Types for tame $p$-adic groups

Jessica Fintzen

Abstract

Let $k$ be a non-archimedean local field with residual characteristic $p$. Let $G$ be a connected reductive group over $k$ that splits over a tamely ramified field extension of $k$. Suppose $p$ does not divide the order of the Weyl group of $G$. Then we show that every smooth irreducible complex representation of $G(k)$ contains an $s$-type of the form constructed by Kim–Yu and that every irreducible supercuspidal representation arises from Yu’s construction. This improves an earlier result of Kim, which held only in characteristic zero and with a very large and ineffective bound on $p$. By contrast, our bound on $p$ is explicit and tight, and our result holds in positive characteristic as well. Moreover, our approach is more explicit in extracting an input for Yu’s construction from a given representation.

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1 Introduction

The aim of the theory of types is to classify, up to some natural equivalence, the smooth irreducible complex representations of a $p$-adic group in terms of representations of compact open subgroups. For $\text{GL}_n$ it is known that every irreducible representation contains an $s$-type. This theorem lies at the heart of many results in the representation theory of $\text{GL}_n$ and plays a key role in the construction of an explicit local Langlands correspondence for $\text{GL}_n$ as well as in the study of its fine structure. One of the main results of this paper is the existence of $s$-types for general $p$-adic groups and the related exhaustion of supercuspidal representations under minimal tameness assumptions. These tameness assumptions arise from the nature of the available constructions of supercuspidal representations for general $p$-adic groups.

To explain our results in more detail, let $k$ denote a non-archimedean local field with residual characteristic $p$ and let $G$ be a (connected) reductive group over $k$. Before introducing the notion of a type, let us first discuss the case of supercuspidal representations, the building blocks of all other representations. Since the constructions below of supercuspidal representations for general reductive groups $G$ assume that $G$ splits over a tamely ramified extension of $k$, we will impose this condition from now on. Under this assumption, Yu (Yu01) gave a construction of supercuspidal representations as representations induced from compact mod center, open subgroups of $G(k)$ generalizing an earlier construction of Adler (Adl98). Yu’s construction is the most general construction of supercuspidal representations for general reductive groups known at present and it has been widely used to study representations of $p$-adic groups, e.g. to obtain results about distinction, to calculate character formulas, to suggest an explicit local Langlands correspondence and to investigate the theta correspondence. However, all these results only apply to representations obtained from Yu’s construction. In this article, we prove that all supercuspidal representations of $G(k)$ are obtained from Yu’s construction if $p$ does not divide the order of the Weyl group $W$ of $G$. This result was previously shown by Kim (Kim07) under the assumption that $k$ has characteristic zero and that $p$ is “very large”. Note that Kim’s hypotheses on $p$ depend on the field $k$ and are much stronger than our requirement that $p \nmid |W|$, see [Kim07, § 3.4]. The few primes that divide the order of the Weyl group of $G$ are listed in Table 1 and we expect that this assumption is optimal in general for the following reason. Yu’s construction is limited to tori that split over a tamely ramified field extension of $k$. If $p$ does not divide the order of the Weyl group $W$ of $G$ and $G$ splits over a tamely ramified extension (our assumptions), then all tori split over a tame extension. However, if one of these assumptions is violated, then, in general, the group $G$ contains tori that do not split over a tame extension (for some non-split inner forms of split groups of type $A_n, n \geq 2, D_l, l \geq 4$ prime, or $E_6$ the condition on the prime number is slightly weaker, see [Fin18, Theorem 2.4 and Corollary 2.6] for the details). We expect that we can use these tori to produce supercuspidal representations that were not constructed by Yu. Examples of such representations are provided by the construction of Reeder and Yu (RY14), whose ingredients exist also when $p \mid |W|$ (whenever they exist for some large prime $p$), see [PR17] and [Fin17].
In order to study arbitrary smooth irreducible representations, we recall the theory of types introduced by Bushnell and Kutzko \([\text{BK}98]\): By Bernstein \([\text{Ber}84]\) the category \(\mathcal{R}(G)\) of smooth complex representations of \(G(k)\) decomposes into a product of subcategories \(\mathcal{R}^s(G)\) indexed by the set of inertial equivalence classes \(\mathcal{I}\) of pairs \((L,\sigma)\) consisting of a Levi subgroup \(L\) of (a parabolic subgroup of) \(G\) together with a smooth irreducible supercuspidal representation \(\sigma\) of \(L(k)\):

\[
\mathcal{R}(G) = \prod_{s \in \mathcal{I}} \mathcal{R}^s(G).
\]

Let \(s \in \mathcal{I}\). Following Bushnell–Kutzko \([\text{BK}98]\), we call a pair \((K,\rho)\) consisting of a compact open subgroup \(K\) of \(G(k)\) and an irreducible smooth representation \(\rho\) of \(K\) an \(s\)-type if for every irreducible smooth representation \(\pi\) of \(G(k)\) the following holds:

\[
\pi \text{ lies in } \mathcal{R}^s(G) \text{ if and only if } \pi|_K \text{ contains } \rho.
\]

In this case the category \(\mathcal{R}^s(G)\) is isomorphic to the category of (unital left) modules of the Hecke algebra of compactly supported \(\rho\)-spherical functions on \(G(k)\). Thus, if we know that there exists an \(s\)-type for a given \(s \in \mathcal{I}\), then we can study the corresponding representations \(\mathcal{R}^s(G)\) using the corresponding Hecke algebra. We say that a smooth irreducible representation \((\pi,V)\) of \(G(k)\) contains a type if there exists an \(s\)-type \((K,\rho)\) for the class \(s \in \mathcal{I}\) that satisfies \((\pi,V) \in \mathcal{R}^s(G)\), i.e. \(\pi|_K \text{ contains } \rho\).

Using the theory of \(G\)-covers introduced by Bushnell and Kutzko in \([\text{BK}98]\), Kim and Yu \([\text{KY}17]\) showed that Yu’s construction of supercuspidal representations can also be used to obtain types by omitting some of the conditions that Yu imposed on his input data. In this paper we prove that every smooth irreducible representation of \(G(k)\) contains such a type if \(k\) is a non-archimedian local field of arbitrary characteristic whose residual characteristic \(p\) does not divide the order of the Weyl group of \(G(k)\). This excludes only a few residual characteristics, and we expect the restriction to be optimal in general as explained above. If \(k\) has characteristic zero and \(p\) is “very large”, then Kim and Yu deduced this result already from Kim’s work \([\text{Kim}07]\).

Our approach is very different from Kim’s approach. While Kim proves statements about a measure one subset of all smooth irreducible representations of \(G(k)\) by matching summands of the Plancherel formula for the group and the Lie algebra, we use a more explicit approach involving the action of one parameter subgroups on the Bruhat–Tits building. This means that even though we have formulated some statements and proofs as existence results, the interested reader can use our approach to extract the input for the construction of a type from a given representation.

To indicate the rough idea of our approach, we assume from now that \(p\) does not divide the order of the Weyl group of \(G\), and we denote by \((\pi,V)\) an irreducible smooth representation of \(G(k)\). Recall that Moy and Prasad \([\text{MP}94,\text{MP}96]\) defined for every point \(x\) in the Bruhat–Tits building \(\mathcal{B}(G,k)\) of \(G\) and every non-negative real number a compact open subgroup \(G_{x,r} \subset G(k)\) and a lattice \(g_{x,r} \subset \mathfrak{g}\) in the Lie algebra \(\mathfrak{g} = \text{Lie}(G)(k)\) of \(G\) such that \(G_{x,r} \leq G_{x,s}\)
and $\mathfrak{g}_{x,r} \subseteq \mathfrak{g}_{x,s}$ for $r > s$. Moy and Prasad defined the depth of $(\pi, V)$ to be the smallest nonnegative real number $r_1$ such that there exists a point $x \in \mathcal{B}(G, k)$ so that the space of fixed vectors $V^{G_{x,r_1}}$ under the action of the subgroup $G_{x,r_1} := \bigcup_{s > r_1} G_{x,s}$ is non-zero. In [MP96], they showed that every irreducible depth-zero representation contains a type. A different proof using Hecke algebras was given by Morris ([Mor99], announcement in [Mor93]). More generally, Moy and Prasad showed that $(\pi, V)$ contains an unrefined minimal $K$-type, and all unrefined minimal $K$-types are associates of each other. For $r_1 = 0$, an unrefined minimal $K$-type is a pair $(G_{x,0}, \chi)$, where $\chi$ is a cuspidal representation of the finite (reductive) group $G_{x,0}/G_{x,0+}$. If $r_1 > 0$, then an unrefined minimal $K$-type is a pair $(G_{x,r_1}, \chi)$, where $\chi$ is a nondegenerate character of the abelian quotient $G_{x,r_1}/G_{x,r_1+}$. While the work of Moy and Prasad revolutionized the study of representations of $p$-adic groups, the unrefined minimal $K$-type itself determines the representation only in some special cases. Our first main result in this paper (Theorem 6.1) shows that every smooth irreducible representation of $G(k)$ contains a much more refined invariant, which we call a datum. A datum is a tuple

