# Applications of Algebraic Methods in Combinatorics

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December 9th, 2020

**Abstract**

The field of combinatorics has produced important results and applications in probability, graph theory, and topology. Some of these took years to derive with many pages describing their proofs, even when studied by the greatest minds in the field. However, these results can be made easier to derive and understand by making use of tools borrowed from linear algebra. In this paper, we introduce proofs for key properties (rank-unimodality and the Sperner property) of several important partially ordered sets (Boolean, subspace, and quotient posets) using vector spaces, linear operators, and their properties.

## Contents

# 1 Introduction

The application of algebraic methods in combinatorics has been a widely studied field in recent decades. Analyzing combinatorial problems from an algebraic perspective allows us to gain insights that are very difficult to do using pure combinatorics, and opens up new applications of these concepts in the areas of graph theory, number theory, and topology. The aim of this paper is to present some of the most important algebraic techniques used in combinatorial proofs and their applications in other fields.

The primary functionality of applying algebra to combinatorics is making it easier to describe mappings from one set of elements to another. Linear algebra in particular allows us to represent a mapping from one vector space to another as a matrix. The properties of matrices allow for us to classify mappings as injective (one-to-one) or surjective (onto) relatively easily. Our paper will focus on establishing the existence of injective and surjective mappings between specific sets of elements, as this technique is used to demonstrate important properties of what are known as partially ordered sets (defined in section 1.1). If we can reduce these mappings to a linear mapping instead, we reduce our problem in combinatorics to one of linear algebra, a new problem which is much easier to solve than the former.

These specific sets of elements with which we are concerned are called partially ordered sets, commonly referred to as posets. We are particularly interested in how we can map certain groups of elements in a poset to each other, as we can use these kinds of mappings to establish a key property of posets known as the Sperner Property.

Sections 1.1 and 1.2 serve to formally define posets and ranked posets, respectively. This will give the necessary background needed to study the key properties of posets. Section 1.3 will formally define these key properties such as the Sperner Property, as well as other properties such as rank unimodality and rank symmetry. Section 2 will provide preliminary concepts in both combinatorics and algebra used throughout the paper. Section 3 will explain how to connect these concepts so that we may reduce the problem of demonstrating the key properties defined in section 1.3 in a given poset to a problem in linear algebra. Finally, section 4 will focus on solving this algebra problem for 3 fundamental types of posets, making use of the ideas presented in section 3.

## 1.1 Posets

A poset is a set which has a notion of order among its elements. That is, a poset consists of two parts: the set of elements, and the way we order those elements. An example of a normal set would be the set $P = \{1, 2, 3, 4, 5\}$. An example of a poset would be the set $P$ along with the fact that for this specific set, we choose to order elements by their size. That is, we establish that the element 1 comes before the element 2, 2 comes before 3, etc.

However, this is intuitive, since, if we order the integers from 1 to 5, 1 comes before 2, 2 comes before 3, and so on. Why are posets useful other than for stating the obvious? The answer becomes clear when we consider sets whose elements are not just numbers. For example, consider the power set $P = \mathcal{P}(\{1, 2, 3\})$ (the set of subsets of the set $\{1,\ 2,\ 3\}$). Its elements are sets themselves, not numbers, meaning there is much less of a sense of order in this set than with the set $\{1, 2, 3, 4, 5\}$. To make $P$ into a poset, we need to choose to order these elements somehow. One way to do this is to establish that for two elements $x, y \in P$, if $x \subseteq y$, then x comes before y in the order. A poset whose set of elements is a power set and whose binary relation is inclusion is called a Boolean Poset, sometimes referred to as a Boolean Algebra or a Boolean Lattice. We will discuss these posets in more detail in section 2.1.

Note that to establish our notion of order in $P$, all we need to do is provide the binary relation " $\subseteq$ ", as it is what we use to determine if one element comes before another in the ordering. Our binary relation is a partial order, which means it satisfies three properties: Reflexivity, Antisymmetry, and Transitivity. Along with these three terms, a partial order is defined as follows:

**Definition 1.** *A partial order is a binary relation, denoted by $\leq$, that has the following properties, for all $x, y, z$:*

1. *Reflexivity: $x \leq x$.*
2. *Antisymmetry: If $x \leq y$ and $y \leq x$, then $x = y$.*
3. *Transitivity: If $x \leq y$ and $y \leq z$, then $x \leq z$.*

Thus, we have the following definition for a poset:

**Definition 2.** *A poset is a set together with a binary relation which is reflexive, antisymmetric, and transitive [3]. The binary relation is denoted by " $\leq$ ".*

With notation for the binary relation in mind, we briefly discuss the notion of an element covering another.

**Definition 3.** *For two elements $x$ and $y$ in a poset $P$, we say that $y$ covers $x$ if $x \leq y$ and there exists no element $z \in P$ such that $x \leq z \leq y$. We denote the relation "$y$ covers $x$" as $x \lessdot y$.*

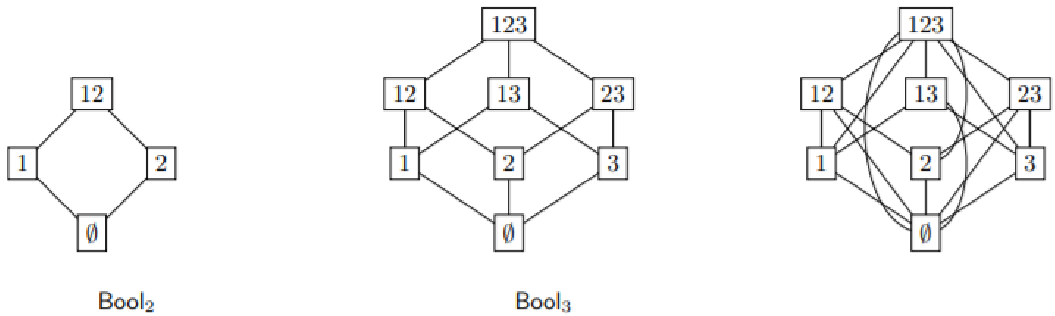


Figure 1: The figure shows the Hasse Diagrams for 2 Boolean posets: $B_2$, $B_3$, and the implied connections (ordered by inclusion) of $B_3$

Reflexivity, antisymmetry, and transitivity will be laid out in the next section of the paper. The binary relation is called a *partial order* to indicate that not every pair of elements needs to be comparable. That is, there may be pairs of elements for which neither element precedes the other in the poset. For example, referring back to $P = \mathcal{P}(1, 2, 3)$, consider the elements $x = \{1\}$ and $y = \{2\}$. Clearly $x \not\subset y$ and $y \not\subset x$. We say that x and y are not comparable, which means that neither element comes before or after each other in the ordering.

