

- ① An algebraic function of deg. n on k -variables is a polynomial map

$$\mathbb{C}^k \setminus \Sigma \xrightarrow{F} \text{Poly}_n(\mathbb{C})$$

where $(a_1, \dots, a_k) \mapsto F(y, \bar{a}) \in \text{Poly}_n(\mathbb{C})$

$$\Sigma = F^{-1}(\text{disc}=0) = \{\bar{a} : \text{disc}_y(F(y, \bar{a})) = 0\}$$

- ② For every algebraic function there corresponds a cover = "the solution"

$$N = \{(y, \bar{a}) : F(y, \bar{a}) = 0\}$$

$$\downarrow$$

$$\mathbb{C}^k \setminus \Sigma = \{\bar{a}\}$$

with the obvious projection map.

- ③ This covering is natural w.r.t. maps between alg. functions, i.e.

$$\text{if } \mathbb{C}^k \setminus \Sigma \xrightarrow{H} \mathbb{C}^l \setminus \Sigma'$$

$$\begin{array}{ccc} & & \downarrow G \\ & & \text{Poly}_n \end{array}$$

H is a polynomial map, and the diagram commutes, then

$$N = H^*(N') \dashrightarrow N'$$

$$\downarrow \quad \quad \quad \downarrow$$

$$\mathbb{C}^k \setminus \Sigma \xrightarrow{H} \mathbb{C}^l \setminus \Sigma'$$

N is the pull-back of N' along H , or alternatively it is the fibered product

$$N = (\mathbb{C}^k \setminus \Sigma) \times_{(\mathbb{C}^l \setminus \Sigma')} N'$$

- * There always exists a tautological projection map

$$N = H^*(N') \xrightarrow{\tau_H} N'$$

$$\downarrow \quad \quad \quad \downarrow$$

$$\mathbb{C}^k \setminus \Sigma \rightarrow \mathbb{C}^l \setminus \Sigma'$$

s.t. τ_H makes the diagram commute.

- ④ The universal alg. function of deg. n is the map
- $$\text{Poly}_n(\mathbb{C}) \xrightarrow{\text{id}} \text{Poly}_n(\mathbb{C})$$

For every algebraic function

$$\mathbb{C}^k \setminus \Sigma \xrightarrow{F} \text{Poly}_n(\mathbb{C})$$

there exists a unique map of algebraic functions

$$\mathbb{C}^k \setminus \Sigma \dashrightarrow \text{Poly}_n(\mathbb{C})$$

$$\begin{array}{ccc} & \exists! & \\ & \dashrightarrow & \\ & \downarrow F & \swarrow \text{id} \\ & \text{Poly}_n(\mathbb{C}) & \end{array}$$

namely F itself.

- Denote the solution to the universal function by $UF_n = \{(y, x_0, \dots, x_{n-1}) : y^n + x_{n-1}y^{n-1} + \dots + x_0 = 0\}$
- $$\downarrow$$
- $$\text{Poly}_n(\mathbb{C})$$

Note: UF_n is a non-normal cover of degree n of $\text{Poly}_n(\mathbb{C})$. It's fibers are the n solutions to the equation $y^n + x_{n-1}y^{n-1} + \dots + x_0 = 0$

In terms of the cover $\text{Cen}_n(\mathbb{C})$

the subcover UF_n corresponds to the subgroup $S_{n-1} < S_n$ and the different choices of conjugations of $S_{n-1} \hookrightarrow S_n$ correspond to the different choice of roots.

- ⑤ Cen. The naturality of the solution $\left(\begin{array}{c} N \\ \downarrow \\ \mathbb{C}^k \setminus \Sigma \end{array} \right)$ together with the universality of the universal function $\left(\begin{array}{c} UF_n \\ \downarrow \\ \text{Poly}_n \end{array} \right)$ imply that $N = F^*(UF_n)$ and $\exists N \xrightarrow{\tau_F} UF_n$.

i.e. there exists a universal tautological map $N \xrightarrow{\tau_H} UF_n$ s.t. the diagram

$$\begin{array}{ccc} N & \xrightarrow{\tau_H} & UF_n \\ \downarrow & & \downarrow \\ \mathbb{C}^k \setminus \Sigma & \xrightarrow{F} & \text{Poly}_n \end{array}$$

