I. Concluding Topics in Hypothesis Testing

A. Testing Nonlinear Restrictions

1. Format (Wald Strategy):

$$\mathrm{H}_0\text{: } f(\boldsymbol{\beta}) = q \Rightarrow z = \frac{f(\mathbf{b}) - q}{s.e.\left(f(\mathbf{b})\right)} \sim^A N(0,1) \quad \text{; n.b., usually use } t_{n-k}$$

2. Issues:

a) Calculating $f(\mathbf{b})$ given estimates $\mathbf{b}$ is no problem

b) But how do we calculate $V\left(f(\mathbf{b})\right)$ for $f(\mathbf{b})$ nonlinear?

3. Taylor-Series Approximation (Linearization) $\Rightarrow$ "Delta Method"

a) $f(\mathbf{b}) \approx f(\boldsymbol{\beta}) + \nabla_{\boldsymbol{\beta}} f(\boldsymbol{\beta}) \cdot (\mathbf{b} - \boldsymbol{\beta})$; recall $\nabla_{\boldsymbol{\beta}} f(\boldsymbol{\beta}) \equiv \dfrac{\partial f(\boldsymbol{\beta})}{\partial \boldsymbol{\beta}}$ is vector

b) $V\left\{\nabla_{\boldsymbol{\beta}} f(\boldsymbol{\beta}) \cdot (\mathbf{b} - \boldsymbol{\beta})\right\} = \text{"}\left[\nabla_{\boldsymbol{\beta}} f(\boldsymbol{\beta})\right]^2 V(\mathbf{b})\text{"} = \left[\nabla_{\boldsymbol{\beta}} f(\boldsymbol{\beta})\right]' V(\mathbf{b}) \left[\nabla_{\boldsymbol{\beta}} f(\boldsymbol{\beta})\right]$

4. So our test statistic:

$$t = \left\{ \begin{array}{c} \dfrac{f(\mathbf{b}) - q}{\sqrt{\left[\nabla_{\boldsymbol{\beta}} f(\boldsymbol{\beta})\right]' \left[V(\mathbf{b})\right] \left[\nabla_{\boldsymbol{\beta}} f(\boldsymbol{\beta})\right]}} = \dfrac{f(\mathbf{b}) - q}{\sqrt{\left[\dfrac{\partial f(\boldsymbol{\beta})}{\partial \boldsymbol{\beta}}\right]' \left[V(\mathbf{b})\right] \left[\dfrac{\partial f(\boldsymbol{\beta})}{\partial \boldsymbol{\beta}}\right]}} = \\ \dfrac{f(\mathbf{b}) - q}{\sqrt{\left[\dfrac{\partial f(\boldsymbol{\beta})}{\partial \beta_0} \quad \dfrac{\partial f(\boldsymbol{\beta})}{\partial \beta_1} \quad \cdots \quad \dfrac{\partial f(\boldsymbol{\beta})}{\partial \beta_k}\right] \left[V(\mathbf{b})\right] \left[\dfrac{\partial f(\boldsymbol{\beta})}{\partial \beta_0} \quad \dfrac{\partial f(\boldsymbol{\beta})}{\partial \beta_1} \quad \cdots \quad \dfrac{\partial f(\boldsymbol{\beta})}{\partial \beta_k}\right]'}} \end{array} \right\} \sim^A N(0,1) \quad \text{; (often use } t_{n-k}\text{)}$$

5. Joint Nonlinear Hypotheses:

$$\mathrm{H}_0\text{: } f(\boldsymbol{\beta}) = \mathbf{q} \Rightarrow z = \left[f(\mathbf{b}) - \mathbf{q}\right]' \left[V\left(f(\mathbf{b})\right)\right]^{-1} \left[f(\mathbf{b}) - \mathbf{q}\right] \sim^A \chi^2_J \quad \text{; } J = \#\text{ restricts/hypoths}$$

6. Example/Illustration! (See handwritten notes, next page)

$$y_t = \beta_0 + \beta_1 y_{t-1} + \beta_2 X_t + \varepsilon_t$$

let $\beta_0 = \varepsilon_t = 0$,

if $X_t = 0$ & $X_{t+i} = 1 \quad \forall i > 0$

$$\Rightarrow y_{t+1} = \beta_2 X_{t+1}$$

$$y_{t+2} = \beta_1 \beta_2 X_{t+1} + \beta_2 X_{t+2}$$

$$y_{t+3} = \beta_1 (\beta_1 \beta_2 X_{t+1} + \beta_2 X_{t+2}) + \beta_2 X_{t+3}$$

$$y_{t+4} = \beta_1^3 \beta_2 X_{t+1} + \beta_1^2 \beta_2 X_{t+2} + \beta_1 \beta_2 X_{t+3} + \beta_2 X_{t+4}$$

$$y_{t+\infty} = \sum_{i=0}^{\infty} \beta_1^i \beta_2 X_{t+4}$$

$$= \frac{1}{1-\beta_1} \cdot \beta_2 X_{t+4} \quad \text{for } 0 < |\beta_1| < 1$$

$$\text{LR effect of } \hat{X}_t = \frac{\beta_2}{1-\beta_1} \equiv LRE$$

$$V\left(\hat{\beta}_2 \cdot (1-\hat{\beta}_1)^{-1}\right) \approx \left[\frac{\partial LRE}{\partial \beta}\right]' V(\hat{\beta}) \left[\frac{\partial LRE}{\partial \beta}\right]$$

$$\approx \begin{bmatrix} \frac{\hat{\beta}_2}{(1-\hat{\beta}_1)^2} \\ \frac{1}{1-\hat{\beta}_1} \end{bmatrix}' V(\hat{\beta}) \begin{bmatrix} \frac{\beta_2}{(1-\beta_1)^2} \\ \frac{1}{1-\beta_1} \end{bmatrix}$$