EXPLICIT FORMAL-THEORETICAL MODELS of INTERDEPENDENCE in TAX COMPETITION and of INTERNATIONAL CONFLICT

Essex Summer School Social Science Data Collection and Analysis (Franzese, Specification, Lecture 7b)

drawn from the joint work of

Robert J. Franzese, Jr., The University of Michigan
Jude C. Hays, The University of Illinois
(Aya Kachi, The University of Illinois)
OVERVIEW

• Explicit models of interdependence:
  – *Tax Competition*
  – *International Conflict*
    • [Learning & Diffusion and Micro-Preferences also, but we’ll not cover here.]

• Using the theoretical models…
  – …to demonstrate interdependence;
  – …to establish/illustrate its explanators, sources, and mechanisms;
  – …to specify empirical models: esp., *how* covariates enter—in $X$, in $W$, in $\rho$…

• Outline of this session:
  – Tax Competition:
    • Recall the generic spillover/resource-flow models (Brueckner ’03).
    • Brueckner (’03) / Brueckner & Saavedra (’01) simple model.
    • Persson & Tabellini (’00: ch. 12.4) model.
    • P&T Extensions (problems 3 & 4)
  – International Conflict:
    • As illustration of how general strategic interdependence in S-Econometric frame.
    • Follow Signorino (’99) and Signorino & Tarar (’06).
Brueckner’s Resource-Flow Theoretical Model\(^1\)

- Interdependence: \(i\)’s utility depends on \(p_i \& p_j\):
  \[
  U^i = U^i \left( p_i, H^i (p_i, p_j; x_i); x_i \right)
  \]

- Accordingly, \(i\)’s optimal \(p_i^*\) depends \(j\)’s, \(p_j\):
  \[
  \text{Max}_{p_i} \left|_{p_j} U^i \left( p_i, H^i (p_i, p_j; x_i); x_i \right) \right.
  \]
  \[
  \Rightarrow U^i_{p_i} (\cdot) + U^i_H (\cdot) H^i_{p_i} (\cdot) = 0
  \]
  \[
  \Rightarrow p_i^* = U^{-1}_i \left( -U^i_H (\cdot) H^i_{p_i} (\cdot) \right) \equiv R(p_j; x_i)
  \]

- Slope of this reaction-function depends on how \(p_j\) affects \(i\)’s marginal utility (2\(^{nd}\) line of above):
  \[
  \frac{\partial U^i}{\partial p_i} = U^i_{p_i} (\cdot) + U^i_H (\cdot) H^i_{p_i} (\cdot) \Rightarrow \frac{\partial \frac{\partial U^i}{\partial p_i}}{\partial p_j} = U^i_{p_i H} H^i_{p_j} + U^i_{HH} H^i_{p_j} H^i_{p_i} + U^i H^i_{p_j p_j}
  \]
Brueckner’s Resource-Flow Theoretical Model\(^2\)

- A bit more specifically in tax-competition context:
- First-order Condition: 
  \[ U^i = U^i \left( \tau_i, k^i (\tau_i, \tau_j; x_i); x_i \right) \]
  \( U^i_\tau (\cdot) + U^i_k (\cdot) k^i_\tau (\cdot) = 0 \quad \Rightarrow \quad \tau^*_i = U^i_{\tau_i} \left( -U^i_k (\cdot) k^i_\tau (\cdot) \right) \equiv R \left( \tau_j; x_i \right) \)

- Slope reaction-function:
  \[
  \frac{\partial U^i}{\partial \tau_i} = U^i_k k^i \Rightarrow \frac{\partial U^i}{\partial \tau_i} = U^i_{\tau_i \tau_j} + U^i_k k^i_{\tau_i} + U^i_k k^i_{\tau_i \tau_j} 
  \]

- Notes:
  - In reaction-function, \( x_i \) gen’ly enters intercept & slope. This should inform empirical-model specification.
  - \( R(\cdot) \) depends \( U(\cdot) \) & \( k(\cdot) \), but intuitions re: pos/neg externalities hold if care to sign reversals via \( k(\cdot) \)
    - E.g., tax-competition: negative externalities, so strategic complements…
    \[ \downarrow \tau_j \Rightarrow \uparrow \text{ marg cost } \tau_i: \text{ i.e., } \downarrow U^i_{\tau_i}, \text{ i.e., } U^i_{\tau_i \tau_j} > 0 \Rightarrow \downarrow \tau_i \]
Brueckner & Saavedra Model of Municipal Tax-Competition