$$(x, (X_i)_{1 \leq i \leq n}, (\rho_0, V_{\rho_0}))$$

for some integer $n$, where $x \in \mathcal{B}(G, k)$, $X_i \in \mathfrak{g}^*$ for $1 \leq i \leq n$ satisfying certain conditions and $(\rho_0, V_{\rho_0})$ is an irreducible representation of a finite group (which is the reductive quotient of the derived group of a twisted Levi subgroup of $G$), see Definition 4.1 and Definition 4.2 for the details. Our datum can be viewed as a refinement of the unrefined minimal $K$-type of Moy and Prasad as follows. To a datum $(x, (X_i)_{1 \leq i \leq n}, (\rho_0, V_{\rho_0}))$ we associate a sequence of subgroups $G \supset H_1 \supset H_2 \supset \ldots \supset H_n \supset H_{n+1}$, which are (apart from allowing $H_1 = G_1$) the derived groups of twisted Levi subgroups $G \supset G_1 \supset G_2 \supset \ldots \supset G_n \supset G_{n+1}$, and real numbers $r_1 > r_2 > \ldots > r_n > 0$ such that for $1 \leq i \leq n$ the element $X_i$ corresponds to a character $\chi_i$ of

$$(H_i)_{x_i,r_i}/(H_i)_{x_i,r_i+} \simeq \text{Lie}(H_i)(k)_{x_i,r_i}/\text{Lie}(H_i)(k)_{x_i,r_i+} \subset \mathfrak{g}_{x,r_i}/\mathfrak{g}_{x,r_i+}$$

for a suitable point $x_i \in \mathcal{B}(H_i, k)$. Then the pair $((H_i)_{x_i,r_i}, \chi_i)$ is an unrefined minimal $K$-type of depth $r_i$ for some subrepresentation of $(\pi|_{H_i}, V)$. The existence of a datum for any irreducible representation of $G(k)$ is a key ingredient for producing the input that is needed for the construction of types as in Kim–Yu ([KY17]). In order to exhibit a datum in a given representation, we require the elements $(X_i)_{1 \leq i \leq n}$ in the datum $(x, (X_i)_{1 \leq i \leq n}, (\rho_0, V_{\rho_0}))$ to satisfy slightly stronger conditions than the non-degeneracy necessary for an unrefined minimal $K$-type. We call our conditions almost strongly stable and generic, see Definition 3.1 and Definition 3.5. These conditions ensure that the deduced input for Yu’s construction is generic in the sense of Yu ([Yu01] § 15) and at the same time they are crucial for the proof of the existence of the datum. The existence of a datum in $(\pi, V)$ is proved recursively, i.e. by first showing the existence of a suitable element $X_1$, then finding a compatible element $X_2$, and then $X_3$, etc., until we obtain a tuple $(X_i)_{1 \leq i \leq n}$ and finally exhibit the representation $(\rho_0, V_{\rho_0})$. The existence of $X_1$ can be considered as a refinement of the existence of an unrefined minimal $K$-type by Moy and Prasad and relies.
on the existence result of almost strongly stable and generic elements proved in Proposition 3.10. When constructing the remaining part of the datum we need to ensure its compatibility with $X_1$ and rely on several preparatory results proved in Section 5. At this step the imposed conditions on the elements $X_i$ become essential.

The final crucial part of this paper is concerned with deducing from the existence of a datum in Theorem 7.12 that every smooth irreducible representation of $G(k)$ contains one of the types constructed by Kim–Yu and, similarly, that every irreducible supercuspidal representation of $G(k)$ arises from Yu’s construction, see Theorem 8.1. This requires using the elements $X_i$ from the datum to provide appropriate characters of the twisted Levi subgroups $G_i$ ($2 \leq i \leq n+1$) and using $((\rho_0, V_{\rho_0}))$ to produce a depth-zero supercuspidal representation $\pi_0$ of $G_{n+1}(k)$. We warn the reader that the depth-zero representation of $G_{n+1}(k)$ is in general not simply obtained by extending and inducing $((\rho_0, V_{\rho_0}))$. The relationship between $\rho_0$ and $\pi_0$ requires the study of Weil representations and can be found in Section 7, in particular in Lemma 7.8. The main difficulty lies in showing that a potential candidate for the depth-zero representation $\pi_0$ of $G_{n+1}(k)$ is supercuspidal, which is the content of Lemma 7.10.

We conclude the paper by mentioning in Corollary 8.3 how to read off from a datum for $(\pi, V)$ if the representation $(\pi, V)$ is supercuspidal or not.

We would like to point out that the exhaustion of supercuspidal representations and the existence of types for arbitrary smooth irreducible representations have already been extensively studied for special classes of reductive groups for which other case-specific tools are available, e.g. a lattice theoretic description of the Bruhat–Tits building and a better understanding of the involved Hecke algebras. In 1979, Carayol ([Car79]) gave a construction of all supercuspidal representations of $GL_n(k)$ for $n$ a prime number different from $p$. In 1986, Moy ([Moy86]) proved that Howe’s construction ([How77]) exhausts all supercuspidal representations of $GL_n(k)$ if $n$ is coprime to $p$. Bushnell and Kutzko extended the construction to $GL_n(k)$ for arbitrary $n$ and proved that every irreducible representation of $GL_n(k)$ contains a type ([BK93][BK98][BK99]). As mentioned above, these results play a crucial role in the representation theory of $GL_n(k)$. Based on the work for $GL_n(k)$, Bushnell and Kutzko ([BK94]) together with Goldberg and Roche ([GR02]) provide types for all Bernstein components for $SL_n(k)$. For classical groups Stevens ([Ste08]) has recently provided a construction of supercuspidal representations for $p \neq 2$ and proved that all supercuspidal representations arise in this way. A few years later, Miyauchi and Stevens ([MS14]) provided types for all Bernstein components in that setting. The case of inner forms of $GL_n(k)$ was completed by Sécherre and Stevens ([SS08][SS12]) around the same time, subsequent to earlier results of others for special cases (e.g. Zink ([Zin92]) treated division algebras over non-archimedean local fields of characteristic zero and Broussous ([Bro98]) treated division algebras without restriction on the characteristic). The existence of types for inner forms of $GL_n(k)$ plays a key role in the explicit description of the local Jacquet–Langlands correspondence.

**Structure of the paper.** In Section 2 we collect some consequences of the assumption that the residual field characteristic $p$ does not divide the order of the Weyl group of $G$. Section 3 concerns the definition and properties of almost strongly stable and generic elements and
includes an existence result for almost strongly stable and generic elements. In Section 4, we introduce the notion of a datum and define what it means for a representation to contain a datum and for a datum to be a datum for a representation. The proof that every smooth irreducible representation of G(k) contains a datum is the subject of Section 6. Several results that are repeatedly used in this proof are shown in the preceding section, Section 3. In Section 7, we use the result about the existence of a datum to derive that every smooth irreducible representation of G(k) contains one of the types constructed by Kim and Yu, and, in Section 8, we prove analogously that every smooth irreducible supercuspidal representation of G(k) arises from Yu’s construction.

**Conventions and notation.** Throughout the paper, we require reductive groups to be connected and all representations are smooth complex representations unless mentioned otherwise. We do not distinguish between a representation and its isomorphism class. As explained in the introduction, by type we mean an s-type for some inertial equivalence class s.

We will use the following notation throughout the paper: k is a non-archimedean local field (of arbitrary characteristic) and G is a reductive group over k that will be assumed to split over a tamely ramified field extension of k. We write f for the residue field of k and denote its characteristic by p. We fix an additive character Φ of k, then ˇΦ(·) denotes the coroots. We might abbreviate Φ(H,S) for the roots of H with respect to S, and if S is a maximal torus, then Φ(H,S) ⊂ X∗(S) denotes the coroots. We might abbreviate Φ(HF,T) by Φ(H) for a maximal torus T of HF if the choice of torus T does not matter. We use the notation − : X∗(S) × X∗(S) → Z for the standard pairing, and if S is a maximal torus, then we write A(H) for the adjoint action, denoted by Ad, unless specified otherwise. If X is a scheme defined over the field k, then we denote by X(k) the k-points of X. The action of the group on its derived group. Moreover, if S is a maximal torus, then we let it also act on A∗ via the contragredient action.

