Special values of Riemann’s zeta function

Cameron Franc

UC Santa Cruz

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If $s > 1$ is a real number, then the series

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}$$

converges.

**Proof:** Compare the partial sum to an integral,

$$\sum_{n=1}^{N} \frac{1}{n^s} \leq 1 + \int_{1}^{N} \frac{dx}{x^s} = 1 + \frac{1}{s-1} \left(1 - \frac{1}{N^{s-1}}\right) \leq 1 + \frac{1}{s-1}.$$
The resulting function \( \zeta(s) \) is called *Riemann's zeta function*.

Was studied in depth by Euler and others before Riemann.

\( \zeta(s) \) is named after Riemann for two reasons:

1. He was the first to consider allowing the \( s \) in \( \zeta(s) \) to be a complex number \( \neq 1 \).
2. His deep 1859 paper *"Ueber die Anzahl der Primzahlen unter einer gegebenen Grössen"* (*"On the number of primes less than a given quantity"*) made remarkable connections between \( \zeta(s) \) and prime numbers.
In this talk we will discuss certain special values of $\zeta(s)$ for integer values of $s$.

In particular, we will discuss what happens at $s = 1, 2$ and $-1$. 
Overview

1. The divergence of $\zeta(1)$

2. The identity $\zeta(2) = \pi^2/6$

3. The identity $\zeta(-1) = -1/12$
What happens as $s \to 1$?

The value $\zeta(s)$ diverges to $\infty$ as $s$ approaches 1.

To see this, use an integral to bound the partial sums from below for $s > 1$:

$$\sum_{n=1}^{N} \frac{1}{n^s} \geq \int_{1}^{N+1} \frac{dx}{x^s} = \frac{1}{s-1} \left(1 - \frac{1}{(N+1)^{s-1}}\right).$$

It follows that $\zeta(s) \geq (s-1)^{-1}$ for $s > 1$. 
In summary, so far we’ve seen that for $s > 1$,

$$\frac{1}{s - 1} \leq \zeta(s) \leq \frac{1}{s - 1} + 1.$$  

Since the lower bound diverges as $s \to 1$, so does $\zeta(s)$.

This is related to the fact that the \textit{Harmonic series}

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

diverges.
The divergence of $\zeta(1)$
The identity $\zeta(2) = \pi^2/6$
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Cute proof that the Harmonic series diverges

We consider the partial sum involving $2^k$ terms:

$$\sum_{n=1}^{2^k} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2^{k-1} + 1} + \frac{1}{2^{k-1} + 2} + \cdots + \frac{1}{2^k}$$

$$\geq 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \cdots + \frac{1}{2^k} + \frac{1}{2^k} + \cdots + \frac{1}{2^k}$$

$$= 1 + \frac{1}{2} + \frac{2}{4} + \cdots + \frac{2^{k-1}}{2^k}$$

$$= 1 + \frac{k}{2}$$
The divergence of \( \zeta(1) \)

The identity \( \zeta(2) = \pi^2 / 6 \)

The identity \( \zeta(-1) = -1/12 \)

Overview

1. The divergence of \( \zeta(1) \)
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The identity $\zeta(2) = \pi^2/6$

If we apply the bounds

$$\frac{1}{s-1} \leq \zeta(s) \leq \frac{1}{s-1} + 1$$

from the previous part to $s = 2$ we deduce that

$$1 \leq \zeta(2) \leq 2.$$ 

But what number in this interval is

$$\zeta(2) = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \cdots?!$$
It turns out that

\[ \zeta(2) = \frac{\pi^2}{6}. \]

In fact, more generally if \( k \geq 1 \) is any positive integer, then

\[ \zeta(2k) = (-1)^{k+1} \frac{B_{2k}(2\pi)^{2k}}{2(2k)!}. \]

Here \( B_n \) is a rational number, the \textit{nth Bernoulli number}, defined to be the coefficient of \( X^n/n! \) in the series

\[ \frac{X}{e^X - 1} = \sum_{n=0}^{\infty} B_n \frac{X^n}{n!}. \]
Thus, each value $\zeta(2k)$ is a rational multiple of $\pi^{2k}$.

If that isn’t surprising to you, be aware of the following: the odd values $\zeta(2k + 1)$ are not expected to be related to $\pi$ in any significant algebraic way.

