Multilinear algebra

- Read appendix B in FH book.

Tensor products

1) Motivation: We want to find the 'right' notion of a product of two vector spaces V, W.

Does the cartesian product work? $V \times W$?

- $\dim V \times W = \dim V + \dim W$
- $(v, w) + (v', w') = (v + v', w + w')$
- $\lambda(v, w) = (\lambda v, \lambda w)$

looks more like a sum: $V \oplus W$

- $(v \oplus w) + (v' \oplus w') = (v + v') \oplus (w + w')$
- $\lambda(v \oplus w) = \lambda v \oplus \lambda w$

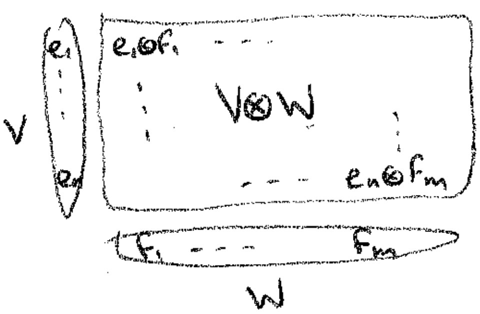
$[v \oplus w \sim (v, w)]$

We want something that acts like multiplication: $V \otimes W$

- $(v + v') \otimes w = v \otimes w + v' \otimes w$
- $\lambda(v \otimes w) = (\lambda v) \otimes w = v \otimes (\lambda w)$.

2) How will such a product look? Say $V = \langle e_1, \dots, e_n \rangle$, $W = \langle f_1, \dots, f_m \rangle$

then



3 We define $V \otimes W$ in the following equivalent ways -

ⓐ If V has a basis $(e_i)_{i \in I}$ and W has a basis $(f_j)_{j \in J}$ then $V \otimes W$ will be the vec. space with basis $(e_i \otimes f_j)_{(i,j) \in I \times J}$

ⓑ By the universal property of tensor product.

<u>Universal properties</u>: Some object can be essentially defined by the map going in and out of them.

A universal property is such a definition.

<u>Example</u>: Quotien object.

Let G be a group and $N \trianglelefteq G$ a normal subgroup.

The quotient $q : G \to G/N$ has the following universal property - $N \leq \ker q$ and for every group H and a map $r : G \to H$ s.t. $N \leq \ker(r)$ there <u>exists</u> a <u>unique</u> map $\bar{r} : G/N \to H$ s.t. the diagram

$$\begin{tikzcd} G \arrow[r, "q"] \arrow[dr, "r"'] & G/N \arrow[d, "\bar{r}"] \\ & H \end{tikzcd}$$

commutes (i.e. $r = \bar{r} \circ q$).

We write this as:
$$\left[\begin{tikzcd} G \arrow[r, "q"] \arrow[dr, "r"'] & G/N \arrow[d, dashed, "\exists!\bar{r}"] \\ & H \end{tikzcd}\right]$$

<u>Note</u>: A universal property defines an object completely up to a <u>unique</u> isomorphism.

<u>Pf.</u> If $q' : G \to K$ also satisfies the same property, then

$N \leq \ker q$, $N \leq \ker q'$

$$\Longrightarrow \begin{tikzcd} G \arrow[r] \arrow[dr, "q'"'] & G/N \arrow[d, "\bar{q}'"] \\ & K \end{tikzcd} \quad \exists\, \bar{q}' : G/N \to K$$

$$\begin{tikzcd} G \arrow[r] \arrow[dr, "q"'] & K \arrow[d, "\bar{q}"] \\ & G/N \end{tikzcd} \quad \exists\, \bar{q} : K \to G/N$$

s.t. the diagrams commute

$$q = \bar{q} \circ q' \quad \text{and} \quad q' = \bar{q}' \circ q .$$

Compose the two maps $G/N \underset{\bar{q}}{\overset{\bar{q}'}{\rightleftarrows}} K$

$$\bar{q} \circ \bar{q}' : G/N \to G/N$$

satisfies the property that

$$\begin{array}{ccc} G & \xrightarrow{q} & G/N \\ {\scriptstyle q}\searrow & & \swarrow{\scriptstyle \bar{q}\circ\bar{q}'} \\ & G/N & \end{array} \qquad q = \bar{q} \circ q' = \bar{q} \circ (\bar{q}' \circ q) = (\bar{q} \circ \bar{q}') \circ q$$

so the composition $\bar{q} \circ \bar{q}'$ is the unique map whose existence is guaranteed by the universal property!

<u>But!</u> $Id : G/N \to G/N$ satisfies the same relation $q = Id \circ q$

$\longrightarrow$ By uniqueness, $Id = \bar{q} \circ \bar{q}'$.

The same argument shows that

$$Id_K = \bar{q}' \circ \bar{q}$$

and $\bar{q}$ is an iso. $G/N \cong K$.