In general, we use upper case roman letters, e.g. G, H, G1, T, ..., to denote linear algebraic groups defined over a field F, and we denote the F-points of their Lie algebras by the corresponding lower case fractur letters, e.g. g, h, g_i, t. The action of the group on its Lie algebra is the adjoint action, denoted by Ad, unless specified otherwise. If H is a reductive group over F, then we denote by Hdet its derived group. Moreover, if S is a split torus contained in H (defined over F), then we write X∗(S) = HomF(S, Gm) for the characters of S defined over F, X∗(S) = HomF(Gm, S) for the cocharacters of S (defined over F), Φ(H,S) ⊂ X∗(S) for the roots of H with respect to S, and if S is a maximal torus, then Φ(H,S) ⊂ X∗(S) denotes the coroots. We might abbreviate Φ(HF,T) by Φ(H) for a maximal torus T of HF if the choice of torus T does not matter. We use the notation − : X∗(S) × X∗(S) → Z for the standard pairing, and if S is a maximal torus, then we
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denote by $\hat{\alpha} \in \hat{\Phi}(H, S)$ the dual root of $\alpha \in \Phi(H, S)$. For a subset $\Phi$ of $X^*(S) \otimes_\mathbb{R} \mathbb{R}$ (or $X_*(S) \otimes_\mathbb{Z} \mathbb{R}$ and $R$ a subring of $\mathbb{R}$, we denote by $R\Phi$ the smallest $R$-submodule of $X^*(S) \otimes_\mathbb{R} \mathbb{R}$ (or $X_*(S) \otimes_\mathbb{Z} \mathbb{R}$, respectively) that contains $\Phi$. For $\chi \in X^*(S)$ and $\lambda \in X_*(S)$, we denote by $d\chi \in \text{Hom}_F(\text{Lie}(G_m), \text{Lie}(S))$ and $d\lambda \in \text{Hom}_F(\text{Lie}(G_m), \text{Lie}(S))$ the induced morphisms of Lie algebras.

If $(\pi, V)$ is a representation of a group $H$, then we denote by $V^H$ the elements of $V$ that are fixed by $H$. If $H'$ is a group containing $H$ as a subgroup and $h' \in H'$, then we define the representation $(h'\pi, V)$ of $h'Hh'^{-1}$ by $h'\pi(h) = \pi(h'^{-1}hh')$ for all $h \in h'Hh'^{-1}$.

Finally, we let $\tilde{\mathbb{R}} = \mathbb{R} \cup \{r + | r \in \mathbb{R} \}$ with its usual order, i.e. for $r$ and $s \in \mathbb{R}$ with $r < s$, we have $r < r+ < s < s+$.

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2 Assumption on the residue field characteristic

Recall that $k$ denotes a non-archimedean local field with residual characteristic $p$ and $G$ is a (connected) reductive group over $k$. Since Yu ([Yu01]) works in his construction of supercuspidal representations only with tori that split over a tame extension, we make the following assumption throughout the paper.

**Assumption 2.1.** We assume that $G$ splits over a tamely ramified extension of $k$ and $p \nmid |W|$, where $W$ denotes the Weyl group $W$ of $G(\bar{k})$.

By [Fin18] Assumption 2.1 implies that all tori of $G$ are tame. For absolutely simple groups other than some non-split inner forms of split groups of type $A_n$, $n \geq 2$, $D_l$, $l \geq 4$ prime, or $E_6$ this assumption is also necessary (and in the excluded cases only minor modifications on the assumption on $p$ are necessary), see [Fin18] Theorem 2.4 and Corollary 2.6 for details.

We collect a few consequences of our assumption for later use.

**Lemma 2.2.** The assumption that $p \nmid |W|$ implies the following

(a) The prime $p$ does not divide the order of the Weyl group of any Levi subgroup of (a parabolic subgroup of) $G_{\bar{k}}$.

(b) The prime $p$ is larger than the order of any bond of the Dynkin diagram $\text{Dyn}(G)$ of $G_{\bar{k}}$, i.e. larger than the square of the ratio of two root lengths of roots in $\Phi(G)$.
(c) The prime $p$ is not a bad prime (in the sense of [SS70, 4.1]) for $\Phi := \tilde{\Phi}(G)$, i.e. $\mathbb{Z}\tilde{\Phi}/\mathbb{Z}\tilde{\Phi}_0$ has no $p$-torsion for all closed subsystems $\Phi_0$ in $\tilde{\Phi}$.

(d) The prime $p$ is not a torsion prime (in the sense of [Ste75, 1.3 Definition]) for $\Phi := \Phi(G)$ (and hence also not for $\tilde{\Phi}(G)$), i.e. $\mathbb{Z}\tilde{\Phi}/\mathbb{Z}\tilde{\Phi}_0$ has no $p$-torsion for all closed subsystems $\Phi_0$ in $\tilde{\Phi}$.

(e) The prime $p$ does not divide the index of connection (i.e. the order of the root lattice in the weight lattice) of any root(sub)system generated by a subset of a basis of $\Phi(G)$.

Proof.
Part (a) is obvious, Part (b), (c) and (d) can be read off from Table 1. Part (e) follows from the fact that the index of connection of $\Phi(G)$ divides $|W|$ ([Bou02, VI.2 Proposition 7]).

<table>
<thead>
<tr>
<th>type</th>
<th>$A_n (n \geq 1)$</th>
<th>$B_n (n \geq 3)$</th>
<th>$C_n (n \geq 2)$</th>
<th>$D_n (n \geq 3)$</th>
<th>$E_6$</th>
<th>$E_7$</th>
<th>$E_8$</th>
<th>$F_4$</th>
<th>$G_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>bad</td>
<td>$(n+1)!$</td>
<td>$2^n \cdot n!$</td>
<td>$2^n \cdot n!$</td>
<td>$2^{n-1} \cdot n!$</td>
<td>$2^3 \cdot 3$</td>
<td>$2^4 \cdot 3^2 \cdot 5 \cdot 7$</td>
<td>$2^4 \cdot 3^2 \cdot 5^2 \cdot 7$</td>
<td>$2^7 \cdot 3^3$</td>
<td>$2^4 \cdot 3$</td>
</tr>
<tr>
<td>torsion</td>
<td>-</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2, 3</td>
<td>2, 3</td>
<td>2, 3, 5</td>
<td>2, 3</td>
<td>2, 3</td>
</tr>
</tbody>
</table>

Table 1: Order of Weyl groups ([Bou02, VI.4.5-VI.4.13]); bad primes ([SS70, 4.3]) and torsion primes ([Ste75, 1.13 Corollary]) for irreducible root systems

3 Almost strongly stable and generic elements

Let $E$ be a field extension of $k$. We denote by $\mathcal{B}(G, E)$ the (enlarged) Bruhat–Tits building of $G_E$ over $E$, and we sometimes write $\mathcal{B}$ for $\mathcal{B}(G, k)$. For $x \in \mathcal{B}(G, E)$ and $r \in \mathbb{R}_{>0}$, we write $G(E)_{x,r}$ for the Moy–Prasad filtration subgroup of $G_E$ of depth $r$, which we abbreviate to $G_{x,r}$ for $G(k)_{x,r}$, and we set $G(E)_r = \bigcup_{x \in \mathcal{B}(G, E)} G(E)_{x,r}$. For $r \in \mathbb{R}$, we denote by $(g_E)_{x,r}$ and $(g_E)^*_{x,r}$ the Moy–Prasad filtration of $g_E = \text{Lie}(G_E)(E)$ and its dual $g_E^*$, respectively. We set $(g_E)_r = \bigcup_{x \in \mathcal{B}(G, E)} (g_E)_{x,r}$ and $(g_E)^*_r = \bigcup_{x \in \mathcal{B}(G, E)} (g_E)^*_{x,r}$. Recall that if $X \in (g_E)_{x,r}$, then $X((g_E)_{x,(-r)+}) \subset P_E$ and $X((g_E)_{x,-r}) \subset O_E$. For convenience, we define our Moy–Prasad filtration subgroups and subalgebras with respect to the valuation val of $E$ that extends the normalized valuation val of $k$, i.e. in such a way that by [Adl98 Proposition 1.4.1] (which applies because $G$ splits over a tamely ramified extension) we have

$$ (g_E)_{x,r} \cap g = g_{x,r} $$

for all $r \in \mathbb{R}$. We denote by $G_x$ the reductive quotient of the special fiber of the (connected) parahoric group scheme attached to $G$ at $x$, i.e. $G_x$ is a reductive group defined over $\mathcal{O}$ satisfying $G_x(\mathcal{O}_F) = G(F)_{x,0}/G(F)_{x,0+}$ for every unramified extension $F$ of $k$ with residue field $k_F$. For any $r \in \mathbb{R}$, the adjoint action of $G(k^ur)_{x,0}$ on $(g_E^*)_{x,r}$ induces a linear action of the algebraic group $G_x$ on $V_{x,r} := g_{x,r}/g_{x,r+}$ and therefore also on its dual $V^*_{x,r}$.

