① An algebraic function of deg. n on k-variables is a polynomial map

$$\mathbb{C}^k \setminus \Sigma \xrightarrow[F]{} \mathrm{Poly}_n(\mathbb{C})$$

where $(a_1,\dots,a_k) \longmapsto F(y,\bar{a}) \in \mathrm{Poly}_n(\mathbb{C})$.

$$\Sigma = F^{-1}(disc=0) = \{\bar{a} : disc_y(F(y,\bar{a}))=0\}.$$

② For every algebraic function there corresponds a cover = "the solution"

$$\begin{array}{cl} N & = \{(y,\bar{a}) : F(y,\bar{a})=0\} \\ \downarrow & \\ \mathbb{C}^k\setminus\Sigma & = \{\bar{a}\} \end{array}$$

with the obvious projection map.

③ This covering is natural w.r.t. maps between alg. functions, i.e. if

$$\begin{tikzcd} \mathbb{C}^k\setminus\Sigma \arrow[rr, "H"] \arrow[dr, "F"'] & & \mathbb{C}^\ell\setminus\Sigma' \arrow[dl, "G"] \\ & \mathrm{Poly}_n & \end{tikzcd}$$

H is a polynomial map, and the diagram commutes, then

$$\begin{tikzcd} N = H^*(N') \arrow[r, dashed] \arrow[d] & N' \arrow[d] \\ \mathbb{C}^k\setminus\Sigma \arrow[r, "H"] & \mathbb{C}^\ell\setminus\Sigma' \end{tikzcd}$$

N is the N is the pull-back of N' along H, or alternatively it is the fibered product

$$N = (\mathbb{C}^k\setminus\Sigma) \times_{(\mathbb{C}^\ell\setminus\Sigma')} N'$$

⊛ There always exists a tautological projection map

$$\begin{tikzcd} N = H^*(N') \arrow[r, "T_H"] \arrow[d] & N' \arrow[d] \\ \mathbb{C}^k\setminus\Sigma \arrow[r] & \mathbb{C}^\ell\setminus\Sigma' \end{tikzcd}$$

s.t. T_H makes the diagram commute.

④ The universal alg. function of deg. n is the map

$$\mathrm{Poly}_n(\mathbb{C}) \xrightarrow{id} \mathrm{Poly}_n(\mathbb{C}).$$

For every algebraic function

$$\mathbb{C}^k\setminus\Sigma \xrightarrow{F} \mathrm{Poly}_n(\mathbb{C})$$

there exists a unique map of algebraic functions

$$\begin{tikzcd} \mathbb{C}^k\setminus\Sigma \arrow[rr, dashed, "\exists!"] \arrow[dr, "F"'] & & \mathrm{Poly}_n(\mathbb{C}) \arrow[dl, "id"] \\ & \mathrm{Poly}_n(\mathbb{C}) & \end{tikzcd}$$

namely F itself.

• Denote the solution to the universal function by

$$\begin{array}{cl} UF_n & = \{(y,x_0,\dots,x_{n-1}) : y^n + x_{n-1}y^{n-1} + \dots + x_0 = 0\} \\ \downarrow & \\ \mathrm{Poly}_n(\mathbb{C}) & \end{array}$$

Note: UF_n is a non-normal cover of degree n of $\mathrm{Poly}_n(\mathbb{C})$. It's fibers are the n solutions to the equation

$$y^n + x_{n-1}y^{n-1} + \dots + x_0 = 0$$

In terms of the cover $\begin{array}{c} \mathrm{Conf}_n(\mathbb{C}) \\ \downarrow S_n \\ \mathrm{Poly}_n(\mathbb{C}) \end{array}$ the subcover UF_n corresponds to the subgroup $S_{n-1} < S_n$ and the different choices of conjugations of $S_{n-1} \hookrightarrow S_n$ correspond to the different choice of roots.

⑤ Cor. The naturality of the solution $\begin{pmatrix} N \\ \downarrow \\ \mathbb{C}^k\setminus\Sigma \end{pmatrix}$ together with the universality of the universal function $\begin{pmatrix} UF_n \\ \downarrow \\ \mathrm{Poly}_n \end{pmatrix}$ imply that $N = F^*(UF_n)$ and $\exists\, N \xrightarrow{T_F} UF_n$.

i.e. there exists a universal tautological map $N \xrightarrow{\tau_H} UF_n$ s.t. the diagram

$$\begin{tikzcd} N \arrow[r, "\tau_H"] \arrow[d] & UF_n \arrow[d] \\ \mathbb{C}^k\setminus\Sigma \arrow[r, "F"] & Poly_n \end{tikzcd}$$

commutes.

