Homology

Motivation – • π_1 is a great tool for analysing spaces up to homotopy and most importantly, it is computable (Using covering space theory and Van-Kampen's thm.)

- However, π_1 is only sensitive to low dim. information: the 2-skeleton of the space. For example, one can't tell S^n from S^m for $n \neq m > 1$ using π_1 alone.
- We define π_n for $n>1$ similarly, but these groups are very hard to compute, and are not what one might expect.

Ex. There are non-trivial maps $S^3 \to S^2$ and in fact $\pi_3(S^2) = \mathbb{Z}$ generated by the quotient map $S^3 \to S^3/S^1 = \mathbb{C}P^1 \cong S^2$ where $S^1 \subseteq \mathbb{C}$ acts on $S^3 \subseteq \mathbb{R}^4 = \mathbb{C}^2$ by scalar multiplication.
$\lambda \in \mathbb{C}$, $|z|=1$ $\quad (z,w) \in \mathbb{C}^2$, $|z|^2+|w|^2=1$

Homology

- We now define another sequence of groups for every space $X \rightsquigarrow \{H_n(X) : n \geq 0\}$ called the homology groups of X.
- Homology is a useful and readily computable tool that will fill-in this gap in higher dimensions.

  The idea is to linearize the problem of understanding spaces up to homotopy. Allowing us to apply linear algebra to solve the problem.
- [Functoriality] For every continuous map $f: X \to Y$, there will be induced group homomorphisms $f_*: H_n(X) \to H_n(Y)$ $\forall n \geq 0$, and later we will prove that $f \simeq g \implies f_* = g_*$ (homotopy invariance).
- $H_n(X)$ will be determined by the $(n+1)$-skeleton of the space X, much like π_1 being determined by the 2-skeleton – a property not shared by π_n for $n>1$.
- Lastly, there are multiple different ways for defining H_n, but these all end-up giving the same groups on CW-complexes.

  This means that $H_n(X)$ "exists" at a deeper level then the original definition might seem to imply. The various definitions are only tools for accessing this deep information.

Simplicial homology

We with an example and later turn this to a definition.

$X =$  we want to capture the "holes" of X in a linear-algebraic way:

- Consider formal sums of edges - $n\bar{a} + m\bar{b} + t\bar{c} + s\bar{d}$, $n,m,t,s \in \mathbb{Z}$
- The boundary of an edge is the formal difference of it's ends:
$$\partial\bar{a} = \partial\bar{b} = \partial\bar{c} = \partial\bar{d} = y - x.$$
- Identify sums that represent closed loops, i.e. have no boundary
$$\partial(\bar{a} - \bar{b}) = \partial\bar{a} - \partial\bar{b} = y - x - (x - y) = 0$$ 
- A loop does not count as a hole if it is itself the boundary of something -  $\partial U = c - d \Rightarrow c - d$ does not represent a hole
- We thus define $H_1(X) =$ "the 1-dim. holes in X" $= \ker\partial / \mathrm{Im}\,\partial$

  ($\ker\partial$: loops with no boundary; $\mathrm{Im}\,\partial$: boundaries of 2-cells.)

There are now two new concepts that we need to make this formal:

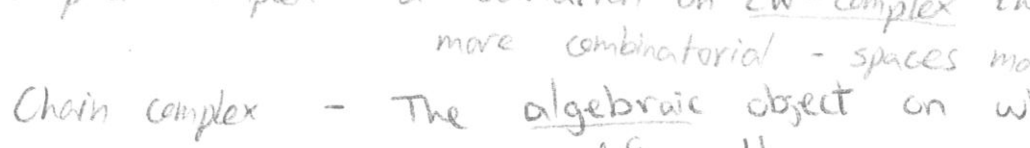
1) (Topology side) Simplicial complex - a variation on CW-complex that is more combinatorial - spaces made-up of triangles
2) (Algebra side) Chain complex - The algebraic object on which we define H_n.

Simplicial complex

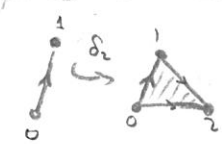
To have a clear definition of the boundary operator ($\partial\bar{a} = y - x$) in higher dimensions, we need to restrict the way we construct CW-complexes.

Def. • The standard n-simplex is the topological space
$$\Delta^n = \{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^{n} t_i = 1 \}$$

Note - 1) These are generalized n-dim. triangles.



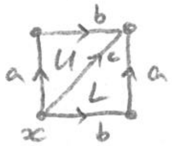
2) The boundary of Δ^n is made up of $(n+1)$ copies of Δ^{n-1}, corresponding to the inclusions $\delta_i : \Delta^{n-1} \to \Delta^n$
$$(t_0, \dots, t_{n-1}) \longmapsto (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1}) \in \mathbb{R}^{n+1}$$



- A Δ-complex is a topological space obtaind from a collection of disjoint simplices $\{\Delta^{n_\alpha}_\alpha\}_{\alpha\in A}$ by glueing them along common faces using the face maps δ_i.