Furthermore the universal property asserts that $\bar{q}$ is <u>unique</u>, is K and G/N are iso. in exactly <u>one</u> way! ☒

Back to tensor products -

ⓑ $\beta : V \times W \to V \otimes W$ satisfies the following universal property:

- β is bilinear,
- For every bilinear map $\alpha : V \times W \to Z$ there <u>exists</u> a <u>unique</u> linear map $\bar{\alpha} : V \otimes W \to Z$ s.t. the diagram

$$\begin{array}{ccc} V \times W & \xrightarrow{\beta} & V \otimes W \\ {\scriptstyle \alpha}\searrow & & \swarrow{\scriptstyle \bar{\alpha}} \\ & Z & \end{array}$$

commutes.

i.e.

$$\begin{array}{ccc} V \times W & \xrightarrow{\beta} & V \otimes W \\ {\scriptstyle \alpha}\searrow & & \swarrow{\scriptstyle \exists!} \\ & Z & \end{array}$$

<u>Ex.</u> Find the right bilinear map $\beta : V \times W \to V \otimes W$ that makes def. (a) satisfy the universal property. Prove that it does. (Hint: There is only one natural way to do this. Your first guess should work!)

4) Basic structure and properties:

Denote the image of β by $\beta(v, w) = v \otimes w$.

- By bilinearity, $(v + v') \otimes w = v \otimes w + v' \otimes w$ and similarly for w.

$$\lambda (v \otimes w) = (\lambda v) \otimes w = v \otimes (\lambda w).$$

- Every element of $V \otimes W$ is a finite sum $\sum_i v_i \otimes w_i$, where if (e_i) and (f_j) are bases for V and W resp. then $(e_i \otimes f_j)$ is a basis for $V \otimes W$.

<u>Rem:</u> Not every element of $V \otimes W$ has the form $v \otimes w$!! The sums are necessary.

- If $T : V \to V'$, $S : W \to W'$ are linear maps, we can define a bilinear map $\alpha : V \times W \to V' \otimes W'$

$$\alpha(v, w) = (Tv) \otimes (Sw).$$

(by the bilinearity of $\otimes$)

$\Longrightarrow \exists!\ T \otimes S : V \otimes W \to V' \otimes W'$ defined by $T \otimes S\,(v \otimes w) = Tv \otimes Sw$.

Exc. What is the matrix rep. of $T \otimes S$ w.r.t. to the basis $(e_i \otimes f_j)$ and the matrix rep. of T and S?

5) Similarly, we have a multiple product $\beta: V_1 \times \dots \times V_n \to V_1 \otimes \dots \otimes V_n$ defined by the universal property for n-multilinear maps.

- $V_1 \otimes \dots \otimes V_n$ is spanned by elements of the form $v_1 \otimes \dots \otimes v_n$.

6) Prove the following isomorphisms using the universal property:

- $V \otimes W \cong W \otimes V$
- $(V \otimes W) \otimes U \cong V \otimes W \otimes U \cong V \otimes (W \otimes U)$
- $(V \oplus V') \otimes W \cong (V \otimes W) \oplus (V' \otimes W)$.

7) Map $V^* \otimes W \to \mathrm{Hom}(V, W)$ via the map
$\varphi \otimes w \mapsto T_{\varphi \otimes w}$ where

$$T_{\varphi \otimes w}(v) = \underbrace{\varphi(v)}_{\text{(a scalar)}} \cdot w.$$

This gives an isomorphism
$$V^* \otimes W \xrightarrow{\sim} \mathrm{Hom}(V, W)$$
for fin. dim. spaces.

Exterior & Symmetric powers

① The k-th exterior power of V is defined by a similar universal property
$$\beta: \underbrace{V \times \dots \times V}_{k} \to \Lambda^k V$$
where β is bilinear and alternating i.e. $\beta(v_1, \dots, v_k) = 0$ if $v_i = v_j$ for any $i \neq j$,

and for any alternating bilinear map $\alpha: V \times \dots \times V \to W$

$$\begin{tikzcd} V \times \dots \times V \arrow[r, "\beta"] \arrow[dr, "\alpha"'] & \Lambda^k V \arrow[d, dashed, "\exists!"] \\ & W \end{tikzcd}$$

We denote $v_1 \wedge \dots \wedge v_k = \beta(v_1, \dots, v_k)$ and notice that $v_1 \wedge \dots \wedge v_k = 0$ whenever $v_i = v_j$ for $i \neq j$.

Cor. $v \wedge v' = -v' \wedge v$
and moreover $\forall \sigma \in S_n$
$v_{\sigma(1)} \wedge \dots \wedge v_{\sigma(n)} = \mathrm{sign}(\sigma)\, v_1 \wedge \dots \wedge v_n$.