Explicitly, $N = \{(y,\bar{a}) : F(y,\bar{a}) = 0\}$

where $F(\bar{a}) = "y^n + g_{n-1}(\bar{a})y^{n-1} + \dots + g_0(\bar{a})"$,
$= (g_0(\bar{a}), \dots, g_{n-1}(\bar{a})) \in \mathbb{C}^n$

$\tau_H(y,\bar{a}) = (y, g_0(\bar{a}), \dots, g_{n-1}(\bar{a})) \in UF_n$.

(6) Substitutions of algebraic functions:

⊛ Given an algebraic function

$$\begin{tikzcd} M \arrow[r, "\tau_n"] \arrow[d] & UF_n \arrow[d] \\ \mathbb{C}^l\setminus\Sigma \arrow[r, "F"] & Poly_n(\mathbb{C}) \end{tikzcd}$$

and any polynomial map

$$\begin{tikzcd} \mathbb{C}^l\setminus\Sigma' \arrow[r, "R"] & \mathbb{C}^k\setminus\Sigma \arrow[d, "F"] \\ & Poly_n \end{tikzcd}$$

we get a pull-back algebraic function $(F \circ R)$ with a solution N:

$$\begin{tikzcd} N \arrow[r, "\tau_R"] \arrow[dr, dashed] \arrow[dd] & M \arrow[d, "\tau_F"'] \arrow[dd] \\ & UF_n \arrow[dd] \\ \mathbb{C}^l\setminus\Sigma \arrow[r] \arrow[dr, dashed] & \mathbb{C}^k\setminus\Sigma \arrow[d] \\ & Poly \end{tikzcd}$$

This is the substitution -

$F(y,\bar{b}) = 0 \rightsquigarrow F(y, R(\bar{a})) = 0$.

Ex. $F(y,x) = y^2 - x$, i.e. $y = \sqrt{x}$.

$R(a,b,c) = b^2 - 4ac$, then

$F \circ R$ is the algebraic function

$y^2 - (b^2 - 4ac) = 0$, i.e. $y = \sqrt{b^2 - 4ac}$

⊛ Given two algebraic functions in the same coefficients $\mathbb{C}^k\setminus\Sigma \overset{F}{\underset{G}{\rightrightarrows}} Poly_n(\mathbb{C})$

we get two coverings

$$\begin{tikzcd} N_F \arrow[dr] & & N_G \arrow[dl] \\ & \mathbb{C}^k\setminus\Sigma & \end{tikzcd}$$

where N_F is the solution to $F(y,\bar{a}) = 0$ and N_G is the solution to $G(y,\bar{a}) = 0$.

A polynomial bundle map

$$\begin{tikzcd} N_F \arrow[rr, "H"] \arrow[dr] & & N_G \arrow[dl] \\ & \mathbb{C}^k\setminus\Sigma & \end{tikzcd}$$

is a solution of G in terms of F.

Why? $(y,\bar{a}) \in N_F$ iff $F(y,\bar{a}) = 0$.

H sends $(y,\bar{a})$ to a point $(H'(y,\bar{a}), \bar{a})$ that satisfies $G(H'(y,\bar{a}), \bar{a}) = 0$

i.e. $H'(y,\bar{a})$ is an explicit solution to $G(\cdot,\bar{a}) = 0$ using only a solution y to $F(y,\bar{a}) = 0$.

Example: "Solving the universal quadratic equation using the square root".

Consider the square root function -

$\mathbb{C}\setminus\{0\} = \mathbb{C}^\times \xrightarrow{F} Poly_2$

$(x) \mapsto (y^2 - x) =$

with solution

$$\begin{tikzcd} N_{\sqrt{x}} = \{(y,x) : y^2 = x,\ x \neq 0\} \arrow[d] \\ \mathbb{C}^\times \end{tikzcd}$$

substitute x for $\frac{b^2}{4} - c$ by mapping

$\mathbb{C}^\times \xleftarrow{\Delta} Poly_2 = \{(b,c) : b^2 - 4c \neq 0\}$

$\frac{b^2}{4} - c \longleftarrow (b,c)$

The solution "$y = \sqrt{\frac{b^2}{4} - c}$" is the pull-back

$$\begin{tikzcd} N_{\sqrt{x}} \arrow[d] & \Delta^*(N_{\sqrt{x}}) \arrow[l, "\tau_\Delta"'] \arrow[d] \\ \mathbb{C}^\times & Poly_2(\mathbb{C}) \arrow[l, "\Delta"'] \end{tikzcd}$$

Now, $Poly_2$ has two covers:

$$\begin{tikzcd} UF_2 \arrow[dr] & & \Delta^*(N_{\sqrt{x}}) \arrow[dl] \\ & Poly_2 & \end{tikzcd}$$

and completing the square

$$y^2+by+c=\left(y+\tfrac{b}{2}\right)^2-\left(\tfrac{b^2}{4}-c\right)$$

provides a bundle map

$$\begin{tikzcd}
UF_2 \arrow[dr] & & \Delta^*(N_{\sqrt{x}}) \arrow[ll, "Sol"'] \arrow[dl] \\
& Poly_2 &
\end{tikzcd}$$

given by

$$(y,b,c) \text{ s.t. } y^2=\tfrac{b^2}{4}-c \mapsto \left(y-\tfrac{b}{2},b,c\right) \in UF_2$$