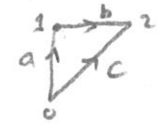
(This is a restricted version of CW-complexes, where the gluing information is fully determined by choosing which $(n-1)$-simplex forms what boundary. No need to choose gluing maps.)

Ex. 1) The Torus



0 - (x)
1 - (a, b, c)
2 - (U, L)

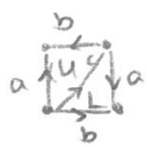
The boundaries of U are as follows -



- The arrows go from the lower index to the higher one.
- The i-th face is the one that does not contain the i-th vertex.

$\sigma_0(U) =$  $= b$ $\quad \sigma_1(U) =$  $= c$ $\quad \sigma_2(U) =$  $= a$

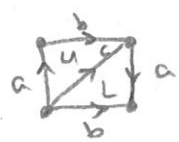
2) $\mathbb{R}P^2$



$\sigma_0(U) = b$
$\sigma_1(U) = a$
$\sigma_2(U) = c$

switched relative to the torus.

3) Klein bottle



$\sigma_i(U) =$ like the torus,
$\sigma_i(L)$ is different.

Chain complexes

Def. A chain complex is a sequence of abelian groups $\dots \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$ with homomorphisms $\partial_n : C_{n+1} \to C_n$ satisfying $\partial_n \circ \partial_{n+1} = 0 \quad \forall n \geq 0$.

Terminology:
- The elements $\sigma \in C_n$ are called n-chains,
- The operators ∂_n are called the boundary operators
- The element $\partial_n(\sigma)$ is called the boundary of σ
- An element with trivial boundary is called a cycle ($\partial_n(\sigma) = 0$ or $\sigma \in \ker \partial_n$)
- An element in the image of ∂_n is called a boundary.

Note: Since $\partial_n \circ \partial_{n+1} = 0$, every boundary is a cycle, i.e. $\operatorname{Im} \partial_{n+1} \subseteq \ker \partial_n$.

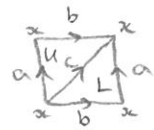
Def. The n-th homology group $H_n(C)$ of a complex $C_\bullet$ is the quotient $\ker \partial_n / \operatorname{Im} \partial_{n+1}$ - the group of cycles modulo boundaries.

For a simplicial complex X, define its <u>simplicial chain complex</u> by

$$\Delta_n(X) = \text{formal sums of } n\text{-simplices in } X$$
$$= \text{The free abelian group generated by the } n\text{-simplices of } X$$
$$= \bigoplus_{\substack{\sigma \ n\text{-simplex} \\ \text{in } X}} \mathbb{Z} \cdot \sigma$$

with boundary maps $\partial_n : \Delta_n(X) \to \Delta_{n-1}(X)$ defined on the generators by $\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i \delta_i \sigma$ - an alternating sum of the faces of σ.

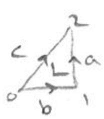
Example: • The torus -



$\Delta_0(X) = \mathbb{Z}x$
$\Delta_1(X) = \mathbb{Z}a \oplus \mathbb{Z}b \oplus \mathbb{Z}c$
$\Delta_2(X) = \mathbb{Z}U \oplus \mathbb{Z}L$
$\Delta_n(X) = 0 \quad \forall n > 2.$

with boundary $\partial_0 x = 0$, $\partial_1(a) = \partial_1(b) = \partial_1(c) = x - x = 0$

and lastly $\partial_2(U) = b - c + a$
$\partial_2(L) = a - c + b$.



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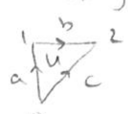

Def. The <u>simplicial homology</u> of a Δ-complex X is defined to be the homology of its simplicial complex

$$H_n^\Delta(X) := H_n(\Delta_\bullet(X)) = \ker \partial_n / \operatorname{Im} \partial_{n+1}.$$

Example: Back to our torus. $\ker \partial_0 = \mathbb{Z}x$, $\operatorname{Im} \partial_1 = 0$

$\Longrightarrow H_0^\Delta(X) = \mathbb{Z}x$.

$\ker \partial_1 = \mathbb{Z}a \oplus \mathbb{Z}b \oplus \mathbb{Z}c$, $\operatorname{Im} \partial_2 = \mathbb{Z}(a+b-c)$

$\Longrightarrow H_1^\Delta(X) = \mathbb{Z}a \oplus \mathbb{Z}b \oplus \mathbb{Z}c / (a+b=c) \cong \mathbb{Z}a \oplus \mathbb{Z}b$

We compute $\ker \partial_2$: $mU + nL \in \ker \partial_2 \Longleftrightarrow \partial_2(mU + nL) = m(b-c+a) + n(a-c+b)$

$= (m+n)a + (m+n)b - (m+n)c = 0 \Longleftrightarrow m = -n$

so $H_2^\Delta(X) = \ker \partial_2 = \mathbb{Z}(U - L)$.

To summarize - $H_n^\Delta(X) = \begin{cases} \mathbb{Z} & n = 0, 2 \\ \mathbb{Z} \oplus \mathbb{Z} & n = 1 \\ 0 & n > 2 \end{cases}$.