We now discuss how to visualize posets. A normal set is visualized simply as a collection of elements. But now that our elements have a notion of order, how can we incorporate this into visualizing them? The answer is through the use of Hasse Diagrams. Hasse Diagrams are graphs whose vertices are the elements of the poset and whose edges represent the covering relations, which are enough to generate all the relations in the poset by transitivity. Examples of Hasse Diagrams are depicted above. Note that for any two elements connected by a line, it is implied that the upper element covers the lower one.

With the ideas of comparability, incomparability, and Hasse Diagrams in mind, we introduce the concept of chains and antichans.

**Definition 4.** *A subset $C \subseteq P($ or $P$ itself $)$ is called a* **chain** *if all of its elements are pairwise comparable.*

Thus every chain is of the form $C = \{x_0, \ldots, x_n\}$, where $x_0 \leq \ldots \leq x_n$ where number $n$ is the length of the chain (note that the length is one less than the cardinality of the chain). This brings us to the idea of a saturated chain:

**Definition 5.** *Let $C$ be a length $n$ chain $\{x_0, \ldots, x_n\}$ of a poset $P$. $C$ is called a* saturated chain *if $x_0 \lessdot \cdots \lessdot x_n$, where $x \lessdot y$ denotes that element $y$ covers element $x$.*

But not all subsets of a poset will be chains. For example, what happens when instead no two elements can be compared at all? This brings us to the definition of an antichain:

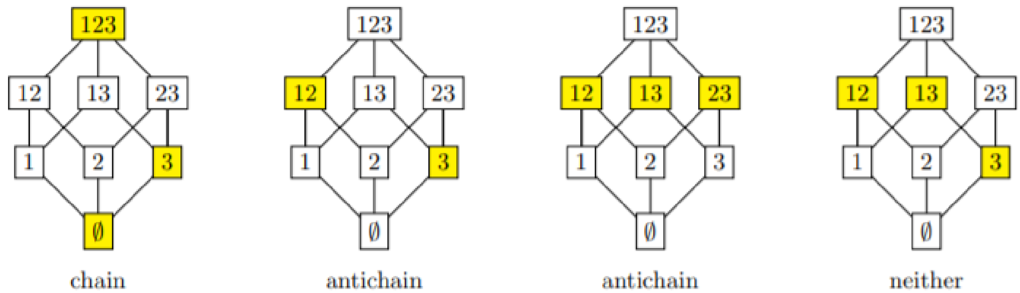


Figure 2: The figure shows examples of a chain, two antichains and a configuration which is not considered either of them, all of the same poset Bool $_3$.

**Definition 6.** *An antichain is a subset of $P$ (or, again, $P$ itself) in which no two of its elements are comparable.*

For example, in the Boolean algebra $B_3$, {{1,2},{3}} and {{1, 2}, {1, 3}, {2, 3}} are antichains. On the other hand, the subset $\{\emptyset, \{3\}, \{1, 2, 3\}\}$ is a chain of length 2 (note that it is not saturated). While we have discussed chains and antichains, is it possible for a subset of a poset to be neither? The answer is yes, as the subset {{1, 2}, {1, 3}, {3}} is neither a chain nor an antichain: {1, 3} is comparable to {3} but not to {1, 2}. These examples are shown in Figure 1.1.

## 1.2 Ranked Posets

Before we are able to understand properties such as the Sperner Property, rank unimodality, and rank symmetry, we must first establish the notion of *rank* in a poset. Consider our first poset from the previous subsection, $P = \{1, 2, 3, 4, 5\}$ where $P$ is ordered by the size of the elements. Our elements are clearly ordered $1 < 2 < 3 < 4 < 5$. If we were to classify each element by its position in the order, we would say that the element 1 is the "lowest" in the order and the element 5 is the "highest". The rest of the elements are somewhere in between, with each element's position becoming higher and higher as we move to the next element to the right. Suppose we assign a number to each element as a way of quantifying its position. That is, say we assign a low number to elements which are "low" in the order (like the element 1 or 2) and a high number to elements which are "high" in the order (like the element 4 or 5).

A natural way to classify each element's position in the ordering would be to assign the number 0 to the lowest element, 1 to the element immediately succeeding that element, 2 to the element succeeding that one, until we reach the "highest" element in our poset and assign it the number n - 1. This means that for our poset $P$, the elements 1, 2, 3, 4, and 5 are respectively assigned the numbers 0, 1, 2, 3, and 4. This number which quantifies the position of an element in the

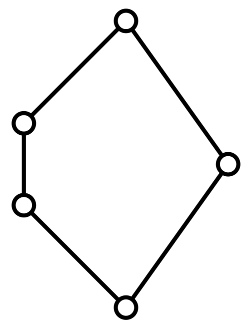

Figure 3: The figure shows examples of a non-ranked poset, where the "left" saturated chain containing the top and bottom elements has a different length than that on the right.

ordering of a poset is what we call the rank of the element. Since each element in our poset $P$ can be assigned a rank, we say that $P$ is a *ranked poset*.

How can we apply the idea of rank to a poset like the Boolean Algebra? It seems as if the only feasible way for a poset to be ranked is for all elements in the poset to form one large, saturated chain like in $P$. In an arbitrary poset, there are an arbitrary number of saturated chains which contain a given element x. For example, considering $B_3$, we see that the element $\{1\}$ appears in the saturated chains $\{\emptyset, \{1\}, \{1,2\}, \{1,2,3\}\}$ and $\{\emptyset, \{1\}, \{1,3\}, \{1,2,3\}\}$. But note that in both of these chains, the element $\{1\}$ appears in the same position (position 2). The natural course of action is to assign the rank of $\{1\}$ to be the rank relative to either of these chains. Since the element is in the second position of both chains, we assign the rank of $\{1\}$ to be 1.