Why the even zeta values $\zeta(2k)$ are algebraically related to $\pi$ and the odd values $\zeta(2k + 1)$ are (probably) not is one unsolved problem in mathematics.
We’ll now offer seven proofs that $\zeta(2) = \pi^2/6$, one for every day of the week.
First proof: An elementary trigonometric argument

First we note that for $\theta = \pi/(2N + 1)$ one has

$$\cot^2(\theta) + \cot^2(2\theta) + \cdots + \cot^2(N\theta) = \frac{N(2N - 1)}{3}.$$  

For $x$ in $(0, \pi/2)$ the inequality $\sin x < x < \tan x$ implies

$$\cot^2 x < \frac{1}{x^2} < \cot^2 x + 1.$$  

Apply this to each of $x = \theta, 2\theta, 3\theta$, etc, and sum to deduce

$$\frac{N(2N - 1)}{3} < \frac{1}{\theta^2} \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{N^2}\right) < \frac{N(2N - 1)}{3} + N.$$  

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Multiply by $\theta^2 = \pi^2/(2N + 1)^2$ to deduce

$$\frac{\pi^2}{3} \frac{N(2N - 1)}{(2N + 1)^2} < \sum_{n=1}^{N} \frac{1}{n^2} < \frac{\pi^2}{3} \frac{N(2N - 1)}{(2N + 1)^2} + \frac{\pi^2}{(2N + 1)^2}$$

Since the upper and lower bounds both converge to the same limit as $N$ grows, and the middle one converges to $\zeta(2)$, we deduce that

$$\zeta(2) = \frac{\pi^2}{3} \lim_{N \to \infty} \frac{N(2N - 1)}{(2N + 1)^2} = \frac{\pi^2}{3} \lim_{N \to \infty} \frac{1 - \frac{1}{2N^2}}{2(1 + \frac{1}{2N})^2} = \frac{\pi^2}{6}.$$
The Fourier series expansion of $x^2$ is

$$x^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} (-1)^n \frac{\cos(nx)}{n^2}.$$ 

Since $\cos(n\pi) = (-1)^n$ for integers $n$, evaluating at $x = \pi$ gives

$$\pi^2 = \frac{\pi^2}{3} + 4 \sum_{k=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{3} + 4\zeta(2).$$

Hence $\zeta(2) = \pi^2/6$. 

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We evaluate a certain double integral two ways. First,

\[
I = \int_0^1 \int_0^1 \frac{dx\,dy}{1 - xy} \\
= \sum_{n \geq 0} \int_0^1 \int_0^1 (xy)^n \, dx\,dy \\
= \sum_{n \geq 1} \int_0^1 y^{n-1} \frac{dy}{n} \\
= \sum_{n \geq 1} \frac{1}{n^2} = \zeta(2).
\]
On the other hand, the substitutions $x = (\sqrt{2}/2)(u - v)$ and $y = (\sqrt{2}/2)(u + v)$ allow one to write

$$I = 4 \int_0^{\sqrt{2}/2} \int_0^{u} \frac{dudv}{2 - u^2 + v^2} + 4 \int_{\sqrt{2}/2}^{\sqrt{2}} \int_0^{\sqrt{2}-u} \frac{dudv}{2 - u^2 + v^2}.$$ 

Persistance and some trig substitutions allow one to evaluate both of the above integrals and show that

$$\zeta(2) = I = \frac{\pi^2}{18} + \frac{\pi^2}{9} = \frac{\pi^2}{6}.$$
Fourth proof: the residue theorem

The following can be proved using the residue theorem from complex analysis.

**Theorem (Summation of rational functions)**

Let $P$ and $Q$ be polynomials with $\deg Q \geq \deg P + 2$ and let $f(z) = \frac{P(z)}{Q(z)}$. Let $S \subseteq \mathbb{C}$ be the finite set of poles of $f$. Then

$$
\lim_{N \to \infty} \sum_{k=-N}^{N} f(k) = - \sum_{p \in S} \text{residue}_{z=p}(\pi f(z) \cot(\pi z)).
$$
Let’s take \( f(z) = 1/z^2 \). In this case \( S = \{0\} \) and the theorem gives a formula for the sum

\[
\sum_{k=-\infty}^{\infty} \frac{1}{k^2} = 2 \sum_{k=1}^{\infty} \frac{1}{k^2} = 2\zeta(2).
\]

Since the polar set \( S \) consists only of 0, the preceding summation theorem shows us that this sum is nothing but

\[-\text{residue}_{z=0} \left( \pi \cot(\pi z)/z^2 \right).\]