For $X \in g_E - \{0\}$ and $x \in \mathcal{B}(G, E)$, we denote by $d_E(x, X) \in \mathbb{R}$ the largest real number $d$ such that $X \in (g_E^*)_{x,d}$, and set $d_E(x, 0) = \infty$. We call $d_E(x, X)$ the depth of $X$ at $x$.
define the depth of $X$ (over $E$) to be $d_E(X) := \sup_{x \in \mathcal{B}(G,E)} d_E(x, X) \in \mathbb{R} \cup \infty$. If $E = k$, then we often write $d(x, X)$ for $d_k(x, X)$ and $d(X)$ for $d_k(X)$. Note that if $X \in \mathfrak{g}^*$, then $d_k(x, X) = d_E(x, X)$ and $d_k(X) = d_E(X)$ by our choice of normalization.

Recall that if $V$ is a finite dimensional linear algebraic representation of a reductive group $H$ defined over some field $F$, then $X \in V(F)$ is called semistable under the action of $H$ if the Zariski-closure of the orbit $H(F).X \subset V(F)$ does not contain zero, and is called unstable otherwise. We introduce two slightly stronger notions for our setting.

**Definition 3.1.** Let $X \in \mathfrak{g}^*$. We denote by $\overline{X}$ the map $V_{x, -d(x,X)} := \mathfrak{g}_{x, -d(x,X)}/\mathfrak{g}_{x, -d(x,X)+} \to \mathfrak{f}$ induced from $X : \mathfrak{g}_{x, -d(x,X)} \to \mathcal{O}$.

- We say that $X$ is almost strongly stable (at $x$) if the $G$-orbit of $X$ is closed and $\overline{X} \in (V_{x, -d(x,X)})^*$ is semistable under the action of $G_x$.
- We say that $X$ is almost stable if the $G$-orbit of $X$ is closed.

**Lemma 3.2.** Let $X \in \mathfrak{g}^* - \{0\}$ be almost strongly stable at $x$. Then $d(x, X) = d(X)$.

**Proof.**
Suppose $d(x, X) < d(X)$, and write $r = d(x, X)$. Then by [AD02, Corollary 3.2.6] (together with their remark at the beginning of Section 3) the coset $X + \mathfrak{g}_{x,r+}^*$ is degenerate, i.e. contains an unstable element. Hence $\overline{X}$ is unstable by [MP94, 4.3. Proposition] (while Moy and Prasad assume simply connectedness throughout their paper [MP94], it is not needed for this claim). A contradiction.

**Definition 3.3.** Let $H$ be a reductive group over some field $F$. A (smooth closed) subgroup $H'$ of $H$ is called a twisted Levi subgroup if there exists a finite field extension $E$ over $F$ such that $H' \times_F E$ is a Levi subgroup of (a parabolic subgroup of) $H \times_F E$.

**Lemma 3.4.** Let $X \in \mathfrak{g}^*$ be almost stable (under the contragredient of the adjoint action of $G$). Then the centralizer $\text{Cent}_G(X)$ of $X$ in $G$ is a twisted Levi subgroup of $G$.

It suffices to show that $\text{Cent}_{G_{\overline{k}}}F(X)$ is a Levi subgroup of $G_{\overline{k}}$, because $\text{Cent}_G(X) \times_k \overline{k} = \text{Cent}_{G_{\overline{k}}}F(X)$. Since $p$ does not divide the order of the Weyl group of $G$, we can $G_{\overline{k}}$-equivariantly identify $\mathfrak{g}_{\overline{k}}^*$ with $\mathfrak{g}_{F}$ (see [AR00, Proposition 4.1], Lemma 2.2(c) and Lemma 2.2(d)). Using this identification to view $X$ in $\mathfrak{g}$, by [Bor91, 14.25 Proposition], every $X$ is contained in the Lie algebra of a Borel subgroup $B = TU$ for $T$ a maximal torus and $U$ the unipotent radical of $B$ (defined over $\overline{k}$). Hence we can write $X = X_s + X_n$, where $X_s \in \text{Lie}(T)(\overline{k})$ and $X_n \in \text{Lie}(U)(\overline{k})$, and there exists a one parameter subgroup $\lambda : \mathbb{G}_m \to T \subset \text{Cent}_{G_{\overline{k}}}F(X_s)$ such that $\lim_{t \to 0} \lambda(t).X_n = 0$, and therefore $\lim_{t \to 0} \lambda(t).X_s = X_s$. Since $X$ is almost stable, this implies that $X_s$ is contained in the $G(\overline{k})$-orbit of $X$, and it suffices to consider $X = X_s$. In other words, we may assume that $X$ is in the zero eigenspace in $\mathfrak{g}_{\overline{k}}^*$ of $T$. Since $p$ is not a torsion prime for $\Phi(G)$ (Lemma 2.2(d)) and $p$ does
Moreover, for all $\alpha \in \Phi(G)$ (Lemma 2.2(e)), we obtain by [Yu01 Proposition 7.1. and 7.2] (which is based on [Ste75]) that the centralizer $\text{Cent}_{G_{\overline{k}}}(X)$ of $X$ in $G_{\overline{k}}$ is a connected reductive group whose root datum is given by $(X^*(T), \Phi_X, X_\ast(T), \check{\Phi}_X)$ with $\Phi_X = \{ \alpha \in \Phi(G_{\overline{k}}, T) \mid X(d\check{\alpha}(1)) = 0 \}$ and $\check{\Phi}_X = \{ \check{\alpha} \mid \alpha \in \Phi_X \}$. Note that $\check{\Phi}_X$ is a closed subsystem of $\check{\Phi}$ (i.e. $\mathbb{Z}\check{\Phi}_X \cap \check{\Phi} = \check{\Phi}_X$). Since $\mathbb{Z}\check{\Phi}/\check{\Phi}_X$ is $p$-torsion free by Lemma 2.2(e), we have $\check{\Phi}_X = \mathbb{Q}\check{\Phi}_X \cap \check{\Phi}$ and hence $\Phi_X = \mathbb{Q}\Phi_X \cap \check{\Phi}$. By [Bou02] VI.1, Proposition 24 there exists a basis $\Delta$ for $\check{\Phi}$ containing a basis $\Delta_X$ for $\Phi_X$. Thus $\text{Cent}_{G_{\overline{k}}}(X)$ is a Levi subgroup of $G_{\overline{k}}$. 

**Definition 3.5.** We say that an almost stable (at $x$) element $X \in g^*$ is *generic of depth $r$ (at $x$)* if there exists a tamely ramified extension $E$ over $k$ and a split maximal torus $T \subset \text{Cent}_G(X) \ltimes_k E$ such that

- $x \in \mathcal{A}(T, E) \cap \mathcal{B}(G, k)$, where $\mathcal{A}(T, E)$ denotes the apartment of $T$ in $\mathcal{B}(G, E)$
- $X \in g_{x,r}^*$ (i.e. $X(g_{x,(-r)+}) \subseteq P$),
- for every $\alpha \in \Phi(G, T)$ we have $X(H_\alpha) = 0$ or $\text{val}(X(H_\alpha)) = r$, where $H_\alpha = d\check{\alpha}(1)$, and
- if $X(H_\alpha) = 0$ for all $\alpha \in \Phi(G, T)$, then $d(x, X) = r$.

Note that $H_\alpha = d\check{\alpha}(1) \neq 0$, because $p$ does not divide the index of connection of $\Phi(G)$ by Lemma 2.2(e).

**Lemma 3.6.** Let $X \in g^*$ be almost stable and generic of depth $r$ at $x$. Then for every (split) maximal torus $T \subset \text{Cent}_G(X) \ltimes_k \overline{k}$ we have

- $X(H_\alpha) = 0$ for all $\alpha \in \Phi(\text{Cent}_G(X), T)$, and
- $\text{val}(X(H_\alpha)) = r$ for all $\alpha \in \Phi(G, T) - \Phi(\text{Cent}_G(X), T)$.

Moreover, for all $\alpha \in \Phi(G, T)$ we have $X((g_{\overline{k}})_\alpha) = 0$, where $(g_{\overline{k}})_\alpha$ denotes the $\alpha$-root subspace of $g_{\overline{k}}$.

**Proof.**

Choose a Chevalley system $\{ x_\alpha : G_a \to G_{\overline{k}} \mid \alpha \in \Phi(G, T) \}$ with corresponding Lie algebra elements $\{ X_\alpha = dx_\alpha(1) \mid \alpha \in \Phi(G, T) \}$. Since $T \subset \text{Cent}_G(X) \ltimes_k \overline{k}$, we have $X(X_\alpha) = X(\text{Ad}(t)(X_\alpha)) = \alpha(t)X(X_\alpha)$ for all $t \in T(\overline{k})$, and hence $X(X_\alpha) = 0$ for all $\alpha \in \Phi(G, T)$. Thus $X((g_{\overline{k}})_\alpha) = 0$.

Since the split maximal tori of $\text{Cent}_G(X) \ltimes_k \overline{k}$ are conjugate in $\text{Cent}_G(X) \ltimes_k \overline{k}$, we have $X(H_\alpha) = 0$ or $\text{val}(X(H_\alpha)) = r$ for $\alpha \in \Phi(G, T)$. By [Yu01 Propostion 7.1], we have $\alpha \in \Phi(\text{Cent}_G(X), T)$ if and only if $X(H_\alpha) = 0$, see also the proof of Lemma 3.4.