Explicitly, $N = \{(y, \bar{a}) : F(y, \bar{a}) = 0\}$

where $F(\bar{a}) = "y^n + g_{n-1}(\bar{a})y^{n-1} + \dots + g_0(\bar{a})"$,
 $= (g_0(\bar{a}), \dots, g_{n-1}(\bar{a})) \in \mathbb{C}^n$

$\tau_H(y, \bar{a}) = (y, g_0(\bar{a}), \dots, g_{n-1}(\bar{a})) \in UF_n$

ⓐ Substitutions of algebraic functions:

* Given an algebraic function

$$\begin{array}{ccc} M & \xrightarrow{\tau_M} & UF_n \\ \downarrow & & \downarrow \\ \mathbb{C}^k \setminus \Sigma & \xrightarrow{F} & \text{Poly}_n(\mathbb{C}) \end{array}$$

and any polynomial map

$$\begin{array}{ccc} \mathbb{C}^l \setminus \Sigma' & \xrightarrow{R} & \mathbb{C}^k \setminus \Sigma \\ & & \downarrow F \\ & & \text{Poly}_n \end{array}$$

we get a pull-back algebraic function $(F \circ R)$ with a solution N :

$$\begin{array}{ccc} N & \xrightarrow{\tau_N} & M \\ \downarrow & \searrow \tau_M & \downarrow \\ \mathbb{C}^l \setminus \Sigma' & \xrightarrow{R} & \mathbb{C}^k \setminus \Sigma \\ & \searrow F & \downarrow \\ & & \text{Poly}_n \end{array}$$

This is the substitution -

$F(y, \bar{b}) = 0 \rightsquigarrow F(y, R(\bar{a})) = 0$

Ex. $F(y, x) = y^2 - x$, i.e. $y = \sqrt{x}$.

$R(a, b, c) = b^2 - 4ac$, then

$F \circ R$ is the algebraic function

$y^2 - (b^2 - 4ac) = 0$, i.e. $y = \sqrt{b^2 - 4ac}$

* Given two algebraic functions in the same coefficients

$$\mathbb{C}^k \setminus \Sigma \xrightarrow{F} \text{Poly}_n(\mathbb{C})$$

we get two coverings

$$\begin{array}{ccc} N_F & & N_G \\ & \searrow & \swarrow \\ & \mathbb{C}^k \setminus \Sigma & \end{array}$$

where N_F is the solution to $F(y, \bar{a}) = 0$ and N_G is the solution to $G(y, \bar{a}) = 0$.

A polynomial bundle map $N_F \xrightarrow{H} N_G$ is a

$$\begin{array}{ccc} & & \\ & \searrow & \swarrow \\ & \mathbb{C}^k \setminus \Sigma & \end{array}$$

solution of G in terms of F :

Why? $(y, \bar{a}) \in N_F$ iff $F(y, \bar{a}) = 0$.

H sends (y, \bar{a}) to a point $(H(y, \bar{a}), \bar{a})$

that satisfies $G(H(y, \bar{a}), \bar{a}) = 0$

i.e. $H(y, \bar{a})$ is an explicit solution to $G(\cdot, \bar{a}) = 0$ using only a solution y to $F(y, \bar{a}) = 0$.

Example: "Solving the universal quadratic equation using the square root".

Consider the square root function -

$$\mathbb{C} \setminus \{0\} = \mathbb{C}^x \xrightarrow{F} \text{Poly}_2$$

$(x) \mapsto (y^2 - x) = 0$

with solution

$$N_{\sqrt{x}} = \{(y, x) : y^2 = x, x \neq 0\}$$

$$\downarrow$$

\mathbb{C}^x

substitute x for $\frac{b^2}{4} - c$ by mapping

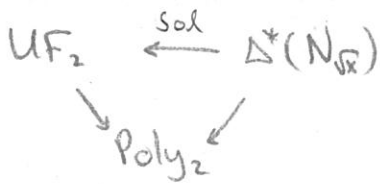
$$\begin{array}{ccc} \mathbb{C}^x & \xleftarrow{\Delta} & \text{Poly}_2 = \{(b, c) : b^2 - 4c \neq 0\} \\ \frac{b^2}{4} - c & \longleftarrow & (b, c) \end{array}$$