- Production:
  \[- F(K_i, P_i) = f(k_i), K \text{ perfectly mobile, } P \text{ perfectly immobile} \]
- Perfect competition for fixed total \( K \Rightarrow f'(k_i) - t_i = \rho \)
  \[- \Rightarrow k_1, k_2, \rho \text{ as functions of } t_1 \text{ & } t_2, \text{ with } \partial k_i/\partial t_i < 0 \]
- Residential land, \( L_i \), fixed: \( q_i^* = L_i/P_i \). Mrkt-clear \( \Rightarrow q_i = q_i^* \)
- Private good, \( x_i \), public, \( z_i \), housing, \( q_i \); perfect Tiebout sorting \( \Rightarrow \) cities of homogenous citizens with \( U(x_i, q_i, z_i) \).
- Income:
  \[- w_i + \rho k^* + r_i q_i^*, \text{ where } w_i = f(k_i) - k_i f'(k_i) \text{ & (assume even-distrib cap)} \]
  \[- k^* = (K_1 + K_2)/(P_1 + P_2) \]
- Tax: \( t_i \) levied on housing and capital; housing price: \( r_i + t_i \)
Brueckner & Saavedra Model of Municipal Tax-Competition

- **Budget Constraints:**
  - Citizens’ BC:
    - Consumption= wage inc + net cap inc + net house inc:
      \[ x_i = f(k_i) - k_i f'(k_i) + \rho k^* - t_i q_i^* (r_i + t_i) q_i \]
      which, since \( q_i = q_i^* \), is
    - Consumption =
      \[ x_i = f(k_i) - k_i f'(k_i) + \rho k^* - t_i q_i^* \]
  - Governments’ BC: \( z_i = t_i (q_i^* + k_i) \)
- **Utilities:**
  \[ U(x_i, q_i, z_i) = U[f(k_i) - k_i f'(k_i) + \rho k^* - t_i q_i^*, q_i^*, z_i, t_i (q_i^* + k_i)] \]
- Rep cit maxʼs \( U[\cdot] \) over \( t_i \), taking \( t_j \) as fixed (Nash)

\[
U_x \times \left( f' \frac{\partial k_i}{\partial t_i} - f' \frac{\partial k_i}{\partial t_i} - k_i f'' \frac{\partial k_i}{\partial t_i} + \frac{\partial \rho}{\partial t_i} k^* + \rho \frac{\partial k^*}{\partial t_i} - q^* \right) + U_z \times \left( (q^* + k_i) + t_i \frac{\partial k_i}{\partial t_i} \right) = 0,
\]

and recall & note: \( k^* = \frac{\bar{K}}{(P_1 + P_2)} \) is fixed, and \( f' = \rho + t_i \Rightarrow f'' = \frac{\partial \rho}{\partial k_i} + \frac{\partial t_i}{\partial k_i} \),

so reduces to:
\[
U_x \times \left( -k_i \left( \frac{\partial \rho}{\partial t_i} + 1 \right) + \frac{\partial \rho}{\partial t_i} k^* - q^* \right) + U_z \times \left( (q^* + k_i) + t_i \frac{\partial k_i}{\partial t_i} \right) = 0
\]

\[
\frac{U_z}{U_x} = \frac{k_i + q^* + (k_i - k^*) \frac{\partial \rho}{\partial t_i}}{k_i + q^* + t_i \frac{\partial k_i}{\partial t_i}}. \quad \text{Call this equilibrium: equation [6].}
\]

- In symmetric case, \( t_1 = t_2 \), so \( k_1 = k_2 = k^* \), implying that:
  \[
  \frac{U_z}{U_x} = \frac{k^* + q^*}{k^* + q^* + t_i \frac{\partial k_i}{\partial t_i}} > 1
  \]

- I.e., \( z \) is under-provided, & \( t \) too low. Classic result.
Brueckner & Saavedra Model of Municipal Tax-Competition

• Consider also an asymmetric eqbm, with city 1 having greater relative preference public good, \( z \).
  – City 1 then has incentive increase \( t_1 \) to raise \( z_1 \).
  – Capital flows 1 to 2, but, at margin, revenue & \( z_1 \) in city 1 rises.
  – (Nash) Equilibrium has \( t_1 > t_2 \) and \( k_1 < k_2 \).