**Corollary 3.7.** Let $X \in g^* - \{0\}$ be almost stable and generic of depth $r$ at $x$. Then $d(x, X) = d(X) = r$. 

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Proof.
Let $E$ be a tame extension of $k$ and $T$ a split maximal torus of $\text{Cent}_G(X) \times_k E$ such that $x \in \mathcal{A}(T, E)$. By Lemma 2.2[1], the element $H_\alpha$ is of depth zero for all $\alpha \in \Phi(G, T)$. Hence $d(x, X) = d_E(x, X) = r$ by Lemma 3.6 (or by definition if $X(H_\alpha) = 0$ for all $\alpha \in \Phi(G, T)$). If $y \in B(G_E, E)$, $s \in \mathbb{R}$ and $X \in (\mathfrak{g}_E)^{y,s}$, then [Yu01, Lemma 8.2] implies that $X$ restricted to $\text{Lie}(T)(E)$ lies in $\text{Lie}(T)^*(E)_x$. Since $X$ has depth $d(x, X) = r$ when restricted to $\text{Lie}(T)(E)$, we deduce that $d_E(y, X) \leq r$. Hence $d(X) = d(x, X) = r$. 

\[\square\]

Corollary 3.8. Let $X \in \mathfrak{g}^* - \{0\}$ be almost stable and generic of depth $r$ at $x$. Then $X$ is almost strongly stable at $x$.

Proof.
Suppose $X$ is not almost strongly stable at $x$. Then $\overline{X} \in \mathfrak{g}^*_{x,r}/\mathfrak{g}^*_{x,r+}$ is unstable. Since $\mathfrak{g}$ is perfect, by Kem78, Corollary 4.3 there exists a non-trivial one parameter subgroup $\lambda : \mathbb{G}_m \rightarrow \mathbb{G}_x$ in the reductive quotient $G_x$ of $G$ at $x$ (defined over $f$) such that $\lim_{t \rightarrow 0} \lambda(t).X = 0$. Let $S$ be a maximal split torus of $G_x$ containing $\overline{\lambda}(\mathbb{G}_m)$. Then there exists a split torus $S$ (defined over $O_k$) in the parahoric group scheme $\mathbb{P}_x$ of $G$ at $x$ whose special fiber is $S$ and whose generic fiber $S$ is a split torus in $G$. This allows us to lift $\lambda$ to a one parameter subgroup $\lambda : \mathbb{G}_m \rightarrow S \subset G$. Let $\mathcal{A}(S, k)$ be the apartment of $S$ (i.e. the apartment of a maximal torus in $G$ that contains $S$). Then $\mathcal{A}(S, k)$ contains $x$ and is the affine space underlying the real vector space $X_*(S) \otimes_{\mathbb{Z}} \mathbb{R}$. If $\epsilon > 0$ is sufficiently small, we obtain $X \in \mathfrak{g}^*_{x+\epsilon}\alpha,r+$. Hence $d(X) > r$, which contradicts Corollary 3.7.

Recall that if $G'$ is a Levi subgroup of (a parabolic subgroup of) $G$, then we have an embedding of the corresponding Bruhat–Tits buildings $\mathcal{B}(G', k) \hookrightarrow \mathcal{B}(G, k)$. Even though this embedding is only unique up to some translation, its image is unique. Since we assume that all tori of $G$ split over tamely ramified extensions of $k$, every twisted Levi subgroup of $G$ becomes a Levi subgroup over a finite tamely ramified extension of $k$. Hence using (tame) Galois descent, we obtain a well defined image of $\mathcal{B}(G', k)$ in $\mathcal{B}(G, k) = \mathcal{B}$ for every twisted Levi subgroup $G'$ of $G$. In the sequel, we might identify $\mathcal{B}(G', k)$ with its image in $\mathcal{B}$.

Note that since $p$ does not divide the index of connection of $\Phi(G)$ (Lemma 2.2[1]), Adler and Roche ([AR00, Proposition 4.1]) provide a non-degenerate, $G$-equivariant, symmetric bilinear form $B : \mathfrak{g} \times \mathfrak{g} \rightarrow k$ such that the induced identification of $\mathfrak{g}$ with $\mathfrak{g}^*$ identifies $\mathfrak{g}_{x,r}$ with $\mathfrak{g}^*_{x,r}$ for all $x \in \mathcal{B}(G, k), r \in \mathbb{R}$. Using the bilinear form to view $(\mathfrak{g}')^* = (\text{Lie}(G')(k))^*$ as a subset of $\mathfrak{g}^*$, we have the following lemma, which is a translation of a result by Kim–Murnaghan ([KM03, Lemma 2.3.3]) into the dual setting.

Lemma 3.9 (Kim–Murnaghan). Let $r \in \mathbb{R}$, $x \in \mathcal{B}$, and let $X \in \mathfrak{g}^*_{x,r} \subset \mathfrak{g}^*$ be almost stable and generic of depth $r$ at $x$. Denote $\text{Cent}_G(X)$ by $G'$ and $\text{Lie}(\text{Cent}_G(X))(k)$ by $\mathfrak{g}'$. If $X' \in (\mathfrak{g}')^*_{r+} \subset \mathfrak{g}^*$ and $y \in \mathcal{B}(G, k) - \mathcal{B}(G', k)$, then $d(y, X + X') < d(X)$.

Proof.
Suppose $X \neq 0$ as the statement is trivial otherwise. Let $E$ be a tame extension of $k$ and $T$ a split maximal torus of $\text{Cent}_G(X) \times_k E$ such that $x \in \mathcal{A}(T, E)$. By the definition of the bilinear form for tame $p$-adic groups Jessica Fintzen
form \( B \) in the proof of \([\text{AR00}, \text{Proposition 4.1}]\) (a sum of scalings of killing forms together with a bilinear form on the center) together with Lemma \(3.6\), the almost stable and generic element \( X \) corresponds to an element \( \tilde{X} \) of \( t = \text{Lie}(T)(k) \subset g \), hence of \( t_r = t \cap g_{x,r} \). Moreover, it follows from the definition of the bilinear form that \( d\alpha(\tilde{X}) = X(H_\alpha) \). Hence Lemma \(3.6\) implies that \( \tilde{X} \) is a good semisimple element of depth \( r \). Since \( \text{Cent}_G(X) = \text{Cent}_G(\tilde{X}) \) and \( X' \) corresponds to an element in \( g'_{x,r} \) under the identification of \( g^* \) with \( g \), the lemma follows from \([\text{KM03}, \text{Lemma 2.3.3}]\), because \( B \) impose in \([\text{KM03}]\) much stronger conditions on \( G \) and \( k \), in particular that \( k \) has characteristic zero. However the required Lemma \(2.3.3\) holds also in our setting by the same proof and from \([\text{KM03}, \text{Lemma 2.3.3}]\), because \( B \) preserves depth. (Note that Kim and Murnaghan impose in \([\text{KM03}]\) much stronger conditions on \( G \) and \( k \), in particular that \( k \) has characteristic zero. However the required Lemma \(2.3.3\) holds also in our setting by the same proof and from \([\text{KM03}, \text{Lemma 2.3.3}]\)) follow from results of Adler and Roche \([\text{AR00}]\) that are valid in our setting.

To state the following main result in this section more conveniently, we fix a \( G \)-equivariant distance function \( d : \mathcal{B}(G, k) \times \mathcal{B}(G, k) \to \mathbb{R}_{>0} \) on the building \( \mathcal{B}(G, k) \), which is the restriction of a distance function \( d_E : \mathcal{B}(G, E) \times \mathcal{B}(G, E) \to \mathbb{R}_{>0} \) for some tame extension \( E \) of \( k \) over which \( G \) splits that satisfies \( |\alpha(x - y)| \leq d_E(x, y) \) for all maximal split tori \( T_E \) of \( G_E \) and all \( x, y \in \mathcal{A}(T, E) \). (This normalization will only become relevant later.)

**Proposition 3.10.** Let \( r \in \mathbb{R} \) and \( x \in \mathcal{B} \). If \( X \in g_{x,r}^* \) is almost strongly stable at \( x \), then for every \( \epsilon > 0 \) there exists \( x' \in \mathcal{B} \) with \( d(x, x') < \epsilon \) such that \( X \in g_{x',r}^* \), the coset \( X + g_{x',r}^* \) contains an element \( \tilde{X} \) that is almost strongly stable and generic of depth \( r \) at \( x \), and the points \( x \) and \( x' \) are contained in \( \mathcal{B}(\text{Cent}_G(X), k) \subset \mathcal{B} \).