That is, the theorem immediately gives us the formula

\[
\zeta(2) = -\frac{\pi}{2} \cdot \text{residue}_{z=0} \left( \frac{\cot(\pi z)}{z^2} \right).
\]
We have

\[
\frac{\cot(\pi z)}{z^2} = \frac{1}{z^2} \left( \frac{a}{z} + b + cz + dz^2 + \cdots \right) = \frac{a}{z^3} + \frac{b}{z^2} + \frac{c}{z} + d + \cdots
\]

and hence

\[
\text{residue}_{z=0} \left( \frac{\cot(\pi z)}{z^2} \right) = \frac{1}{2} \cdot \frac{d^2}{dz^2} \left( z^3 \cdot \cot(\pi z) \right) \bigg|_{z=0} = -\frac{\pi}{3}.
\]

Putting everything together shows that

\[
\zeta(2) = -\frac{\pi}{2} \cdot \text{residue}_{z=0} \left( \frac{\cot(\pi z)}{z^2} \right) = \left( -\frac{\pi}{2} \right) \cdot \left( -\frac{\pi}{3} \right) = \frac{\pi^2}{6}.
\]
Fifth proof: Weierstrass product

Let $P(X)$ be a polynomial of the form

$$P(X) = (1 + r_1X)(1 + r_2X) \cdots (1 + r_nX).$$

Then the coefficient of $X$ in $P(X)$ is equal to

$$r_1 + r_2 + \cdots + r_n.$$
The divergence of $\zeta(1)$

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The sine function is like a polynomial: it has a Taylor series

$$\sin(X) = X - \frac{X^3}{3!} + \frac{X^5}{5!} - \frac{X^7}{7!} + \cdots$$

and a Weierstrass product

$$\sin(X) = X \prod_{n=1}^{\infty} \left(1 - \frac{X^2}{(\pi n)^2}\right).$$

If we cancel $X$ and let $Z = X^2$ then we deduce that

$$1 - \frac{Z}{3!} + \frac{Z^2}{5!} - \frac{Z^3}{7!} + \cdots = \prod_{n=1}^{\infty} \left(1 + \left(-\frac{1}{(\pi n)^2}\right) Z\right).$$
In analogy with polynomials, the identity

\[ 1 - \frac{Z}{3!} + \frac{Z^2}{5!} - \frac{Z^3}{7!} + \cdots = \prod_{n=1}^{\infty} \left( 1 + \left( -\frac{1}{(\pi n)^2} \right) Z \right). \]

suggests that the coefficient of \( Z \) should be the sum of the reciprocal roots on the right. That is:

\[ -\frac{1}{3!} = \sum_{n \geq 1} \frac{-1}{(\pi n)^2} \]

and hence \( \zeta(2) = \frac{\pi^2}{6} \).
The divergence of $\zeta(1)$

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**Sixth proof: moduli of elliptic curves**

Let

$$\mathcal{H} = \{ z \in \mathbb{C} \mid \Im(z) > 0 \}$$

and let $\text{SL}_2(\mathbb{Z})$ act on $\mathcal{H}$ via fractional linear transformation. Then

$$\text{SL}_2(\mathbb{Z}) \backslash \mathcal{H}$$

is the coarse moduli space of elliptic curves, and one can show that

$$\int_{\text{SL}_2(\mathbb{Z}) \backslash \mathcal{H}} \frac{dxdy}{y^2} = \frac{2\zeta(2)}{\pi}.$$

But this integral can be computed explicitly and is equal to $\pi/3$. Hence $\zeta(2) = \pi^2/6$. 

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Euler used unique factorization to prove that

\[ \zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}. \]

This Euler product is taken over all primes \( p \).

The probability that an integer is divisible by \( p \) is \( 1/p \).

This is independent among numbers, so the probability that two integers are simultaneously divisible by \( p \) is \( 1/p^2 \).
Recall: *coprime* integers share no common prime factors.

The probability $P($coprime$)$ that two random integers are coprime is the product over all primes $p$ of the probability that they do not share the prime factor $p$.

Thus, the Euler product for $\zeta(s)$ shows that

$$P($coprime$) = \prod_p \left(1 - \frac{1}{p^2}\right) = \zeta(2)^{-1}.$$ 

So to prove $\zeta(2) = \pi^2/6$, you just need to choose enough random pairs of integers and test whether they’re coprime!
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In what sense does $1 + 2 + 3 + 4 + \cdots = -\frac{1}{12}$?

Figure: Wikipedia talk page for the article Zeta function regularization, February 25, 2013
Of course it’s not literally true that the series

$$1 + 2 + 3 + 4 + 5 + \cdots$$

converges in the conventional sense of convergence.

There is a deeper truth hidden in the seemingly absurd claim that

$$1 + 2 + 3 + 4 + 5 + \cdots = -1/12.$$
Zeta as a function of a complex variable

As observed by Riemann, the sum defining the zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

makes sense for all complex $s$ with $\Re(s) > 1$.