• Perhaps more critically for our concerns:

\[
\frac{U_z}{U_x} = \frac{k_i + q^* + (k_i - k^*) \frac{\partial \rho}{\partial t_i}}{k_i + q^* + t_i \frac{\partial k_i}{\partial t_i}} \text{ implicitly defines reaction functions, } t_i^R = R(t_j)
\]

because \( \frac{\partial \rho}{\partial t_i} = \frac{\partial (f'(k_i) - t_i)}{\partial t_i} \) and \( \frac{\partial k_i}{\partial t_i} \) depend on \( t_j \) [...because \( k_i = g(t_i, t_j) \)].

• Substantively: the effects of tax competition derive from what competitors are doing (duh!). So…
The effects of tax competition derive from what competitors are doing (*duh!*). So…

– Standard empirical strategies problematic:
  - Nonspatial LS or ML or Bayesian estimation inappropriate.
  - SUTVA violated, so ditto std ‘causal inference’ strategies.

– Need to model the interdependence:
  - However, Galton’s problem (+): easy to mistake common exposure (or selection) for contagion because look similar.
  - Specification (incl. measurement, etc.) is everything! EITM.

The first-order conditions in [6] give reaction functions $t_i = R(t_j)$ implicitly. (Nash) Equilibrium occurs at intersection of those $R(\cdot)$. (See next.)
Brueckner & Saavedra Model of Municipal Tax-Competition

- $R(\cdot)$ may slope up or down.
  - Former is typical assumed ‘race to [?]’ case.
  - Latter: $i$ prefers to use cap tax-base $\uparrow$ from $\uparrow t_j$ to $\downarrow t_i$ yet maintain or even $\uparrow$ revenue.

- Can derive some EI of TM by comp statics on gen model.
  - Illustrated might be $\downarrow 1$’s pref for $z$, for instance.
  - Galton’s Problem (+) will plague empirical eval though.
  - Need specify theoretical model adequately for powerful specify of empirical.

- Fuller aim is often to estimate these $R(\cdot)$. Call it EM of TM.
  - For that especially, & esp. given Galton’s Problem (+), want/need specify theoretical model for powerful specification empirical…
B&S’s Example:
- Let \( f \) be quadratic...
  - \( f(k_i) = \beta k_i + \frac{1}{2}(\gamma k_i^2) \), where \( \beta, \gamma > 0 \)
- ...and \( U \) linear
  - \( U(x_i, q_i, z_i) = x_i + \lambda_i q_i + \eta_i z_i \), where \( \lambda_i, \eta_i > 0 \)
- With these specifications:

\[
f'(k_1) - t_1 = \rho = f'(k_2) - t_2 \quad \text{becomes} \quad \beta - \gamma k_1 - t_1 = \rho = \beta - \gamma k_2 - t_2,
\]

which, with \( k_1 = k_2 = k^* \), becomes \( k_1 = k^* + (t_2 - t_1) / 2\gamma \), so that \( \frac{\partial k_1}{\partial t_1} = -\frac{1}{2\gamma} \).

Substituting this into [6] produces...
\[
\frac{U_z}{U_x} = \frac{k_i + q^* + (k_i - k_i^*)\frac{\partial \rho}{\partial t_i}}{k_i + q^* + t_i \frac{\partial k_i}{\partial t_i}} \Rightarrow \eta_1 = \frac{1}{k_i + q^* + t_i \frac{\partial k_i}{\partial t_i}} = \frac{k^* + \frac{(t_2 - t_1)}{2\gamma} + q^* + \left(k^* + \frac{(t_2 - t_1)}{2\gamma} - k^*\right)(-\gamma(-\frac{1}{2\gamma}) - 1)}{k^* + \frac{(t_2 - t_1)}{2\gamma} + q^* - \frac{1}{2\gamma}}
\]