**Proof.**

Let \( T \) be a maximal torus of \( \text{Cent}_G(X) \) and \( E \) a tame extension of \( k \) over which \( T \) splits. Choose a point \( y \) in \( \mathcal{A}(T, E) \cap \mathcal{B}(G, k) \). If \( \alpha \in \Phi(G, T_E) \) and \( X_\alpha \in (g_E)_\alpha \), then \( X_\alpha = X(\text{Ad}(t)X_\alpha) = \alpha(t)X(X_\alpha) \) for all \( t \in T(E) \), hence \( X_\alpha = 0 \). Thus the depth of \( X \) at \( y \) is equal to the depth of \( X \) restricted to \( t = \text{Lie}(T)(k) \). On the other hand, by \([\text{Yu01}, \text{Lemma 8.2}]\), the assumption that \( X \in g_{x,r}^* \) implies that \( X \) restricted to \( t \) lies in \( t_r \). Hence \( d(y, X) \geq r \). Since \( r = d(X) \) by Lemma \(3.2\), we deduce that \( d(y, X) = r \).

**Claim.** \( X + g_{y,r}^* \) contains an almost stable and generic element of depth \( r \).

**Proof of claim.** Let \( \hat{\Phi}_0 \subset \hat{\Phi} := \hat{\Phi}(G, T_E) \) be the collection of coroots \( \hat{\alpha} \) for which \( \text{val}(X(H_\alpha)) > r \). Note that \( \hat{\Phi}_0 \) is a closed subsystem of \( \hat{\Phi} \) (i.e. \( \mathbb{Z}\hat{\Phi}_0 \cap \hat{\Phi} = \hat{\Phi}_0 \)). Since \( \mathbb{Z}\hat{\Phi} / \mathbb{Z}\hat{\Phi}_0 \) is \( p \)-torsion free by Lemma \(2.2[\text{c}]\), we also have \( \hat{\Phi}_0 = \mathbb{Q}\hat{\Phi}_0 \cap \hat{\Phi} \). Moreover, since \( X \) and \( T \) are defined over \( k \), the set \( \hat{\Phi}_0 \) is stable under the action of the Galois group \( \text{Gal}(E/k) \). Let \( Y \subset g_E = \text{Lie}(G)(E) \) be the \( E \)-subspace spanned by \( \{H_\alpha | \hat{\alpha} \in \hat{\Phi}_0\} \). By the above observations about \( \hat{\Phi}_0 \), the subspace \( Y \) is \( \text{Gal}(E/k) \)-stable, and if \( H_\alpha \subset Y \), then \( \hat{\alpha} \in \hat{\Phi}_0 \).

Define

\[
Y_T^\perp := \{ Z \in \text{Lie}(T)(E) | d\alpha(Z) = 0 \ \forall \alpha \in \hat{\Phi}_0 \}.
\]

Then \( Y_T^\perp \) is a \( \text{Gal}(E/k) \)-stable complement to \( Y \) in \( \text{Lie}(T)(E) \), and we set

\[
Y^\perp := Y_T^\perp \oplus \bigoplus_{\alpha \in \Phi(G, T)} (g_E)_\alpha.
\]
Then $Y^\perp$ is a $\text{Gal}(E/k)$-stable complement to $Y$ in $\mathfrak{g}_E$, and we define $X' \in \mathfrak{g}_E^*$ by

$$X'(Z + Z^\perp) = X(Z) \text{ for all } Z \in Y, Z^\perp \in Y^\perp.$$

Since $Y$ and $Y^\perp$ are $\text{Gal}(E/k)$-stable and $X$ is defined over $k$, the linear functional $X'$ is $\text{Gal}(E/k)$-invariant and hence defined over $k$, i.e. we can view $X'$ as an element of $\mathfrak{g}^*$. 

Let $\tilde{\Delta}_0$ be a basis for $\check{\Phi}_0$, and $\tilde{\Delta}$ a basis for $\check{\Phi}$ containing $\tilde{\Delta}_0$ (which is possible by [Bou02, VI.1, Proposition 24]). For $\tilde{\alpha} \in \Delta$, we denote by $\check{\omega}_\alpha \in \mathbb{Q}\check{\Phi}$ the fundamental coweight corresponding to $\alpha$, i.e. $\langle \check{\omega}_\alpha, \alpha \rangle = 1$ and $\langle \check{\omega}_\alpha, \beta \rangle = 0$ for $\beta \notin \bigwedge - \{ \tilde{\alpha} \}$. Similarly, for $\tilde{\alpha} \in \tilde{\Delta}_0$, let $\check{\omega}_\alpha^0 \in \mathbb{Q}\check{\Phi}_0$ be the fundamental coweight with respect to the (co-)root system $\check{\Phi}_0$. By Lemma 2.2(e), we have $\check{\omega}_\alpha \in \mathbb{Z}\left[ \frac{1}{|W|} \right] \check{\Phi}$ and $\check{\omega}_\alpha^0 \in \mathbb{Z}\left[ \frac{1}{|W|} \right] \check{\Phi}_0$. Denote by $H_{\check{\omega}_\alpha} (\tilde{\alpha} \in \tilde{\Delta})$ and $H_{\check{\omega}_\alpha^0} (\tilde{\alpha} \in \tilde{\Delta}_0)$ the image of $\check{\omega}_\alpha$ and $\check{\omega}_\alpha^0$ under the linear map $\mathbb{Z}\left[ \frac{1}{|W|} \right] \check{\Phi} \to \text{Lie}(T)(E)$ obtained by sending $\tilde{\alpha}'' = H_{\check{\omega}_\alpha^0} (\tilde{\alpha} \in \tilde{\Delta}_0)$. Then we have

$$H_{\check{\omega}_\alpha} = \begin{cases} 0 & \text{mod } Y^\perp \text{ for } \tilde{\alpha} \in \tilde{\Delta} - \tilde{\Delta}_0, \\
H_{\check{\omega}_\alpha^0} & \text{mod } Y^\perp \text{ for } \tilde{\alpha} \in \tilde{\Delta}_0. \end{cases}$$

For $\beta \in \Phi$, we have $\check{\beta} = \sum_{\tilde{\alpha} \in \Delta} \langle \check{\beta}, \alpha \rangle \check{\omega}_\alpha$, and hence we obtain

$$H_{\beta} = \sum_{\tilde{\alpha} \in \Delta} \langle \check{\beta}, \alpha \rangle H_{\check{\omega}_\alpha} \equiv \sum_{\tilde{\alpha} \in \tilde{\Delta}_0} \langle \check{\beta}, \alpha \rangle H_{\check{\omega}_\alpha^0} \text{ mod } Y^\perp. \quad (2)$$

Recall that $\langle \check{\beta}, \alpha \rangle$ are integers for $\tilde{\alpha} \in \tilde{\Delta}$ and that the index of the coroot lattice $\mathbb{Z}\check{\Phi}_0$ in the coweight lattice $\mathbb{Z}[\check{\omega}_\alpha | \tilde{\alpha} \in \check{\Phi}_0]$ is coprime to $p$ by Lemma 2.2(e). Hence $\sum_{\tilde{\alpha} \in \tilde{\Delta}_0} \langle \check{\beta}, \alpha \rangle H_{\check{\omega}_\alpha^0}$ is contained in the $\mathcal{O}_E$-span of $\{ H_{\check{\omega}_\alpha} \tilde{\alpha} \in \check{\Phi}_0 \}$. Thus, by the definition of $\check{\Phi}_0$, we obtain $\text{val}(X'(H_{\beta})) > r$ for all $\tilde{\beta} \in \check{\Phi}$. In addition, $X'$ vanishes on the center of $\mathfrak{g}_E$ and on $\bigoplus_{\alpha \in \check{\Phi}(G,T)} (\mathfrak{g}_E)_\alpha$, because these subspaces are contained in $Y^\perp$. Hence, by Lemma 2.2(e), we have $\text{val}(X'( (\mathfrak{g}_E)_{y,0} )) \subset \mathbb{R}_{>r}$. Using that the Moy–Prasad filtration behaves well with respect to base change (Equation 1), we obtain

$$\text{val}(X'( (\mathfrak{g}_{y,-r}) )) \subset \text{val}(X'( (\mathfrak{g}_E)_{y,-r} )) \subset \mathbb{R}_{\geq r} + \text{val}(X'( (\mathfrak{g}_E)_{y,0} )) \subset \mathbb{R}_{>0}.$$  

Thus $X' \in \mathfrak{g}_{y,r+}^*$, and $\check{X} = X - X' \in X + \mathfrak{g}_{y,r+}^*$ with $\text{val}(\check{X}(H_{\alpha})) = r$ for $\tilde{\alpha} \notin \check{\Phi}_0$ and $\check{X}(H_{\alpha}) = 0$ for $\tilde{\alpha} \in \check{\Phi}_0$.