For $B_3$, we can continue this process for all elements to find that $\emptyset$ has rank 0, $\{1\}, \{2\}$, and $\{3\}$ all have rank 1, $\{1,2\}, \{1,3\}$, and $\{2,3\}$ each have rank 2, and $\{1,2,3\}$ has rank 3. We see that multiple elements can have the same rank. It becomes useful to group elements of like rank with each other into subsets, creating *rank levels* of the poset $P$, where the $i^{th}$ rank level is denoted $P_i$. For $P = B_3$, we thus have that $P_0 = \{\emptyset\}$, $P_1 = \{\{1\}, \{2\}, \{3\}\}$, $P_2 = \{\{1,2\}, \{1,3\} \{2,3\}\}$, and $P_3 = \{\{1,2,3\}\}$.

It is important to note that not all posets can be ranked. That is, it is not always possible to assign exactly one number to quantify the position of a given element in the ordering of a poset $P$. For example, in Figure 3, we see that there are two saturated chains which contain both the top and bottom elements, but they have different lengths. This means that the poset cannot be ranked.

We thus give the following formal definitions for rank, ranked posets, and rank levels:

**Definition 7.** *The rank of an element $x$ of the poset $P$ is the output of the rank function $\rho(x)$, where $\rho$ respects the covering relation. That is, we have that for $x, y \in P$, if $x \lessdot y$, then $\rho(y) = \rho(x) + 1$.*

**Definition 8.** *A ranked poset is a poset which has the following property: For any element x, consider all saturated chains whose element with the highest position is x. All of these chains have the same length.*

**Definition 9.** *The $i^{th}$ rank level of the ranked poset $P$, denoted $P_i$, is the collection of all elements in $P$ whose rank is $i$.*

## 1.3 Key Properties of Posets

Now that we have established the notion of rank in a poset, we can define key properties of posets with which we will be working extensively in the rest of the paper.

Our first property is that of rank unimodality, and it pertains to the sizes of each rank level in the poset. As previously defined, the $i^{th}$ rank level $P_i$ of a ranked poset $P$ is the collection of all elements of $P$ with rank $i$. If we consider $B_3$, we see that the $0^{th}$ rank level only has 1 element in it. The first and second rank levels have 3 elements, and the third rank level has 1 element. In other words, the size of the rank level increases as rank increases up to a given value, then after that value, as we increase rank, the size of the rank levels decreases. A ranked poset which possesses this property is said to be *rank unimodal.* Another way to view rank unimodality is that the Hasse Diagram for a ranked poset becomes "fatter" as the rank increases to a given point, and it becomes "thinner" as the rank increases beyond that point. We observe that this is indeed the case for the Boolean Poset $B_3$.

We thus give the following definition for rank unimodality:

**Definition 10.** *A poset $P$ is rank unimodal if for some rank value $k$, $|P_i| \leq |P_{i+1}|$ for $i < k$ and $|P_i| \geq |P_{i+1}|$ for $i > k$.*

Our next key property of posets is that of rank symmetry, which also pertains to the sizes of each rank level in the poset. Referring to $B_3$ again, we see that the 0th and 3rd rank levels are the same size, and so are the 1st and 2nd rank levels. This makes the Hasse Diagram horizontally symmetric. We refer to this symmetry in $B_3$ as *rank symmetry.* It is formally defined as follows:

**Definition 11.** *A poset $P$ is rank symmetric if we have $|P_i| = |P_{n-i}|$, where $n$ is the largest rank of the poset and is often referred to as the rank of the poset itself.*

Our final and most important property of posets is the Sperner Property. The Sperner Property pertains to the sizes of not only the rank levels, but to the antichains as well.

As we have noted, the sizes of the $0^{th}$, $1^{st}$, $2^{nd}$, and $3^{rd}$ rank levels in $B_n$ are 1, 3, 3, and 1, respectively. Now let us consider the different antichains in $B_3$ and determine their sizes. To do this, we first propose the following corollary:

**Corollary 0.1.** *For any ranked poset $P$, each rank level $P_i$ is an antichain as well.*

*Proof.* Assume for the sake of contradiction that two elements in a given rank level are comparable. Then one element would have a higher rank than the other, meaning they could not be in the same rank level. This is a contradiction, so we conclude that each rank level in any ranked poset is also an antichain, as desired. □

Thus, we already know the sizes of 4 antichains in $B_3$. Upon further inspection, we see that the only other antichains in $B_3$ must be of length no more than 2. To see why, consider the elements $\emptyset$ and $\{1, 2, 3\}$. Any antichain containing either of those two elements would just be the element itself, as those elements are comparable with every other element of $B_3$. Since we are now left to consider the 6 elements in the middle two ranks, we consider possible antichains created from elements in both the 1st and 2nd rank levels. As soon as we select one element from one rank level, we can only select one more, as each element in the middle two rank levels either cover or are covered by 2 elements in the other rank.

The point of computing the possible lengths of antichains of $B_3$ in this way is to show that the maximum size of any antichain is the same as the maximum size of any rank level. This means that the poset $B_3$ has the *Sperner Property.* We formally define the Sperner Property as follows:

**Definition 12.** *A poset $P$ has the Sperner Property if the maximum size of all rank levels in $P$ equals the maximum length of all antichains in $P$. That is, $max_i|P_i| = max_{a\in A}|a|$, where $A$ is the set of all antichains in $P$.*

### 1.4 Applying Algebra to These Properties

Our focus for the rest of the paper is to demonstrate these properties in specific posets by making use of algebraic methods. Upon first glance, it is not clear how these properties can be demonstrated through the use of algebra rather than combinatorics alone. As mentioned previously, we will use algebraic methods to convert a problem of mapping elements in one set to another to that of creating a matrix to represent a mapping from one vector space to another. As it turns out, one can demonstrate both rank unimodality and the Sperner Property in a poset $P$ by creating mappings between rank levels in $P$, as will be elaborated in section 3. The next section serves to define preliminary concepts that are essential to understanding how we will convert the problem of demonstrating these properties in a given poset to a problem of linear algebra.