The solution $y = \sqrt{\frac{b^2}{4} - c}$ is the pull-back

$$\begin{array}{ccc} N_{\sqrt{x}} & \xleftarrow{\tau_{\Delta}} & \Delta^*(N_{\sqrt{x}}) \\ \downarrow & & \downarrow \\ \mathbb{C}^x & \xleftarrow{\Delta} & \text{Poly}_2(\mathbb{C}) \end{array}$$

Now, Poly_2 has two covers: $UF_2 \xrightarrow{\Delta^*} \Delta^*(N_{\sqrt{x}})$

and completing the square
 $y^2 + by + c = (y + \frac{b}{2})^2 - (\frac{b^2}{4} - c)$
 provides a bundle map



given by

$$(y, b, c) \text{ s.t. } y^2 = \frac{b^2}{4} - c \mapsto (y - \frac{b}{2}, b, c)$$

$\cong UF_2$

This can be read as:

$$\forall b, c \text{ s.t. } b^2 - 4c \neq 0, \text{ and } y \in \mathbb{C} \text{ s.t. } y^2 = \frac{b^2}{4} - c \quad (y = \pm \sqrt{\frac{b^2}{4} - c})$$

We have the equation

$$(y - \frac{b}{2})^2 + b(y - \frac{b}{2}) + c = 0$$

i.e. $y - \frac{b}{2} = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$ is a formula for the solution of $y^2 + by + c = 0$.

⑦ Composition of alg. functions:

Suppose we want to express $y = \sqrt{a + \sqrt{b-1}}$, how would this be phrased?

- Let $\mathbb{C}^k \setminus \Sigma \xrightarrow{F} \text{Poly}_n$ be a function in k variables, and $\mathbb{C}^{k+1} \setminus \Sigma' \xrightarrow{G} \text{Poly}_m$ is an algebraic function of $k+1$ variables.

The solution N_F includes into $\mathbb{C}^k \setminus \Sigma$

\mathbb{C}^{k+1} and thus defines a polynomial map

$$N_F \xrightarrow{i} \mathbb{C}^{k+1} \xrightarrow{G} \text{Poly}_m$$

and we may consider the pullback

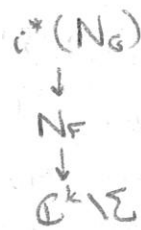
$$i^*(N_G) = \{ (z, y, a) : \begin{matrix} G(z, y, a) = 0 \\ F(y, a) = 0 \end{matrix} \}$$

↓ N_F

$$= \{ (y, a) : F(y, a) = 0 \}$$

So the coordinate z of this cover is the composition of algebraic functions F into G .

- This is again a cover of the coefficient space $\mathbb{C}^k \setminus \Sigma$:

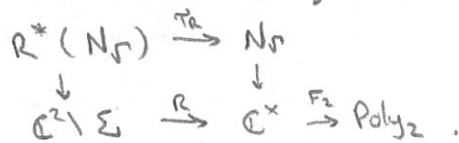


Example: $\mathbb{C}^x \xrightarrow{F_d} \text{Poly}_d(\mathbb{C})$
 $x \mapsto (y^d - x)$ - the d -th root.

Map $\mathbb{C}^2 \setminus \Sigma$ into \mathbb{C}^x by

$$R: (a, b) \mapsto b-1$$

and consider the square root pull-back

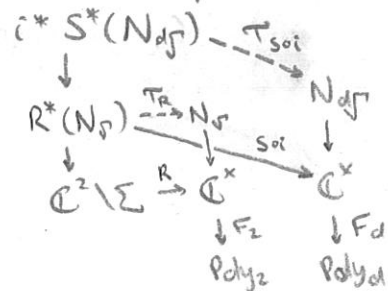


Now map $R^*(N_F) \subseteq \mathbb{C}^3 \setminus \Sigma$ into \mathbb{C}^x by

$$S: (y, a, b) \mapsto a+y$$

$^*y = \sqrt{b-1}$

and consider the pull-back of the d -th root -



A point in the top cover is of the form (z, y, a, b) s.t. $z^d = a+y$ and $y^2 = b-1$

i.e. $z = \sqrt{a + \sqrt{b-1}}$ so indeed

this describes the composition of algebraic functions.

⑧ Conclusions:

Since an expression of an alg. function as some composition of other alg. functions is put in terms of covering maps $A \rightarrow B$, all such solution can be described in terms of covering space theory.