(substing above \(k_i = k^* + \frac{(t_2 - t_1)}{2\gamma}\), \(\frac{\partial k_i}{\partial t_i} = -\frac{1}{2\gamma}\), \(\frac{\partial \rho}{\partial t_i} = -\gamma(-\frac{1}{2\gamma}) - 1 = -\frac{1}{2}\); simplify) \(\Rightarrow\)

\[
\eta_1 = \frac{k^* + q^* + \frac{(t_2 - t_1)}{2\gamma} - \frac{1}{2}\left(\frac{(t_2 - t_1)}{2\gamma}\right)}{k^* + q^* + \frac{(t_2 - t_1)}{2\gamma} - \frac{1}{2\gamma}}, \text{ rearrange } \Rightarrow k^* + q^* + \frac{(t_2 - t_1)}{2\gamma} - \frac{1}{2\gamma} = \frac{k^* + q^* + \frac{(t_2 - t_1)}{2\gamma} - \frac{1}{2}\left(\frac{(t_2 - t_1)}{2\gamma}\right)}{\eta_1}
\]

(multiply both sides by \(2\gamma\)) \(\Rightarrow\)

\[
2\gamma(k^* + q^*) + t_2 - 2t_1 = \frac{2\gamma(k^* + q^*) + \frac{1}{2\eta_1}(t_2 - t_1)}{\eta_1}
\]

(isolate \(t_1\) and gather terms) \(\Rightarrow\)

\[
-2t_1 + \frac{1}{2\eta_1}t_1 = \frac{2\gamma(k^* + q^*) - 2\gamma(k^* + q^*) + \frac{1}{2\eta_1}t_2 - t_2}{\eta_1}
\]

\[
\Rightarrow t_1\left(\frac{1 - 4\eta_1}{2\eta_1}\right) = 2\gamma(k^* + q^*)\left(\frac{1 - \eta_1}{\eta_1}\right) + t_2\left(\frac{1 - 2\eta_1}{2\eta_1}\right) \Rightarrow t_1 = \left(\frac{2\eta_1}{1 - 4\eta_1}\right)\left(\frac{1 - \eta_1}{\eta_1}\right)2\gamma(k^* + q^*) + \left(\frac{2\eta_1}{1 - 4\eta_1}\right)\left(\frac{1 - 2\eta_1}{2\eta_1}\right)t_2
\]

\[
\Rightarrow t_1 = \left(\frac{4\gamma(1 - \eta_1)}{1 - 4\eta_1}\right)(k^* + q^*) + \left(\frac{1 - 2\eta_1}{1 - 4\eta_1}\right)t_2
\]

- For our purposes, especially key to note is how the sensitivity of \(t_1\) depends on the marginal utility of public goods (relative to private). This is \(w_{12}\) in the theoretical model. It also enters the (non-spatial) intercept, where also go terms related to capital and housing endowments and to the diminishing returns in the production function.
**Persson & Tabellini’s (2000: 12.4) International Tax-Competition Model**

- Two jurisdictions, domestic & foreign, levy capital taxes, $\tau_k$ & $\tau_k^*$, to fund exogenously fixed (and wasted) public spend.

- **Individuals:**
  - have labor-capital endowment, $e^i$, and
  - choose labor, $l$, vs. leisure, $x$, and
  - make savings-investment decisions, $s=k+f$, with
    - foreign investment, $f$, paying mobility costs, $M(f)$,
  - to max $\omega=U(c^l)+c^2+V(x)$, over $l$, $c^1$, & $c^2$,
    - s.t. time-constraint: $1+e^i=l+x$, and
    - BC1 & BC2: $1-e^i=c^l+k+f \equiv c^l+s$ and $c^2=(1-\tau_k)k+(1-\tau_k^*)f-M(f)+(1-\tau_l)l$.

- $\Rightarrow$ equilibrium economic choices of citizens […], which $\Rightarrow$ indirect utility, $W$, defined over policy variables, $\tau_l$, $\tau_k$, & $\tau_k^*$, […as indicated on next slide…]
Persson & Tabellini’s (2000: 12.4) International Tax-Competition Model

• ⇒ equilibrium economic choices of citizens:

\[ s = S(\tau_k) = 1 - U_c^{-1}(1 - \tau_k) \]

\[ f = F(\tau_k, \tau^*_k) = M_f^{-1}(\tau_k - \tau^*_k) \]

\[ k = K(\tau_k, \tau^*_k) = S(\tau_k) - F(\tau_k, \tau^*_k) \]