## 2 Preliminaries

### 2.1 The Boolean Poset

Let $[n] = \{1, 2, \ldots, n\}$ (a standard piece of notation in combinatorics) and let $2^{[n]}$ be the power set of $[n]$. We can partially order $2^{[n]}$ by writing $S \leq T$ if $S \subseteq T$. A poset isomorphic to $2^{[n]}$ is called a Boolean algebra of rank $n$, denoted here by the symbol $B_n$. We may also use $B_S$ for the Boolean Poset (or Boolean Algebra or Boolean Lattice) of subsets of any finite set $S$; clearly $B_S \cong B_n$. The cardinality of $S$ is called the rank of $B_S$; it is not hard to see that every Boolean algebra is determined up to isomorphism by its rank [2]. Figure 4 below represents the Hasse Digaram of $B_3$.

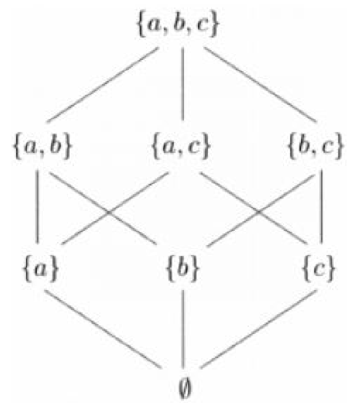


Figure 4: The Hasse diagram of the set of all subsets of a three-element set $a, b, c$, ordered by inclusion. Distinct sets on the same horizontal level are incomparable with each other. Some other pairs, such as $a$ and $b, c$, are also incomparable. [1]

## 2.2 Posets as Vector Spaces

In this subsection, we introduce the notion of treating the elements of a given poset $P$ as the basis of a vector space $\mathbb{R}P$, where $\mathbb{R}P$ denotes the vector space consisting of linear combinations of elements of $P$ over the field $\mathbb{R}$. This concept is one of the primary ideas behind combining linear algebra with combinatorics, and it will be used in the coming sections.

To treat our poset $P$ as the basis of a vector space, we need to treat each of the elements of $P$ as basis vectors. As a concrete example, we will consider the poset $P = B_3$ and create a basis for $\mathbb{R}P = \mathbb{R}B_3$. Since $B_3$ has 8 elements, to turn it into a basis for a vector space, all we need to do is map each element of $B_3$ to a basis vector of $\mathbb{R}^8$. The basis vectors are the eight length-8 vectors $(1, 0, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, 0, 0, \ldots, 1)$. As for which elements to map to which vectors, this is completely arbitrary so long as the mapping is bijective.

Now that we have demonstrated how to represent the elements of a poset as a basis for a vector space, we now explain what the span of these new basis vectors represents. The span of any basis vectors gives the set of all possible linear combinations of those basis vectors. But we just established that the basis vectors in $\mathbb{R}B_3$ each correspond to a member of $B_3$, all of which are sets. This means that the notion of taking linear combinations of its elements is very abstract. For example, when $P = B_3$, how would one evaluate $2 \cdot \emptyset + 3 \cdot \{1\} - \{1, 2\}$?

The answer to this question is to treat each element of $B_3$ as a vector. Suppose that we established that $\emptyset \to (1, 0, 0, \ldots, 0), \{1\} \to (0, 1, 0, \ldots, 0)$, and $\{1, 2\} \to (0, 0, 1, \ldots, 0)$. Then the expression $2 \cdot \emptyset + 3 \cdot \{1\} - \{1, 2\}$ becomes $2 \cdot (1, 0, 0, \ldots, 0) + 3 \cdot (0, 1, 0, \ldots, 0) - (0, 0, 1, \ldots, 0) = (2, 3, -1, \ldots, 0)$.

Why is it useful to consider linear combinations of elements in a poset rather than the elements themselves? As mentioned before, this is a rather abstract concept. The reason will become very clear in section 3. Section 3.1 will introduce a lemma which can be used to demonstrate the Sperner Property in a poset using only methods in combinatorics. Section 3.2 will introduce a theorem which makes use of this concept of linear combinations of poset elements to convert the combinatorics methods into algebraic methods, thus converting the problem into an algebra problem and simplifying it.

# 3 Proof Framework

In the rest of this paper, we present proofs for the rank-unimodality and Sperner property of various posets (classes of posets). These proofs follow a common framework, which we present in this section.

## 3.1 Order-matchings

First, we will define a function for each poset called an order-matching and show that if certain properties hold for this function, then the poset is rank-unimodal and Sperner.

The existence of an order matching between any two consecutive levels is a useful working condition which implies that a rank unimodal poset $P$ is Sperner. We define an order matching as follows. Let $\mu : P_i \to P_{i+1}$ be a one-to-one mapping between consecutive rank levels $P_i$ and $P_{i+1}$. If $\mu$ maps an element $x \in P_i$ to an element $y \in P_{i+1}$ such that y covers x, then $\mu$ is an order matching. We also have that $\mu$ is an order matching if $\mu$ injectively maps $P_{i+1}$ to $P_i$ and $\mu$ maps each element $y \in P_{i+1}$ to an element $x \in P_i$ which is covered by y. We thus give the following formal definition for an order matching:

**Definition 13.** *An order matching $\mu$ is a one-to-one mapping where either $\mu : P_i \to P_{i+1}$ and $x \leq \mu(x)$, or $\mu : P_{i+1} \to P_i$ and $\mu(x) \leq x$.*

Note how in both versions of $\mu$, $\mu(x)$ is always comparable to x. This means that $\mu$ *respects the order* of $P$. Using order matchings, we have the following simple lemma related to the Sperner Property:

**Lemma 1.** *Suppose that in the poset $P$ there exist order matchings $P_0 \to P_1 \to ... \to P_k \leftarrow P_{k+1} \leftarrow ... \leftarrow P_n$. Then $P$ is rank unimodal and Sperner, with $p_j = \max_i p_i$*

*Proof.* The unimodality property is clear from the definition of an order matching. The order matchings between successive ranks give rise to a partition of $P$ into (disjoint) chains, each of which intersects $P_k$. Therefore, the number of chains is $p_k$. On the other hand, every antichain $A$ intersects each chain in at most one point; hence, $|A| \leq p_k$, so $P$ is Sperner. □

This lemma will serve as the basis for using algebraic methods to demonstrate the Sperner Property in posets. Before we move on to these methods, we first introduce a different way to phrase the lemma, as well as some alternative notation. For any mapping $f : A \to B$, if f is one-to-one, then $f^{-1} : B \to A$ is onto. Therefore, the existence of a one-to-one order matching $\mu : P_{i+1} \to P_i$ is equivalent to the existence of an onto order matching $\mu' : P_i \to P_{i+1}$. We thus introduce a new lemma equivalent to Lemma 4.