In order to prove the claim, it remains to show that the orbit of $\check{X}$ is closed. Since $p \nmid |W|$ we can $G$-equivariantly identify $\mathfrak{g}^*$ with $\mathfrak{g}$ as above. Since $T$ is in $\text{Cent}_G(X)$ and acts trivially on $X'$, the torus $T$ also centralizes $\check{X} = X - X'$, and hence $\check{X} \subset \text{Lie}(T)(E)$ under the identification of $\mathfrak{g}^*$ with $\mathfrak{g}$. Thus $\check{X}$ is semisimple, and therefore its $G$-orbit is closed ([Bor91, 9.2]). Hence $\check{X} \in X + \mathfrak{g}_{y,r+}^*$ is almost stable and generic of depth $r$ at $y$.

To finish the proof of the proposition, recall that $d(x, \check{X} + X') = d(x, X) = r = d(y, \check{X}) = d(\check{X})$ (by Corollary 3.7). We write $G' = \text{Cent}_G(\check{X})$ and $\mathfrak{g}' = \text{Lie}(G')(k)$. Since $X'$ has depth
greater $r$ at $y$ and vanishes on $\bigoplus_{\alpha \in \Phi(G,T)}(g_{\alpha})_{\alpha}$, it lies in $(g')_{+} \subset g^\ast$. Hence we deduce from Lemma 3.9 that $x \in \mathcal{B}(G', k)$. Thus there exists a maximal torus $\tilde{T}$ in $G' \subset G$ with $x \in \mathcal{A}(\tilde{T})$, and, by Lemma 3.6, the element $\tilde{X}$ is almost stable and generic of depth $r$ at $x$. If $\tilde{X} \in X + g_{x,r,r}^\ast$, then we are done by choosing $x' = x$ and observing that $X = \tilde{X}$.

Hence it remains to consider the case that $\tilde{X} \notin X + g_{x,r,r}^\ast$. Then $d(x, X') = d(x, X - \tilde{X}) = r < d(y, X') \leq d(X')$. Viewing these as depths for $\mathcal{B}(G', k)$, we deduce from [AD02 Corollary 3.2.6] (together with their remark at the beginning of Section 3) that the coset $X' + (g')_{+}^\ast$ is degenerate, i.e., contains an unstable element. Hence $\tilde{X}' \in (g')_{+}^\ast/(g')_{+}^\ast$ is unstable by [MP94 4.3. Proposition]. Since $f$ is perfect, by [Kem78 Corollary 4.3] there exists a non-trivial one parameter subgroup $\lambda : \mathbb{G}_m \to G_x^\ast$ in the reductive quotient $G_x^\ast$ of $G'$ at $x$ (defined over $\mathbb{F}_q$) such that $\lim_{t \to 0} \lambda_0(t) = 0$. As in the proof of Corollary 3.8, we let $S$ be a maximal split torus of $G'_x$ containing $\tilde{X}(\mathbb{G}_m)$, and $\mathcal{A}$ a split torus (defined over $\mathcal{O}_k$) in the parahoric group scheme $\mathbb{P}_x$ of $G'$ at $x$ whose special fiber is $S$ and whose generic fiber $S$ is a split torus in $G'$. This allows us to consider $\lambda$ as an element $\lambda$ of $X_0(S)$. Let $\mathcal{A}(S, k)$ be the apartment of $S$ (i.e., the apartment of a maximal (maximally split) torus $T_S \subset G'$ containing $S$). Then $\mathcal{A}(S, k)$ contains $x$ and is the affine space underlying the real vector space $X_0(S) \otimes \mathbb{R}$. If $\epsilon > 0$ is sufficiently small, then $X' \in g_{x,\epsilon+\lambda,r,r}^\ast$ and $X \equiv \tilde{X}$ mod $g_{x,\epsilon+\lambda,r,r}^\ast$.

Let $E'$ be a tamely ramified extension of $k$ over which $T_S$ splits. Then $x' := x + \epsilon \lambda \in \mathcal{A}(T_S, E) \cap \mathcal{A}(G, k)$, and since $T_S \subset G' = \text{Cent}_G(S)$, the element $\tilde{X}$ is almost stable and generic at $x'$ of depth $r$ (by Lemma 3.6). Assuming $\epsilon$ is sufficiently small, the quotient $g_{x,r}^\ast/g_{x,r,r}^\ast$ is a quotient of a subspace of $g_{x,r}^\ast/g_{x,r,r}^\ast$, and the reductive quotient $G_{x'}$ is a quotient of a subgroup of $G_x$. Identifying the images of $(t'_x)_{x'}$ in $g_{x',r}^\ast/g_{x',r,r}^\ast$ and in $g_{x,r}^\ast/g_{x,r,r}^\ast$ and the root subspaces of $g_{x,r}^\ast/g_{x,r,r}^\ast$ with the corresponding root subspaces in $g_{x',r}^\ast/g_{x',r,r}^\ast$, we can view $g_{x,r}^\ast/g_{x,r,r}^\ast$ as a subspace of $g_{x',r}^\ast/g_{x',r,r}^\ast$, and similarly view $G_{x'}$ as a subgroup of $G_x$.

Under this identification $\lim_{t \to 0} \lambda(t) = \lim_{t \to 0} \lambda_0(t) = \lim_{t \to 0} \lambda(t)(X + X') = \lim_{t \to 0} \lambda(t)X \in g_{x,r}^\ast/g_{x,r,r}^\ast$ is contained in $g_{x',r}^\ast/g_{x',r,r}^\ast$ and coincides with the image of $\tilde{X}$ in $g_{x',r}^\ast/g_{x',r,r}^\ast$. Since $\tilde{X} \in g_{x,r}^\ast/g_{x,r,r}^\ast$ is semistable under the action of $G_x$, the limit $\lim_{t \to 0} \lambda(t)X$ is also semistable under the action of $G_x$, and hence under the action of $G_{x'} \subset G_x$. Thus $\tilde{X} \subset X + g_{x',r}^\ast$ is almost strongly stable at $x'$.

\begin{aside}

3.11. The claim proved within the proof of Proposition 3.10 is the dual statement of [Fin18 Theorem 3.3] and could be deduced from the latter as well. We decided to give an independent (but analogous) proof so that the reader has the option to see what assumptions on $p$ enter the claim at which point and observe that in many cases slightly weaker assumptions on $p$ suffice.

\end{aside}

4 The datum

In this section we define the notion of a datum of $G$ and what it means for a datum to be contained in a smooth irreducible representation of $G(k)$. In Section 6 (Theorem 6.1) we
will show that every irreducible representation contains such a datum. From this result we will deduce in Section 7 (Theorem 7.12) and Section 8 (Theorem 8.1) that every irreducible representation contains a type of the form constructed by Kim–Yu [KY17] based on Yu’s construction of supercuspidal representations [Yu01] and that Yu’s construction yields all supercuspidal representations.

**Definition 4.1.** Let \( n \in \mathbb{Z}_{\geq 0} \). An extended datum of \( G \) of length \( n \) is a tuple

\[
(x, (r_i)_{1 \leq i \leq n}, (X_i)_{1 \leq i \leq n}, (G_i)_{1 \leq i \leq n+1}, (\rho_0, V_{\rho_0}))
\]

where

(a) \( r_1 > r_2 > \ldots > r_n > 0 \) are real numbers

(b) \( X_i \in \mathfrak{g}^*_{x,-r_i} - \{0\} \) for \( 1 \leq i \leq n \)

(c) \( G = G_1 \supseteq G_2 \supseteq G_3 \supseteq \ldots \supseteq G_{n+1} \) are twisted Levi subgroups of \( G \)

(d) \( x \in \mathcal{B}(G_{n+1}, k) \subset \mathcal{B}(G, k) \)

(e) \( (\rho_0, V_{\rho_0}) \) is an irreducible representation of \( (G^\text{der}_{n+1})_{x,0}/(G^\text{der}_{n+1})_{x,0} \)

satisfying the following conditions for all \( 1 \leq i \leq n \)

(i) \( X_i \in \mathfrak{g}^*_i := \text{Lie}(G_i)(k)^* \subset \mathfrak{g}^* \)

(ii) \( X_i \) is almost strongly stable and generic of depth \(-r_i\) at \( x \in \mathcal{B}(G_i, k) \) as element of \( \mathfrak{g}^*_i \) (under the action of \( G_i \))

(iii) \( G_{i+1} = \text{Cent}_{G_i}(X_i) \)

A truncated extended datum of \( G \) of length \( n \) is a tuple

\[
(x, (r_i)_{1 \leq i \leq n}, (X_i)_{1 \leq i \leq n}, (G_i)_{1 \leq i \leq n+1})
\]

of data as above satisfying (a) through (d) and (i) through (iii).

