have a relation between $|G|$ and the dimensions of the irreps.

$$8 = |G| = \sum_{i=1}^{5}(\dim V_i)^2$$

and since these are integers, the only possibility is $(\dim V_i) = (1,1,1,1,2)$.

⊛ Let's find the 1-dim. reps.

Note: If V_i is 1-dim, then $\rho_{V_i}(g)$ = multiplication by some scalar $\lambda_g \in \mathbb{C}^\times$

In particular,

$\rho_i(gh) = \rho_i(g)\rho_i(h) = \lambda_g\lambda_h = \lambda_h\lambda_g = \rho_i(hg)$.

For D_4 this means,

$\rho_i(\tau^{-1}) = \rho_i(\sigma\tau\sigma) = \rho_i(\sigma\sigma\tau) = \rho_i(\tau)$

$\Longrightarrow \rho_i(\tau^2) = 1$.

Thus ρ_i factors through the quotient $D_4 \to D_4/\langle\tau^2\rangle \to GL(V_i)$

Ex. $D_4/\langle\tau^2\rangle \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

by $\sigma \mapsto (1,0)$
$\tau \mapsto (0,1)$.

And we know exactly 4 non-iso. 1-dim. representations of $(\mathbb{Z}/2\mathbb{Z})^2$:

① $V_{triv} = V_{triv}\otimes V_{triv}$ ③ $V_{triv}\otimes V_{sign}$

② $V_{sign}\otimes V_{triv}$, ④ $V_{sign}\otimes V_{sign}$.

i.e. as D_4 representations these are:

① $\rho_{triv}(\tau) = \rho_{triv}(\sigma) = 1$

② $\rho_{s\otimes t}(\tau) = \rho_{s\otimes t}(0,1) = 1$

$\rho_{s\otimes t}(\sigma) = \rho_{s\otimes t}(1,0) = -1$

③ $\rho_{t\otimes s}(\tau) = -1$, $\rho_{t\otimes s}(\sigma) = 1$

④ $\rho_{s\otimes s}(\tau) = -1$, $\rho_{s\otimes s}(\sigma) = -1$

and on the conjugacy classes

	C_1 (1)	C_τ (2)	C_{τ^2} (1)	C_σ (2)	$C_{\tau\sigma}$ (2)
triv	1	1	1	1	1
t⊗s	1	-1	1	1	-1
s⊗t	1	1	1	-1	-1
s⊗s	1	-1	1	-1	1

⊛ Lastly, we need to find the 2-dim irrep.

Let $v_1, v_2 \in V$ be a diagonalizing basis for τ.

Since $\tau^4 = 1$, the eigenvalues are $\pm i$ or ± 1.

If both e-values are ± 1, then $\tau^2 = 1$ and we reduce back to one of the four reps. above.

Thus we must take $\pm i$.

Suppose $\tau v_1 = i v_1$, then

$$\tau(\sigma v_1) = (\tau\sigma) v_1 = (\sigma\tau^{-1}) v_1 = \sigma(\tau^{-1} v_1) = -i\,\sigma(v_1)$$

$\Longrightarrow$ τ has the diagonal form $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ and $V = V_i \oplus V_{-i}$

σ acts by swithing the two spaces $V_i \underset{\sigma}{\overset{\sigma}{\rightleftarrows}} V_{-i}$.

i.e. $\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

$\tau^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $\tau^3 = \tau^{-1} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$

and $\sigma\tau^i$ switches V_i and V_{-i} so always have trace 0.

$Tr(\tau) = i - i = 0$

$Tr(\tau^2) = -1 - 1 = -2$

$Tr(\tau^3) = -i + i = 0$

and we have completed our character table:

	C_1 (1)	C_τ (2)	C_{τ^2} (1)	C_σ (2)	$C_{\tau\sigma}$ (2)
V	2	0	-2	0	0

Ex. Verify that the rows and columns of this 5×5 table are orthogonal w.r.t. the inner product

$$\frac{1}{8} \sum_{\{C \in \text{conjugacy classes}\}} |C| \cdot \chi_1(C)\overline{\chi_2(C)}$$

$$= \frac{1}{|G|} \sum_{g \in G} \chi_1(g)\overline{\chi_2(g)}$$