**Lemma 2.** *Suppose that in the poset $P$ there exist one-to-one order matchings $P_0 \to P_1 \to \cdots \to P_k$ as well as onto order-matchings $P_k \to P_{k+1} \to \cdots \to P_n$. Then $P$ is rank unimodal and Sperner, with $|P_k| = p_k = max_i p_i$.*

Before we discuss algebraic methods to apply to this lemma, it is useful to introduce notation for dealing with one-to-one and onto mappings. Suppose we have a mapping $f : A \to B$ which is one-to-one. We can specify that f is one-to-one by using the $\hookrightarrow$ symbol rather than the $\to$ symbol. Thus, if we see that $f : A \hookrightarrow B$, we know that f injectively maps A to B. Likewise, if we want to specify that f is onto instead, we use the notation $f : A \twoheadrightarrow B$. Putting this notation in action gives us the following new lemma equivalent to both Lemma 4 and Lemma 5:

**Lemma 3.** *Suppose that in the poset $P$ there exist order matchings $P_0 \hookrightarrow P_1 \hookrightarrow \cdots \hookrightarrow P_k$ as well as order matchings $P_k \twoheadrightarrow P_{k+1} \twoheadrightarrow \cdots \twoheadrightarrow P_n$. Then $P$ is rank unimodal and Sperner, with $|P_k| = p_k = max_i p_i$.*

## 3.2 Connecting Order-Matchings to Linear Algebra

Consider the order matchings in Lemma 3. They map a given element in the poset to exactly one element which covers it. Finding such a series of mappings is not too challenging for $B_3$. But for even a poset such as $B_6$, such a series of mappings becomes hard to find. It becomes even harder to generalize these mappings to the general case $B_n$.

The reason that these mappings are so hard to find is because the conditions are somewhat restrictive. When choosing a mapping, we need to map an element to exactly one element which covers it, all while making sure that the mappings are injective and surjective at the correct times. What would be more convenient would be the ability to map a given element to several elements which cover it rather than just one, as this would give us more freedom when choosing our mappings.

Using the preliminary ideas from section 2.2, we can do exactly that. For every order matching we need to find, we can instead find a linear operator which maps the *vector representation* of an element to a *linear combination* of the vector representation of multiple elements which cover it. This is what will give us more freedom when choosing our mappings.

We will define a theorem to connect order matchings with linear operators. But first, we must define what an order raising operator is. An order raising operator is a mapping $U : \mathbb{R}P_i \to \mathbb{R}P_{i+1}$ such that $\forall x \in P_i,\ \ U(x) = y$ and $y \in \mathbb{R}C^+(x)$. Here we have $C^+(x)$ representing the set of all elements in $P$ which cover element $x$, making $\mathbb{R}C^+(x)$ the set of all linear combination of these elements. Similarly, we define $C^-(x)$ to be the set of elements in $P$ which are covered by $x$.

We combine the previous result with our new notions to get the following theorem:

**Theorem 1.** *If* $\exists\, U$ *where* $U : \mathbb{R}P_i \hookrightarrow \mathbb{R}P_{i+1}\ \forall\, i \leq k$ *and* $U : \mathbb{R}P_i \twoheadrightarrow \mathbb{R}P_{i+1}\ \forall i > k$ *for some* $k$*, and* $U$ *is order raising, then* $\exists$ *an order matching* $\mu$ *with the properties in Lemma 3.*

*Proof.* Note that we will only prove the case where $i < k$. That is, we will demonstrate that if $\exists\, U : \mathbb{R}P_i \hookrightarrow \mathbb{R}P_{i+1}$ for $i < k$, then $\exists\, \mu : P_i \hookrightarrow P_{i+1}$ for $i < k$, for some k. The case where $i > k$ is analogous and uses very similar reasoning.

U is a linear transformation, meaning that we can represent $U$ as a matrix. The matrix $U$ maps vectors from $\mathbb{R}P_i$ to $\mathbb{R}P_{i+1}$, whose bases are $P_i$ and $P_{i+1}$, respectively. Therefore, our matrix has $|P_{i+1}| = p_{i+1}$ rows and $|P_i| = p_i$ columns. Since $U$ is an injective mapping, $U$ is of full rank, meaning $rank(U) = p_i$.

This means that our matrix $U$ has $p_i$ independent rows as well as $p_i$ independent columns. We define $A$ to be the $p_i$ x $p_i$ matrix whose rows are these independent $p_i$ rows of $U$. By definition, this makes all of the rows of $A$ independent, meaning it is non-singular. Equivalently, the determinant of $A$ is nonzero, and $A$ is invertible.

Now let us write out the determinant of $A$ in terms of its elements. Let $a_{i,j}$ be the element of $A$ in row $i$ and column $j$. Normally, one expresses the determinant as a recursive formula containing determinants of submatrices of the original matrix. However, we wish to express the determinant in terms of the elements alone. This is done with the formula

$$det(A) = \textstyle\sum_\pi a_{\pi(1),1} ... a_{\pi(p_i),p_i},$$

where our sum is across $\pi$, the set of permutations of the set $\{1, \ldots, p_i\}$. Note that since $det(A) \neq 0$, at least one of the terms in our summation must be nonzero. In other words, for some permutation $\pi$, we have $a_{\pi(1),1} \cdot ... \cdot a_{\pi(p_i),p_i} \neq 0$, which means that each of the terms in the product must also be nonzero.

Recall that since U is order raising, an element $x \in P_i$ is mapped to a linear combination of elements which cover it. Because each matrix element $a_{\pi(j),j}$ is nonzero, this means that the poset element $x_j \in P_i$ is covered by the poset element $y_{\pi(j)} \in P_{i+1}$. Therefore, we choose our mapping $\mu$ to map $x_i$ to $y_{\pi(j)}$, and we have therefore shown that $\mu$ exists, as desired.