• ⇒ indirect utility, \( W \), over policy vars, \( \tau_l, \tau_k, \& \tau^*_k \):

  • Utility of \( c^1 \)…
  
  • Utility of \( c^2 \) via \( S \& L \)…
  
  • Leisure value…
  
  • Convenient way write \( f \) costs…

• Besley-Coate (97) cit-cands face voters w/ these pref
  
  – Stages: 1) elects, 2) wins set taxes, 3) private actors go
  
  – Ebm winner \( e^P \) s.t. desires enact Modified Ramsey Rule…

\[
W(\tau_l, \tau_k) = U\left(1 - S(\tau_k)\right) + (1 - \tau_k)S(\tau_k) + (1 - \tau_l)L(\tau_l) + V\left(1 - L(\tau_l)\right) + (\tau_k - \tau^*_k)F(\tau_k, \tau^*_k)M\left(F(\tau_k, \tau^*_k)\right)
\]
Persson & Tabellini’s (2000: 12.4) International Tax-Competition Model

- Besley-Coate (‘97) cit-cands winner has endowment \( e^P \) s.t. desires implement this Modified Ramsey Rule:

\[
\frac{S(\tau^p_k) - e^P}{S(\tau^p_k)} \left[ 1 + \varepsilon_l(\tau^p_k) \right] = \frac{L(\tau^p_l) + e^P}{L(\tau^p_l)} \left[ 1 + \frac{S_\tau(\tau^p_k) + 2F^*_\tau(\tau^p_k^*, \tau^p_k)\tau_k}{S(\tau^p_k)} \right]
\]

- Ramsey Rule: choose \( \tau_l \) & \( \tau_k \) to equate elasticities lab & cap supply
  - Modification 1: Because noncoop b/w domestic & foreign, elasticity of savings on RHS inflated by the \( 2F \) term. Shift \( \tau \) toward immobile: \( \tau_l \uparrow \).
  - Modification 2: Because cit-cand has distributional pref’s regarding \( \tau_l \) vs. \( \tau_k \), she shifts \( \tau \) toward \( \tau_k \) by a Meltzer-Richard type consideration in \( e^P \).

- \( \Rightarrow \) Best-Response Functions: \( \tau_k = T(e^P, \tau^*_k) \) & \( \tau^*_k = T^*(e^{P*}, \tau_k) \) for domestic & foreign policymakers.
  - \( \partial T/\partial \tau^*_k \) & \( \partial T^*/\partial \tau_k \), positive or negative as in B&S…
  - \( F_\tau = M_f^{-1}(\tau_k - \tau^*_k) \), so \( \downarrow \) mobility-costs likely \( \downarrow \tau_k \) and \( \uparrow |\partial T/\partial \tau^*_k| \).
  - Likewise, \( e^P \) seems to enter both as \( x \), shift \( R(\cdot) \), and as \( \uparrow w_{ij} \).
    - Next illustrates reaction-functions & “leftward” shift domestic policymaker
Persson & Tabellini’s (2000: 12.4) International Tax-Competition Model

• Globalization as $\downarrow M(f)$ example: Let $M(f) = \mu^B(f^A)^2$
  express $\uparrow/\downarrow$ mobility (from $B$) costs by $\uparrow/\downarrow \mu^B$
  
  – Rest as before:

$$c^A_1 = U^{-1}_c(1 - \tau^A) \equiv C^A(\tau^A); \quad f^A = M^{-1}_f(\tau^A - \tau^B; \mu^B) \equiv F^A(\tau^A, \tau^B; \mu^B)$$

$$k = 1 + e^A - F^A(\tau^A, \tau^B; \mu^B) - U^{-1}_c(1 - \tau^A) \equiv D^A(\tau^A, \tau^B; \mu^B) + e^A$$

  – $\Rightarrow$ Reaction functions like this:

$$\tau^A \left( C^A_{\tau^A} + F^A_{\tau^A} - F^B_{\tau^A} \right) = F^B - e^A$$

$$\tau^B \left( C^B_{\tau^B} + F^B_{\tau^B} - F^A_{\tau^B} \right) = F^A - e^B$$

  – $\Rightarrow \mu^B \& \mu^A$ (and $e^B \& e^A$) in two places:
  