*Proof:* If $s = x + iy$ with $x > 1$, note that

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{e^{i \log(n)y} n^x}.$$ 

Since $|e^{i \log(n)y}| = 1$, this series converges absolutely if $\zeta(x)$ does. Since $x > 1$, we win.
Analyticity of $\zeta(s)$

The resulting complex zeta function is *analytic* (a.k.a. *complex differentiable*).

*Proof:* The partial sums are clearly analytic, being a finite sum of exponentials. It’s not hard to prove that they converge uniformly on regions $\Re(s) \geq 1 + \varepsilon$ for $\varepsilon > 0$. A standard result in complex analysis then implies that $\zeta(s)$ is analytic in the region $\Re(s) > 1$. 
Analytic functions are very rigid — they satisfy the property of \textit{analytic continuation}. More precisely, one proves the following in a first course on complex analysis:

\begin{center}
\textbf{Theorem}
\end{center}

\textit{Let} \( U \subseteq \mathbb{C} \) \textit{be an open subset and let} \( f \) \textit{be analytic on} \( U \). \textit{Let} \( V \supset U \) \textit{denote a larger open subset, and assume further that} \( V \) \textit{is connected. Then there exists at most one analytic function} \( g \) \textit{on} \( V \) \textit{such that} \( g|_U = f \).
Another fundamental contribution of Riemann to the study of $\zeta(s)$ is his proof that $\zeta(s)$ continues analytically to an analytic function on $\mathbb{C} - \{1\}$.

Since $\zeta(s)$ has a pole at $s = 1$, this is as good as it could be!

*Note:* outside the region $\Re(s) > 1$, the function $\zeta(s)$ is not defined by the usual summation. This distinction is crucial!
Still another fundamental contribution of Riemann to the study of \( \zeta(s) \) is his proof of the functional equation:

\[
\zeta(s) = 2^s \pi^{s-1} \sin \left( \frac{\pi s}{2} \right) \Gamma(1 - s) \zeta(1 - s),
\]

where \( \Gamma(s) \) denotes the gamma function defined via the integral

\[
\Gamma(s) = \int_0^\infty t^s e^{-t} \frac{dt}{t}.
\]

Note that the functional equation relates \( \zeta(-1) \) with \( \zeta(2) \)!
The value $\zeta(-1)$

So, if we plug in $s = -1$ to

$$\zeta(s) = 2^s \pi^{s-1} \sin \left( \frac{\pi s}{2} \right) \Gamma(1 - s) \zeta(1 - s)$$

we get

$$\zeta(-1) = 2^{-1} \pi^{-2} \sin(-\pi/2) \Gamma(2) \zeta(2)$$

$$= \left( \frac{-1}{2\pi^2} \right) \cdot 1! \cdot \left( \frac{\pi^2}{6} \right)$$

$$= -\frac{1}{12}.$$
Zeta function regularization

Physicists will often use this sort of technique to assign finite values to divergent series. Let

\[ a_1 + a_2 + a_3 + \cdots \]

denote a possibly divergent series.

To assign it a finite value, define an associated zeta function:

\[ \zeta_A(s) = \sum_{n=1}^{\infty} \frac{1}{a_n^s}. \]

If this converges and continues analytically to \( s = -1 \), then one can think of the value \( \zeta_A(-1) \) as “acting like” the sum of the series \( a_1 + a_2 + a_3 + \cdots \).
Strictly speaking, the sum

\[ a_1 + a_2 + a_3 + \cdots \]

would not necessarily converge to \( \zeta_A(-1) \) in any rigorous sense. Nevertheless, it turns out to be physically useful to assign such “zeta-regularized” values to certain divergent series.
On this note, we’ll end by discussing how to assign a “value” to $\infty!$. Here is a highly suspicious derivation:

\[
\infty! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdots \\
= \exp \left( \sum_{n=1}^{\infty} \log(n) \right) \\
= \exp (-\zeta'(0))
\]

Since $\zeta(s)$ is analytic on $\mathbb{C} - \{1\}$, the value $\zeta'(0)$ is finite!
One can use the functional equation for $\zeta(s)$ to deduce that

$$-\zeta'(0) = \frac{1}{2} \log(2\pi).$$

Hence,

$$\infty! = \exp(-\zeta'(0)) = \exp((1/2) \log(2\pi)) = \sqrt{2\pi}$$

... right?
The divergence of $\zeta(1)$

The identity $\zeta(2) = \frac{\pi^2}{6}$

The identity $\zeta(-1) = -\frac{1}{12}$

Thanks for listening!