□

# 4 Algebraic Methods in Action

## 4.1 Boolean Posets

We will first apply this result to the boolean algebra. Again, although an explicit order matching for $B_n$ can be exhibited, this is not straightforward. In the case of our approach, it will suffice to map each element of $B_n$ of rank $k, k < \frac{n}{2}$, to a linear combination of all the elements which cover it. This is one of the instances where linear algebra will simplify the work for us by supplying an order matching. This is done by ensuring that in the linear combinations associated with different elements of rank $k$, a different element has nonzero coefficient. We need to prove only that the linear transformation is one-to-one if $k < \frac{n}{2}$ and onto if $k \geq \frac{n}{2}$.

Take $U : \mathbb{R}B_n \to \mathbb{R}B_n$ defined by $U(S) = \Sigma_{T \in C^{\cdot}(S)} T$; that is, to each subset $S \in B_n$ we associate the sum of all the subsets which cover it. Let $U_j = U|_{(B_m)_j}$, i.e., $U_j$ is the restriction of $U$ to the $j$ th rank of $B_n$

**Theorem 2.** *With the notation established above, if $k < \frac{n}{2}$, then $U_k$ is one-to-one, and dually, if $k \geq \frac{n}{2}$, then $U_k$ is onto.*

The second part of the theorem follows from the first because $B_n$ is self-dual. That is, there is a bijection $f : B_n \to B_n$, such that $S \subseteq T$ implies $f(T) \subseteq f(S)$; the complementation map serves as $f$.

In linear algebra terms, we can restate this theorem as: the incidence matrix between the $k$-element subsets and the $(k+1)$-element subsets of an $n$-element set has full rank. We present below one of the various proofs that exist for it.

*Proof.* Define a second linear transformation, $D_j : \mathbb{R}(B_n)_j \to \mathbb{R}(B_n)_{j-1}$ ($D$ for down), which maps a subset to the sum of the subsets covered by it and so is dual to $U_j$; namely , $D_j(S) = \sum_{T \in C^{-}(S)} T$, for each $S \in (B_n)_j$. The main detail for this proof is that $U_j$ and $D_{j+1}$ are adjoints (with respect to the bases $P_j$ and $P_{j+1}$ ) since their matrices (with respect to these bases) are transposes of one another. We claim that for each $k$

$$D_{k+1}U_k - U_{k-1}D_k = (n-2k)I_k$$

where $I_k$ is the $p_k$ by $p_k$ identity matrix, and the lincar transformations are multiplied from right to left. We apply the left-hand side to a generic $k$-element set $S$ and single out the coefficient of an arbitrary but fixed set $S' \in (B_n)_k$ in the resulting linear combination of $k$-element subsets. The set $S'$ will have coefficient equal to the number of ways in which it can be obtained from $S$ by first adjoining and then deleting an element, minus the number of ways in which it can be obtained from $S$ by first deleting and then adjoining an element. There are two main situations when this is possible: either $S = S'$, or $S$ and $S'$ have $k-1$ elements in common.

Since there are $n-k$ possibilities for an element to be adjoined to $S$ and then removed, and $k$ possibilities for an element to be removed from $S$ and then added back, in the first situation the coefficient of $S'$ is $(n-k)-k = n-2k$. In the second situation the coefficient is $1-1=0$ because the element to be adjoined and then deleted as well as the element to be deleted and then adjoined are completely determined by $S$ and $S'$. Hence, the claim is true.

Now, because $U_{k-1}$ is the adjoint of $D_k$, the product $U_{k-1}D_k$ is a positive semidefinite matrix, which has only nonnegative eigenvalues. If $k < \frac{n}{2}$, then the matrix $(n-2k)I_k$ is positive definite, so the sum $U_{k-1}D_k + (n-2k)I_k$ has only positive eigenvalues and therefore it is invertible. But, by the claim above, this expression equals $D_{k+1}U_k$. Finally, if the composition of the two operators is invertible, then the first one is one-to-one and the second is onto. Consequently, if $k < \frac{n}{2}$, then

$U_k$ is one-to-one, completing the proof of the first part of the theorem. A dual argument yields the existence of matchings between successive ranks above the middle rank of $B_n$. $\square$

## 4.2 Subspace Posets

The next poset which we will analyze is what is known as the subspace poset or subspace lattice, denoted $L_n(q)$. This poset is the collection of subspaces of a vector space of dimension n defined over the field $\mathbb{F}_q$. The elements are ordered by inclusion, i.e., if A is a subspace of B, then $A \leq B$. $L_n(q)$ is graded, with the rank of an element (subspace) being its dimension.

We first note that $L_n(1)$ is exactly $B_n$. That is, we can view the subspace lattice over the field $\mathbb{F}_q$ as a generalization of the Boolean Lattice. Specifically, we say that $L_n(q)$ is the q-analogue of $B_n$. As a result, we can expect to see many parallels between $B_n$ and $L_n(q)$. An example of this would be determining the size of each rank level. For $B_n$, the size of the $i^{th}$ rank level is $\frac{n\cdot(n-1)\cdot\dots\cdot(n-k+1)}{k\cdot(k-1)\cdot\dots\cdot 1}$, which is known as a binomial coefficient and is denoted as $\binom{n}{i}$. For $L_n(q)$, the size of the $i^{th}$ rank level is $\frac{(q^n-1)\cdot(q^n-q)\cdot\dots\cdot(q^n-q^{i-1})}{(q^i-1)\cdot(q^i-q)\cdot\dots\cdot(q^i-q^{i-1})}$, which is known as a q-binomial coefficient and is denoted as $\begin{bmatrix} n \\ i \end{bmatrix}_q$. Upon further evaluation, one can find that $\lim_{q\to 1} \begin{bmatrix} n \\ i \end{bmatrix}_q = \binom{n}{i}$.

Given the parallels between the Boolean and Subspace Lattices, it is natural to wonder whether the Boolean Lattice shares its properties of rank unimodality, rank symmetry and the Sperner Property with its q-analogue $L_n(q)$. The answer turns out to be yes, and just like with the Boolean Lattice, we will use algebraic methods to demonstrate this fact. However, we keep in mind that while using algebraic methods has allowed us to demonstrate the Sperner Property in posets relatively easily, this is not the only reason to use such methods. Regarding the Subspace Lattice, we wish to emphasize the connections between this poset and the Boolean Lattice, along with how using algebraic methods can make use of and emphasize these connections.