  • Intercept shift (i.e., presence in $x$) essentially RHS of $R(\cdot)$
  
  • Responsiveness of $A$ to $B$ (i.e., in $w_{ij}$).
Persson & Tabellini’s (2000: 12.4) International Tax-Competition Model

\[ T(e^p, \tau_k^*) \]

\[ T(e_0^p, \tau_k^*) \]
Signorino’s (1999) Model of Strategic Interaction in International Conflict

**FIGURE 2. A Typical Bilateral Crisis Game**

Note: States 1 and 2 alternate moves at decision nodes. Actions by state 1 are shown in uppercase, those by state 2 in lowercase. A bar over an action refers to the opposite of the action (e.g., not using force). The equilibrium choice probabilities used in the statistical model are denoted \( p_i \) for state 1 and \( q_j \) for state 2. Nonterminal nodes are numbered to simplify the expected utility notation and to index choice probabilities.

(Strategic Logit with Agent Errors)
The Underlying Behavioral Model: Random Utility via Multinomial Logit

\[ U^*(O_j) = U(O_j) + \varepsilon_j \]

Choose \( O_j \) such that \( U^*(O_j) > U^*(O_k) \) for all \( k \neq j \).

Assuming \( \varepsilon \) i.i.d. type-I extreme value gives

\[ p_j = \frac{e^{U(O_j)}}{\sum_k e^{U(O_k)}}, \text{ where the } U(O_j) \text{ are "observed" utilities.} \]
Quantal Response Equilibrium

Actors play best responses with mistakes from a known distribution of errors.

\[
p_4 = \frac{e^{\lambda U_1(W2)}}{e^{\lambda U_1(W2)} + e^{\lambda U_1(C2)}}
\]

\[
q_3 = \frac{e^{\lambda U_2(W2)}}{e^{\lambda U_2(W2)} + e^{\lambda U_2(C2)}}
\]

\[
q_2 = \frac{e^{\lambda[p_4 U_2(W2) + (1-p_4) U_2(C1)]}}{e^{\lambda[p_4 U_2(W2) + (1-p_4) U_2(C1)]} + e^{\lambda U_2(SQ)}}
\]

\[
p_1 = \frac{e^{\lambda[q_3 U_1(W1) + (1-q_3) U_1(C2)]}}{e^{\lambda[q_3 U_1(W1) + (1-q_3) U_1(C2)]} + e^{\lambda[q_2 U_2(W2) + (1-p_4) U_1(C1)] + (1-q_2) U_1(SQ)}}
\]

Best response functions are interdependent and probabilistic. The probability that State 1 chooses to fight depends on the probability that State 2 will fight.
We can make the observed utilities (and the best responses) a parameterized function of covariates.

\[
U_i(SQ) = \beta_d D_{ij} \\
U_i(Ci) = -\beta_{ai} A_i \\
U_i(Cj) = \beta_{aj} A_j \\
U_i(W) = P_i \beta_{aj} A_j + (1 - P_i)(-\beta_{ai} A_i - \beta_{mi} M_i)
\]

where \( P_i = \frac{M_i}{M_i + M_j} \)
Reduced-form probabilities get a bit complicated...

\[
p_1 = \frac{e^{U_1(W1) + 1 - \frac{e^{U_2(W1)}}{e^{U_2(W2) + 1}} U_1(C2)}}{E^{P1} + E^{P2} + E^{P3}}, \text{ where:}
\]

\[
E^{P1} = e^{-\left[\frac{e^{U_2(W2) - U_2(W2)}}{e^{U_2(W2) + e^{U_2(C2)}}} U_1(W1) + 1 - \frac{e^{U_1(W2)}}{e^{U_1(W2) + e^{U_1(C2)}}} U_1(C1)\right]}
\]

\[
E^{P2} = e^{\left[\frac{e^{U_1(W2) - U_2(W2)}}{e^{U_1(W2) + e^{U_1(C2)}}} U_2(W2) + 1 - \frac{e^{U_1(W2)}}{e^{U_1(W2) + e^{U_1(C2)}}} U_2(C1)\right]} + e^{U_2(SQ)}
\]

