Where we were last time:

(A) Properties of Expectations, (Co)Variances, & Conditioning

1) Laws of Iterated Expectations
   \[ x, y, z = \text{R.V.'s} \]
   \[ \begin{align*}
   &a) \quad E(E(x)) = E(x) \\
   &b) \quad E(E(y|x)) = E(y)
   \end{align*} \]

2) Expectation of a sum is sum of the expectations:
   \[ E(\sum_{i=1}^{n} g(x)) = \sum_{i=1}^{n} E(g(x)) \]

3) Expectation of a constant is the constant
   \[ E(a) = a \quad b) E(X|X) = X \]

   "To "give" something renders it constant (if "conditions on that info")"

4) Rob's Rule of Expectations:
   "E slides on through constants, snagging around random variables"
   \[ \begin{align*}
   &a) \quad E(a + bX + cY) = a + bE(X) + cE(Y) \\
   &b) \quad E[(X'X)^{-1}X'[ee']X(X'X)^{-1}] = (X'X)^{-1}X'E[ee']X(X'X)^{-1}
   \end{align*} \]

(B) Conditioning:

1) Conditioning is distributive:
   \[ E(a + bX + cY|X) = E(a|X) + b E(X|X) + c E(Y|X) \]
   \[ = a + bX + c E(Y|X) \]

2) Conditioning More than Once is Redundant:
   \[ E[E(Y|X)|X] = E(Y|X) \quad \text{(on some info!)} \]

3) Conditioning on X renders X constant (see ABa)

4) Conditioning & Expectation sort of undo each other (see ABb, ABa)
(Co)Variances:

1) Rob's Rules of (Co)Variances:
   a) Constants neither vary nor covary.
      \[ V(a) = 0 \quad C(a, X) = 0 \]
   b) "Variance slides through sums, eliminating constants, squaring on random variabes
      and squaring their coefficients, while splitting out two times all the covariances
      times their associated coefficients."
      \[ V(a + bX + cY) = b^2 V(X) + c^2 V(Y) + 2bc \text{Cov}(X, Y) \]
   c) Notice that in matrix notation, with \( X \) a constant
      vector and \( b \) a random vector, this becomes
      \[ V(X' b) = X' V(b) X \]
      \[ \text{"X squared" in matrix-land} \]

   Special Cases of the Rule:
   - \[ V(X + Y) = V(X) + V(Y) + 2 \text{Cov}(X, Y) \]
   - \[ V(X - Y) = V(X) + V(Y) - 2 \text{Cov}(X, Y) \]

2) The Analogous for Covariances (becomes a kind of FOIL):
   \[ \text{Cov}(a + bX + cY, d + eW + fV) = \]
   \[ \text{be Cov}(X, W) + bf \text{Cov}(X, V) + ce \text{Cov}(Y, W) + df \text{Cov}(V, Y) \]

D) Properties of Note Combining the Above:
   1) Any R.V. \( Y \) can be written: \( Y = E(Y|X) + (Y - E(Y|X)) \)
   \[ = E(Y|X) + \tilde{Y} \]
   a) \( \text{Cov}(X, Y) = \text{Cov}(X, E(Y|X)) \)
   b) \( V(Y) = V(E(Y|X)) + V(E(Y|X)) \frac{\sigma^2}{n-1} \]
   \( \Rightarrow TSS = RSS + ESS = ESS + RSS \)
   \( \Rightarrow \text{SSR}^2 + \text{SSE}^2 = \text{SSE} + \text{SSR} \)
   Note: \( \Rightarrow V(\tilde{Y}) \leq V(Y) \) (and a so that \( V(\tilde{Y}) \approx V(Y) \) though that less notable)
I.D. 2. (cont.) Properties that follow from \( E(y) = E(y|x) + (y - E(y|x)) \)

c) \( E(xy) = E(x \cdot E(y|x)) \)
d) \( E(\epsilon) = E(\epsilon|x) = E(\epsilon|y) = 0 \)

\( i) \ E(\epsilon) = E(y - E(y|x)) = E(y) - E(E(y|x)) = E(y) - E(y|x) = 0 \)

\( ii) \ E(\epsilon|x) = E(y - E(y|x)|x) = E(y|x) - E(y|x|x) = E(y|x) - E(y|x) = 0 \)

\( iii) \ E(\epsilon|y) = E(y - E(y|x)|y) = E(y|x) - E(y|x|y) = y - y = 0 \)
e) \( Cov(x, \epsilon) = 0 \); in fact, \( Cov(h(x), \epsilon) = 0 \) \( \forall h(\cdot) \)

\[
E(x\epsilon) - E(x)E(\epsilon) = E(x(y - E(y|x))) - E(x) \cdot 0 \\
= E(x(y) - E(xE(y|x))) \\
= E(x(y)) - E(xE(y)) = 0
\]
f) \( V(\epsilon) = E(V(y|x)) \)

\[
V(y - E(y|x)) = V(y) + V(E(y|x)) - 2Cov(y, E(y|x)) \\
= V(y) + V(E(y|x)) - 2V(E(y|x)) \\
= V(y) - V(E(y|x))
\]

\( \text{I.D.3.} \) For special case where \( E(y|x) \) is linear, i.e., \( E(y|x) = a + bx \)

a) \( a = E(y) - bE(x) \)
b) \( b = \frac{Cov(x, y)}{V(x)} \)

Proof: \( Cov(x, y) = Cov(x, E(y|x)) = Cov(x, a + bx) = bV(x) \Rightarrow b = \frac{Cov(x, y)}{V(x)} \)

d) \( \rho_{xy} = \frac{b_{xy}}{\sqrt{b_{yy} b_{yx}}} = \frac{\sqrt{Cov(x, y)} \cdot \sqrt{Cov(y, x)}}{\sqrt{V(y)} \sqrt{V(x)}} = \sqrt{\frac{Cov(x, y)^2}{V(y)V(x)}} \)

e) \( E(y) = a + bE(x) \); i.e., \( \bar{y} = a + b\bar{x} \)

\( f) \text{"Coefficient of Determination,\" } R^2, = \frac{\text{\text{explained variance}}}{\text{\text{total variance}}} = \frac{E[(y - \bar{y})]^2}{\text{\text{total variance}}} = \frac{\sum(y_i - \bar{y})^2}{\sum(y_i - \bar{y})^2} \)