As before with the Boolean Lattice, we demonstrate rank unimodality and the Sperner Property in the Subspace Lattice by defining order raising and lowering operators on $\mathbb{R}L_n(q)$ as:

$$U(W) = \sum_{Y \in C^+(W)} Y$$
$$D(W) = \sum_{Y \in C^-(W)} Y$$

We then consider the restriction of U to the $i^{th}$ rank level of $L_n(q)$, $U_i$, and show that $U_i$ is one-to-one for $i < n/2$ and onto for $i > n/2$. By Theorem 3.2, we will have shown that $L_n(q)$ is both Sperner and rank unimodal. As for rank symmetry, as with the Boolean Lattice, this is trivial, as $\begin{bmatrix} n \\ i \end{bmatrix}_q = \begin{bmatrix} n \\ n-i \end{bmatrix}_q$. That is, the size of the $i^{th}$ rank level of $L_n(q)$ is equal to that of the (n - i)th rank level.

To demonstrate the Sperner Property and rank unimodality, we will only demonstrate that $U_i$ is one to one for $i < n/2$, as proving that $U_i$ is onto for $i > n/2$ is analogous. For similar reasons to the case of $B_n$, we see that $D_{i+1} \cdot U_i - U_{i-1} \cdot D_i = c \cdot I_i$, where c is a constant and $D_i$ is the restriction of $D$ to the $i^{th}$ rank level of $L_n(q)$. If we consider a given i-dimensional vector field W, c represents the difference of the number of distinct $(i+1)$-dimensional vector spaces which contain $W$ and the number of distinct $(i-1)$.-dimensional vector spaces contained in $W$.

Our claim is that $c = [n-i]_q - [i]_q$, where $[i]_q = \frac{q^i-1}{q-1}$. To see why, we consider the i-dimensional vector space $W$. The number of vectors we can add to $W$ to make it a $(i+1)$-dimensional vector space is $q^n - q^i$, as there are $q^n$ total vectors and adding the $q^i$ vectors in $W$ does not change the dimension. However, we note that the same subspace can be spanned by different vectors. That is, of the $q^n - q^i$ vector spaces we just created by adding a new independent vector, we have created many repeat vector spaces. As it turns out, for a given W, there are exactly $q^{i+1} - q^i$ vectors we can add to create the same $(i+1)$-dimensional vector space. Thus, the number of distinct $(i+1)$-dimensional subspaces which contain $W$ is $\frac{q^n-q^i}{q^{i+1}-q^i} = [n-i]_q$. As for the number of distinct $(i-1)$.-dimensional subspaces contained in W, to create such a subspace, we can choose each of the i - 1 vectors $(q^i-1) \cdot (q^i-q) \cdot ... \cdot (q^i - q^{i-2})$ ways. However, the same subspace has $(q^{i-1}-1) \cdot (q^{i-1}-q) \cdot ... \cdot (q^{i-1}-q^{i-2})$ different bases which span it. Thus, the number of distinct (k-1)-dimensional subspaces contained in $W$ is the quotient of these two values, which gives us $[i]_q$. Thus, we have that c $= [n-i]_q - [i]_q$, as desired.

Proceeding as we did with $B_n$, we see that $D_{i+1} \cdot U_i = ([n-i]_q - [i]_q) \cdot I_i + U_{i-1} \cdot D_i$. Just as with $B_n$, we have that $U_{i-1}$ and $D_i$ are adjoints of each other, meaning as long as $[n-i]_q - [i]_q$ is positive, we have that $U_i$ must be a one-to-one mapping. $[n-i]_q - [i]_q$ is positive when $i < n/2$, meaning we have shown that $U_i$ is one-to-one when $i < n/2$, as desired.

What purpose did it serve to do this work? As mentioned when proving Theorem 7, to show that a poset is rank unimodal and Sperner, we normally create an order matching $\mu$ which maps elements of the poset to each other. By Theorem 7, we do not have to do this, and instead we can just map linear combinations of poset elements to each other. In the case of $B_n$, while it is quite challenging, order matchings have been found. In the case of $L_n(q)$, however, we have yet to find a direct order matching. In other words, using algebraic methods has allowed us to avoid trying to solve a nearly unsolvable problem in demonstrating the Sperner Property and rank unimodality in $L_n(q)$.

We also see that we connected a lot to the proof for $B_n$ being rank unimodal and Sperner when presenting our analogous proof for $L_n(q)$. Upon inspection, we notice that the proof for $B_n$ is the same as the proof for $L_n(q)$ when q = 1. Using Algebraic Methods has allowed us to discover a relationship between $B_n$ and $L_n(q)$ which otherwise is not obvious. Algebraic methods have also shown us that both posets share an intrinsic structure demonstrated by important properties such as rank symmetry, rank unimodality, and the Sperner Property.

## 4.3 Groups and group actions

Groups are one of the most fundamental concepts in modern algebra. In this paper, we will use groups to help us “extend” rank unimodality and the Sperner property from posets for which these properties can be proven directly to additional posets. We will use groups acting on posets to create new, interesting posets.

**Definition 14.** *A group $G$ is a set (collection of elements) equipped with a binary operation, often denoted with the multiplication sign $\cdot$. The group and its operation have to satisfy the following properties:*

- *Closure:* $\forall\ a, b \in G : a \cdot b \in G$.
- *Associativity:* $\forall\ a, b, c \in G : (a \cdot b) \cdot c = a \cdot (b \cdot c)$.
- *Identity:* $\exists e \in G : \forall a \in G, a \cdot e = e \cdot a = a$. *$e$ is unique and is called the identity element.*
- *Inverses:* $\forall a \in G, \exists b \in G : a \cdot b = b \cdot a = e$. *$b$ is called the inverse of $a$ and denoted $a^{-1}$.*

**Definition 15.** *An automorphism of the poset $P$ is a bijective map of a poset onto itself that preserves the order of the elements, i.e.: $x < y \implies f(x) < f(y)$. An automorphism group $G$ of a poset $P$ is a group whose elements are automorphisms of $P$.*

**Definition 16.** *Elements of an automorphism group $G$ of $P$ act on the poset $P$. The set of all elements of $P$ to which a single element $x \in P$ is mapped by the maps in $G$ is the orbit $Gx$: $Gx = \{gx : g \in G\}$. The poset $P$ is partitioned by $G$ into disjoint orbits.*

**Definition 17.** *The elements of a quotient poset $P/G$ are the orbits $Gx$ of elements $x \in P$ under the automorphism group $G$. They are ordered such that an orbit is less than another if some element of the former is less than some element from the latter: $x \leq y \Rightarrow Gx \leq Gy$.*

## 4.4 Quotient Posets

In Theorem 2, we showed that the order-raising operator $U : \mathbb{R}B_n \to \mathbb{R}B_n$ is one-to-one when restricted to the subspace corresponding to a rank level $i < n/2$ and onto when restricted to a rank level $i > n/2$. Applying Lemma 3 and Theorem 1 yielded that the Boolean poset is rank unimodal and Sperner.