\[
E^{P3} = 1 - e^{-\left[\frac{e^{U_1(W2) - U_2(W2)}}{e^{U_1(W2) + e^{U_1(C2)}}} U_2(W2) + 1 - \frac{e^{U_1(W2)}}{e^{U_1(W2) + e^{U_1(C2)}}} U_2(C1)\right]} + e^{U_2(SQ)}
\]

\[
q_3 = e^{-\left[\frac{e^{U_2(W2) - U_2(W2)}}{e^{U_2(W2) + e^{U_2(C2)}}} U_1(W1) + 1 - \frac{e^{U_2(W2)}}{e^{U_2(W2) + e^{U_2(C2)}}} U_1(C2)\right]}
\]

\[
U_i(SQ) = \beta_d D_j \quad ; \quad U_i(Ci) = -\beta_{ai} A_i \quad ; \quad U_i(Cj) = \beta_{aj} A_j
\]

\[
U_i(W) = P_i \beta_{aj} A_j + (1 - P_i)(-\beta_{ai} A_i - \beta_{mi} M_i)
\]

\[
Pr(W1) = p_1 \times q_3
\]

\[
Pr(War) = Pr(W1) + Pr(W2)
\]
Signorino and Tarar’s (2006) Model of Extended Deterrence

**Figure 1** The Deterrence Model with Uncertainty Concerning Utilities

(Attacker)

- $A$
  - $P_{a}$
  - $P_{\bar{a}}$

(Defender)

- $\overline{D}$
  - $P_{\bar{d}}$
  - $P_{d}$

- $\overline{D}$
  - $U_{d}(SQ) + \pi_{a1}$

- $D$
  - $U_{a}(Cap) + \pi_{a3}$
  - $U_{d}(War) + \pi_{d3}$

- $D$
  - $U_{a}(War) + \pi_{a4}$
  - $U_{d}(War) + \pi_{d4}$

(Strategic Probit with Private Information)
The Underlying Behavioral Model:
Random Utility via Multinomial Probit

The defender chooses \( D \) iff \( U_d^* (War) > U_d^* (Cap) \).

The attacker chooses \( A \) iff \( p_d \times U_a^* (War) + (1 - p_d) \times U_a^* (Cap) > U_a^* (SQ) \).

Assuming \( \pi_{ij} \sim \text{i.i.d. } N(0, \sigma^2) \) gives us

\[
\begin{align*}
    p_d &= \Phi \left[ \frac{U_d (War) - U_d (Cap)}{\sqrt{2\sigma^2}} \right], \text{ and} \\
    p_a &= \Phi \left[ \frac{p_d U_a (War) + (1 - p_d) U_a (Cap) - U_a (Cap)}{\sqrt{\sigma^2 \left( p_d^2 + (1 - p_d)^2 + 1 \right)}} \right].
\end{align*}
\]

Only probabilistic from the analyst’s and attacker’s perspectives.

Two sources of uncertainty.

Three sources of uncertainty here.
Deriving the Probabilities

\[ p_d = \Pr\left[ U_d^*(War) > U_d^*(Cap) \right] \]

\[ = \Pr\left[ U_d(War) + \pi_{d4} > U_d(Cap) + \pi_{d3} \right] \]

\[ = \Pr\left[ U_d(War) - U_d(Cap) > \pi_{d3} - \pi_{d4} \right] \]

\[ p_d = \Phi \left[ \frac{U_d(War) - U_d(Cap)}{\sqrt{\text{var}(\pi_{d3}) + \text{var}(\pi_{d4})}} \right] \]

\[ = \Phi \left[ \frac{U_d(War) - U_d(Cap)}{\sqrt{2\sigma^2}} \right] \]
Deriving the Probabilities

\[ p_a = \Pr \left[ p_d U^*_a(War) + (1 - p_d) U^*_a(Cap) > U^*_a(SQ) \right] \]

\[ = \Pr \left[ p_d [U_a(War) + \pi_{a4}] + (1 - p_d) [U_a(Cap) + \pi_{a3}] > U_a(SQ) + \pi_{a1} \right] \]

\[ = \Pr \left[ p_d U_a(War) + (1 - p_d) U_a(Cap) - U_a(SQ) > \pi_{a1} - p_d \pi_{a4} - (1 - p_d) \pi_{a3} \right] \]

\[ p_d = \Phi \left[ \frac{U_d(War) - U_d(Cap)}{\sqrt{\text{var}(\pi_{a1}) + p_d^2 \text{var}(\pi_{a4}) + (1 - p_d)^2 \text{var}(\pi_{a3})}} \right] \]

\[ = \Phi \left[ \frac{U_d(War) - U_d(Cap)}{\sqrt{\sigma^2 \left( p_d^2 + (1 - p_d)^2 + 1 \right)}} \right] \]
Note that the source of uncertainty matters…