This can be read as:

$\forall b,c$ s.t. $b^2-4c\neq 0$, and

$y\in\mathbb{C}$ s.t. $y^2=\frac{b^2}{4}-c$ $\left(y=\pm\sqrt{\frac{b^2}{4}-c}\right)$

we have the equation

$$\left(y-\tfrac{b}{2}\right)^2+b\left(y-\tfrac{b}{2}\right)+c=0$$

i.e. $y-\frac{b}{2}=\frac{-b\pm\sqrt{b^2-4c}}{2}$ is a formula for the solution of $y^2+by+c=0$.

⑦ Composition of alg. functions:

Suppose we want to express $y=\sqrt[d]{a+\sqrt{b-1}}$, how would this be phrased?

- Let $\mathbb{C}^k\setminus\Sigma \xrightarrow{F} Poly_n$ be a function in k variables, and $\mathbb{C}^{k+1}\setminus\Sigma' \xrightarrow{G} Poly_m$ is an algebraic function of $k+1$ variables.

The solution

$$\begin{matrix} N_F \\ \downarrow \\ \mathbb{C}^k\setminus\Sigma \end{matrix}$$

includes into $\mathbb{C}^{k+1}$ and thus defines a polynomial map

$$\begin{tikzcd}
 & N_G \arrow[d] & \\
N_F \arrow[r, hook, "i"] & \mathbb{C}^{k+1} \arrow[r, "G"] & Poly_m
\end{tikzcd}$$

and we may consider the pullback

$$\begin{aligned}
i^*(N_G) &= \left\{(z,y,a): \begin{matrix} G(z,y,\bar{a})=0 \\ F(y,\bar{a})=0 \end{matrix}\right\} \\
\downarrow \quad & \\
N_F &= \{(y,\bar{a}): F(y,\bar{a})=0\}
\end{aligned}$$

So the coordinate z of this cover is the composition of algebraic functions F into G.

- This is again a cover of the coefficient space $\mathbb{C}^k\setminus\Sigma$:

$$\begin{matrix} i^*(N_G) \\ \downarrow \\ N_F \\ \downarrow \\ \mathbb{C}^k\setminus\Sigma. \end{matrix}$$

<u>Example:</u> $\mathbb{C}^\times \xrightarrow{F_d} Poly_d(\mathbb{C})$

$x \mapsto (y^d-x)$ - the d-th root.

Map $\mathbb{C}^2\setminus\Sigma$ into $\mathbb{C}^\times$ by

$R:(a,b)\mapsto b-1$

and consider the square root pull-back

$$\begin{tikzcd}
R^*(N_{\sqrt{}}) \arrow[r, "T_R"] \arrow[d] & N_{\sqrt{}} \arrow[d] & \\
\mathbb{C}^2\setminus\Sigma \arrow[r, "R"] & \mathbb{C}^\times \arrow[r, "F_2"] & Poly_2.
\end{tikzcd}$$

Now map $R^*(N_{\sqrt{}})\subseteq\mathbb{C}^3\setminus\tilde{\Sigma}$ into $\mathbb{C}^\times$ by

$S:(y,a,b)\mapsto a+y$

"$y=\sqrt{b-1}$"

and consider the pull-back of the d-th root -

$$\begin{tikzcd}
i^*S^*(N_{\sqrt[d]{}}) \arrow[rr, dashed, "T_{S\circ i}"] \arrow[d] & & N_{\sqrt[d]{}} \arrow[dd] \\
R^*(N_{\sqrt{}}) \arrow[r, dashed, "T_R"] \arrow[d] & N_{\sqrt{}} \arrow[d] \arrow[dr, "S\circ i"] & \\
\mathbb{C}^2\setminus\Sigma \arrow[r, "R"] & \mathbb{C}^\times \arrow[d, "F_2"] & \mathbb{C}^\times \arrow[d, "F_d"] \\
 & Poly_2 & Poly_d
\end{tikzcd}$$

A point in the top cover is of the form (z,y,a,b) s.t. $z^d=a+y$ and $y^2=b-1$ i.e. $z=\sqrt[d]{a+\sqrt{b-1}}$ so indeed this describes the composition of algebraic functions.

⑧ Conclusions:

Since an expression of an alg. function as some composition of other alg. functions is put in terms of covering maps

$$\begin{tikzcd}
A \arrow[r, dashed] \arrow[dr] & B \arrow[d] \\
 & M
\end{tikzcd}$$

; all such solution can be described in terms of covering space theory.