We now look at the fixed space of $\mathbb{R}B_n$ under the action of $G$. Define $\mathbb{R}B_n^G$ as:

$$\mathbb{R}B_n^G := \{v \in \mathbb{R}B_n : gv = v \text{ for all } g \in G\}$$

From the definition of the quotient poset $B_n/G$ we can see that the restriction of $U$ to $\mathbb{R}B_n^G$ is an order raising operator on the quotient poset $B_n/G$ (elements just have been grouped together into orbits). We will prove that rank unimodality and the Sperner property also hold for quotient posets $B_n/G$ of the Boolean poset by showing that:

1. There is a subspace of $\mathbb{R}B_n$ that is isomorphic to $\mathbb{R}(B_n/G)$, i.e. there exists a bijection (one-to-one correspondence) between the elements of the two vector spaces that preserves the properties of the vector space (vector addition and scalar multiplication).
2. When we restrict $U$ to this subspace of $\mathbb{R}B_n$, it is still one-to-one when restricted to a rank level $i < n/2$ and onto when restricted to a rank level $i > n/2$.

As before, the order-raising operator $U : \mathbb{R}B_n \to \mathbb{R}B_n$ maps each element of $B_n$ to the sum of all elements that cover it: $U(x) = \sum_{y \in C^+(x)} y$.

In addition, the action of the group $G$ on $B_n$ can be extended by linearity to all of $\mathbb{R}B_n$. For any element $\sum a_x x \in \mathbb{R}B_n$ (where $a_x$ are real coefficients) and any element $g \in G$, we define:

$$g \cdot \sum a_x x = \sum a_x (g \cdot x)$$

With these definitions, we can prove a lemma that will be needed in our main theorem:

**Lemma 4.** *The order-raising operator $U$ commutes with the action of $G$:*

$$\forall v \in \mathbb{R}B_n, \forall g \in G : U(gx) = g(U(x)).$$

*Proof.* Since $B_n$ is a basis for $\mathbb{R}B_n$ and $U$ is linear, it suffices to verify this fact for a single element $x \in B_n$.

If $x \in B_n$ and $g \in G$, it directly follows from the definition above that $C^+(gx) = g(C^+(x))$, that is, the sum of the elements covering $gx$ is the same as the result of applying $g$ to the sum of the elements covering $x$.

We can combine this equality with the definition of $U$ applied to $gx$:

$$U(gx) = \sum_{y \in C^+(gx)} y = \sum_{y \in g(C^+(x))} y$$

If we substitute $y$ for $gz$, where $z = g^{-1}y$, we get the desired relation:

$$\sum_{y \in g(C^+(x))} y = \sum_{gz \in g(C^+(x))} gz = \sum_{z \in C^+(x)} gz = g \cdot \sum_{z \in C^+(x)} z = g(U(x))$$

□

**Theorem 3.** *If $G$ is an automorphism group of $B_n$, then the quotient poset $B_n/G$ is rank unimodal and Sperner.*

*Proof.* As we outlined above, first we need to find a subspace of $\mathbb{R}B_n$ isomorphic to $\mathbb{R}(B_n/G)$. Let us consider the "fixed space" of $\mathbb{R}B_n$ under the action of $G$: $\mathbb{R}B_n^G$. This is the subspace containing the vectors which are not changed when we apply any $g \in G$ to them:

$$\mathbb{R}B_n^G := \{v \in \mathbb{R}B_n : \forall g \in G, gv = v\}$$

By definition, all elements $x \in B_n$ which lie in the same orbit $Gx$ under the action of $G$ must have the same coefficients for any of these vectors $v \in \mathbb{R}B_n^G$. Therefore, a basis for $\mathbb{R}B_n^G$ is

$\{\sum_{y \in Gx} y : Gx \in B_n/G\}$. But note that $\{Gx : Gx \in B_n/G\}$ is trivially a valid basis for $B_n/G$, and these two bases are equivalent under the bijection $Gx \longleftrightarrow \sum_{y \in Gx} y$ that maps each orbit to the sum of its elements and vice versa. Therefore, $\mathbb{R}B_n^G$ and $\mathbb{R}(B_n/G)$ are isomorphic.

Let us turn to proving the desired properties of $\mathbb{R}B_n^G$. By definition, for all $g \in G$ and $v \in \mathbb{R}B_n^G$, $gv = v$. Lemma 4 states that $U$ and $G$ "commute", so $g(U(v)) = U(gv) = U(v)$, which means that $U$ maps $\mathbb{R}B_n^G$ to itself! This means that $U$ is one-to-one when restricted to rank levels $i < n/2$ and onto when restricted to rank levels $i > n/2$, just as for $\mathbb{R}B_n$.

Combining this result with Theorem 1 gives us one-to-one order matchings from rank level 0 up to the middle rank level $n/2$ and from rank level $n$ down to the middle rank level $n/2$. Lemma 1 tells us that this means that the quotient posets $B_n/G$ are rank unimodal and Sperner. $\square$

## 5 Summary and Conclusions

We see that algebraic methods can be used to demonstrate an intricate yet elegant underlying structure for posets, which is by no means obvious. Algebraic methods provide a clear and concise way to demonstrate this structure via the Sperner Property and prevents us from having to create complicated mappings to do so. They reveal and emphasize nontrivial connections between the Boolean and Subspace posets, and combining these methods with methods from modern algebra allows us to generalize these results to more abstract posets such as quotient posets. These methods open many doors to innovation and intuition in the field of combinatorics, which is what makes them so powerful.

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