- If (e_i) is a basis for V, then $(e_{i_1} \wedge \dots \wedge e_{i_k} : i_1 < i_2 < \dots < i_k)$ is a basis for $\Lambda^k V$.
In particular, $\dim(\Lambda^k V) = \binom{n}{k}$ where $n = \dim(V)$.

- The symbol $\wedge$ is a multilinear anti-symmetric "product".
$(v + v') \wedge w = v \wedge w + v' \wedge w$
$v \wedge w = -w \wedge v$

2) If $T: V \to W$ is a linear map, define an alternating multilinear map $V^k \to \Lambda^k W$ by
$(v_1, \dots, v_k) \mapsto (Tv_1) \wedge \dots \wedge (Tv_k) \in \Lambda^k W$.

By the universal property, there exists a unique linear map

$\Lambda^k T : \Lambda^k V \to \Lambda^k W$ s.t.

$$\begin{tikzcd} V^k \arrow[r, "\beta_V"] \arrow[dr] & \Lambda^k V \arrow[d, "(\Lambda^k T)"] \\ & \Lambda^k W \end{tikzcd}$$ commutes,

i.e.

$\Lambda^k T(v_1 \wedge \dots \wedge v_k) = \Lambda^k T \circ \beta_V(v_1, \dots, v_k)$
$= (Tv_1) \wedge \dots \wedge (Tv_k) \in \Lambda^k W$.

Thus $T: V \to W$ induces a map $\Lambda^k T : \Lambda^k V \to \Lambda^k W$ naturally.

⊛ <u>Note</u>: Let $n = \dim V$, and $T: V \to V$. Then $\dim(\Lambda^n V) = \binom{n}{n} = 1$ and the space $\Lambda^n V$ is one dim. spanned by $e_1 \wedge \dots \wedge e_n$ where $\{e_1, \dots, e_n\} \in V$ forms a basis.

Computing $\Lambda^n T : \Lambda^n V \to \Lambda^n V$ we find that $\Lambda^n T$ acts simply by scalar multiplication.

$\Lambda^n T(e_1 \wedge \dots \wedge e_n) = Te_1 \wedge \dots \wedge Te_n$
$= \sum_{\sigma \in S_n} \operatorname{sign}(\sigma)\, a_{1\sigma(1)} \cdots a_{n\sigma(n)} (e_1 \wedge \dots \wedge e_n)$
$= \underline{\det(T)} \cdot (e_1 \wedge \dots \wedge e_n)$.

This is where the determinant comes from, and why it has it's familiar form.

Note that the definition of $\Lambda^k T$ never involved a choice of basis, which is why $\det(T)$ is independent of the basis for V.

Moreover, $\Lambda^k(T \circ S) = \Lambda^k T \circ \Lambda^k S$ for all T and S s.t. $T \circ S$ is defined.

This is why det is multiplicative:

$\det(T \circ S) = \Lambda^n(T \circ S) = \Lambda^n T \circ \Lambda^n S$
$= \det T \cdot \det S$.

3) <u>Symmetric powers</u>

Similarly to $\Lambda^k V$, we define $\beta : V \times \dots \times V \to \operatorname{Sym}^k V$ to be the universal object satisfying the universal property:

① β is symmetric and multilinear;

i.e. $\beta(v_1, \dots, v_i, \dots, v_j, \dots, v_n) = \beta(v_1, \dots, v_j, \dots, v_i, \dots, v_n)$

② $\forall\, \alpha : V^k \to Z$ symmetric,

$$\begin{tikzcd} V^k \arrow[r, "\beta"] \arrow[dr, "\alpha"'] & \operatorname{Sym}^k V \arrow[d, dashed, "\exists!"] \\ & Z \end{tikzcd}$$

⊛ Given a basis $(e_i)_{i=1}^n$ for V, Check that the vec. space whose basis is $(e_{i_1} \cdot e_{i_2} \cdot \ldots \cdot e_{i_k} : i_1 \le i_2 \le \dots \le i_k)$ admits a natural map β from V^k which satisfies the universal prop.

⊛ We denote $\beta(v_1, \dots, v_k)$ simply by $v_1 \cdots v_k$ or $v_1 \cdot v_2 \cdot \ldots \cdot v_k$ as if we are multiplying the vectors together.

⊛ $\operatorname{Sym}^k V$ is additively spanned by vectors of the form $v_1 \cdot \ldots \cdot v_k$ for $v_i \in V$.

⊛ A map $T: V \to W$ induces a map $\operatorname{Sym}^k T : \operatorname{Sym}^k V \to \operatorname{Sym}^k W$ by $\operatorname{Sym}^k T(v_1 \cdots v_k) = Tv_1 \cdots Tv_k$ as we has in the alternating case.

<u>Ex.</u> Compute the dimension of $\operatorname{Sym}^k V$ assuming $\dim V = n$.