If the uncertainty were due to agent error, the denominators in $p_d$ and $p_a$ would be the same. (Attackers and defenders make mistakes at same rate.)

\[ p_d = \Phi \left[ \frac{U_d(War) - U_d(Cap)}{\sqrt{2\sigma^2}} \right], \text{ and} \]

Uncertainty due to defender’s mistakes.

\[ p_a = \Phi \left[ \frac{p_d U_a(War) + (1 - p_d)U_a(Cap) - U_a(Cap)}{\sqrt{2\sigma^2}} \right] \]

Uncertainty due to attacker’s mistakes.
Well…Signorino’s games don’t exactly lead to spatial lag models. Do they? Could they? What about multilateral crises and war-joining behavior?

Some interdependent discrete choice models do lead directly to this kind of empirical representation.

E.g.,…
Beron et al.'s (2003) Model of International Environmental Cooperation

The Behavioral Model under *Independence*

\[ NB_i = x_i \beta + \varepsilon_i \]

Choose to cooperate if \( NB_i > 0 \) and not cooperate if \( NB_i \leq 0 \).

\[ \Pr[NB_i > 0] = \Pr[\varepsilon_i < x_i \beta] = \Phi(x_i \beta) \]
Beron et al.'s (2003) Model of International Cooperation

Model under **Interdependence**: The Role of Power

Power = \( P \times \text{NB} \), where the \( p_{ij} \) are measures of export dependence.

\[
\text{NB} = \alpha P \times \text{NB} + X\beta + \varepsilon \quad \Rightarrow \quad \text{NB} = (I - \alpha P)^{-1} X\beta + (I - \alpha P)^{-1} \varepsilon
\]

\[
\Pr[NB_i > 0] = \Pr \left[ \left( (I - \alpha P)^{-1} \varepsilon \right)_i < \left( (I - \alpha P)^{-1} X\beta \right)_i \right]
\]

\[
= \Pr \left[ \varepsilon_i < \frac{\left( (I - \alpha P)^{-1} X\beta \right)_i}{\sigma_i} \right]
\]

\[
= \Phi \left[ \frac{\left( (I - \alpha P)^{-1} X\beta \right)_i}{\sigma_i} \right]
\]

C.d.f. for the marginal distribution after integrating out the other \( n-1 \) dimensions.
The Estimation Problem

\[ \mathbf{\epsilon}^* = \left( \mathbf{I} - \alpha \mathbf{P} \right)^{-1} \mathbf{\epsilon} \]

\[ \mathbf{\epsilon}^* \sim N(0, \Sigma), \text{ with } \Sigma = \left[ \left( \mathbf{I} - \alpha \mathbf{P} \right)' \left( \mathbf{I} - \alpha \mathbf{P} \right) \right]^{-1} \]

\[ \mathbf{\epsilon}^{**} = \Theta \mathbf{\epsilon}^*, \text{ with elements } \theta_{ij} = 1 - 2y_i \text{ for } i = j \text{ and } \theta_{ij} = 0 \text{ for } i \neq j \]

\[ \mathbf{\epsilon}^{**} \sim N(0, \Theta \Sigma \Theta) \]

\[ \mathbf{z} = \Theta (\mathbf{I} - \alpha \mathbf{P})^{-1} \mathbf{X} \beta \]

\[ \text{Pr}[\mathbf{y}] = \text{Pr}\left[ \mathbf{\epsilon}^{**} < \mathbf{z} \right] \]

\[ = \int_{-\infty}^{\mathbf{z}} \phi_n \left( \mathbf{\epsilon}^{**} ; 0, \Theta \Sigma \Theta \right) d\mathbf{\epsilon}^{**} \]

Calculating the joint likelihood requires us to integrate over an N-dimensional multivariate normal distribution.