**Definition 4.2.** Let \( n \in \mathbb{Z}_{\geq 0} \). A datum of \( G \) of length \( n \) is a tuple \( (x, (X_i)_{1 \leq i \leq n}, (\rho_0, V_{\rho_0})) \) consisting of a point \( x \in \mathcal{B}(G, k) \), elements \( X_i \in \mathfrak{g}^* \) for \( 1 \leq i \leq n \) and an irreducible representation \( (\rho_0, V_{\rho_0}) \) of \( \text{Cent}_G(\sum_{i=1}^n X_i)^\text{der}(k) \cap G_{x,0}/\text{Cent}_G(\sum_{i=1}^n X_i)^\text{der}(k) \cap G_{x,0}^+ \) for which there exist real numbers \( r_1 > r_2 > \ldots > r_n > 0 \) and a sequence of twisted Levi subgroups \( G = G_1 \supseteq G_2 \supseteq G_3 \supseteq \ldots \supseteq G_{n+1} \) of \( G \) such that \( (x, (r_i)_{1 \leq i \leq n}, (X_i)_{1 \leq i \leq n}, (G_i)_{1 \leq i \leq n+1}, (\rho_0, V_{\rho_0})) \) is an extended datum.

A truncated datum of \( G \) of length \( n \) is a tuple \( (x, (X_i)_{1 \leq i \leq n}) \) consisting of a point \( x \in \mathcal{B}(G, k) \) and elements \( X_i \in \mathfrak{g}^* \) for \( 1 \leq i \leq n \) for which there exist real numbers \( r_1 > r_2 > \ldots > r_n > 0 \) and a sequence of twisted Levi subgroups \( G = G_1 \supseteq G_2 \supseteq G_3 \supseteq \ldots \supseteq G_{n+1} \) of \( G \) such that \( (x, (r_i)_{1 \leq i \leq n}, (X_i)_{1 \leq i \leq n}, (G_i)_{1 \leq i \leq n+1}) \) is a truncated extended datum.
Given a truncated datum \((x, (X_i)_{1 \leq i \leq n})\) or a datum \((x, (X_i)_{1 \leq i \leq n}, (\rho_0, V_{\rho_0}))\) of \(G\) of length \(n\), we denote by \((x, (r_i)_{1 \leq i \leq n}, (X_i)_{1 \leq i \leq n}, (G_i)_{1 \leq i \leq n+1})\) the unique truncated extended datum containing it or by \((x, (r_i)_{1 \leq i \leq n}, (X_i)_{1 \leq i \leq n}, (G_i)_{1 \leq i \leq n+1}, (\rho_0, V_{\rho_0}))\) the unique extended datum containing it, respectively, as in Definition 4.2. For convenience, we set \(r_{n+1} = 0\). In addition, we define for \(1 \leq i \leq n + 1\) the following associated subgroups

- \(H_1 := G_1\) if \(G_1 = G_2\) and \(H_1 := G_1^{\text{der}}\) if \(G_1 \neq G_2\)
- \(H_i := G_i^{\text{der}}\) for \(i > 1\)
- \((H_i)_{x,\overline{r}} := G_{x,\overline{r}} \cap H_i(k) = (H_i)_{x,\overline{r}}\) for \(\overline{r} \in \overline{\mathbb{R}}_{\geq 0} := \mathbb{R}_{\geq 0} \cup \{r + \mid r \in \mathbb{R}_{\geq 0}\}\),

where \(x_i\) denotes the image of \(x \in \mathcal{B}(G_i, k)\) in \(\mathcal{B}(H_i, k)\). In order to define another subgroup \((H_i)_{x,\overline{r},r'}\) of \(G(k)\) for \(\overline{r} \geq r' \geq \frac{\nu}{2} > 0\) \((\overline{r}, r' \in \overline{\mathbb{R}})\) and \(1 \leq i \leq n\), we choose a maximal torus \(T\) of \(G_{i+1}\) such that \(x \in \mathcal{A}(T, E)\), where \(E\) denotes a finite tamely ramified extension of \(k\) over which \(T\) splits. Then we define

\[
(H_i)_{x,\overline{r},r'} := (G_i)_{x,\overline{r}} \cap (G_i)_{x,\overline{r},r'}
\]

Note that \((H_i)_{x,\overline{r},r'}\) is denoted \((H_{i+1}, H_i)(k)_{x,\overline{r},r'}\) in [Yu01]. Yu ([Yu01, p. 585 and p. 586]) shows that this definition is independent of the choice of \(T\) and \(E\). We define the subalgebras \(h_i\), \((h_i)_{x,\overline{r}}\) and \((h_i)_{x,\overline{r},r'}\) of \(g\) analogously.

**Definition 4.3.** Let \((\pi, V_\pi)\) be a smooth irreducible representation of \(G(k)\). A datum \((x, (X_i)_{1 \leq i \leq n}, (\rho_0, V_{\rho_0}))\) of \(G\) is said to be contained in \((\pi, V_\pi)\) if \(V_{\pi}^{(H_i)_{x,\overline{r},r'+1}}\) contains a subspace \(V'\) such that

- \((\pi|_{(H_{n+1})_{x,0}}/V')\) is isomorphic to \((\rho_0, V_{\rho_0})\) as a representation of \((H_{n+1})_{x,0}/(H_{n+1})_{x,0+}\)
- \((H_i)_{x,r_i,\overline{r_i}'+1}/(H_i)_{x,r_i} \simeq (h_i)_{x,r_i,\overline{r_i}'+1}/(h_i)_{x,r_i}\) acts on \(V'\) via the character \(\varphi \circ X_i\) for \(1 \leq i \leq n\),

where \(\varphi : k \to \mathbb{C}^*\) is an additive character of \(k\) of conductor \(\mathcal{P}\) that is fixed throughout the paper.

Similarly, \((\pi, V_\pi)\) is said to contain a truncated datum \((x, (X_i)_{1 \leq i \leq n})\) if \(V_{\pi}^{(H_i)_{x,\overline{r_i}+1}}\) contains a one dimensional subspace on which \((H_i)_{x,r_i,\overline{r_i}+1}/(H_i)_{x,r_i} \simeq (h_i)_{x,r_i,\overline{r_i}+1}/(h_i)_{x,r_i}\) acts via \(\varphi \circ X_i\) for \(1 \leq i \leq n\).
Definition 4.4. Let \((\pi, V_{n})\) be a smooth irreducible representation of \(G(k)\). We say that a tuple \((x, (X_{i})_{1 \leq i \leq n}, (\rho_{0}, V_{r_{0}}))\) is a datum for \((\pi, V_{n})\) if \((x, (X_{i})_{1 \leq i \leq n}, (\rho_{0}, V_{r_{0}}))\) is a datum of \(G\) that is contained in \((\pi, V_{n})\) such that if \((x', (X'_{i})_{1 \leq i \leq n'}, (\rho'_{0}, V'_{r_{0}}))\) is another datum of \(G\) contained in \((\pi, V_{n})\), then the dimension of the facet of \(\mathcal{B}(G_{n+1}, k)\) that contains \(x\) is at least the dimension of the facet of \(\mathcal{B}(\text{Cent}_{G}(\sum_{i=1}^{n'} X'_{i}), k)\) that contains \(x'\).

5 Some results used to exhibit data

In order to prove that every irreducible representations of \(G(k)\) contains a datum, we first prove a lemma and derive some corollaries that we are going to repeatedly use in the proof of the existence of a datum in Section 6.1 (The reader might skip this section at first reading and come back to it when the results are used in the proof of Theorem 6.1.)

Lemma 5.1. Let \(x \in \mathcal{B}(G, k), r \in \mathbb{R}_{\geq 0}\), and let \(X \in g^{*} \setminus \{0\}\) be almost stable and generic of depth \(-r\) at \(x\). Write \(G' = \text{Cent}_{G}(X), g' = \text{Lie}(G')(k)\), and \(T\) be a maximal torus of \(G'\) that splits over a tamely ramified extension \(E\) of \(k\). We set \(t = \text{Lie}(T)(k)\), \(\mathfrak{r}' = g \cap (\bigoplus_{\alpha \in \Phi(G', T_{E})} (g_{E})_{\alpha}) \subset g'\) and \(\mathfrak{r}'' = g \cap (\bigoplus_{\alpha \in \Phi(G, T_{E}) - \Phi(G', T_{E})} (g_{E})_{\alpha})\), and we denote by \(j^{*}\) the subspace of elements in \(g^{*}\) that vanish on \(t \oplus \mathfrak{r}' = g'\). Then

(a) The map \(f_{X} : \mathfrak{r}'' \to g^{*}\) defined by \(Y \mapsto (Z \mapsto X([Y, Z]))\) is a vector space isomorphism of \(\mathfrak{r}''\) onto \(j^{*}\), and \(f_{X}^{-1}(\mathfrak{g}_{x, r'}) = j^{*} \cap \mathfrak{g}_{x, r' + r}^{*}\) for \(r' \in \mathbb{R}\).

(b) Let \(d\) be a real number such that \(\frac{r}{2} \leq d < r\). For every \(0 < \epsilon < \frac{r-d}{2}\), if \(C \in j^{*} \cap \mathfrak{g}_{x, -(d + \epsilon)}^{*}\), then there exists \(g \in G_{x, r-d-\epsilon} \subset G_{x, 0}\) such that

(i) \(\text{Ad}(g)(X + C)|_{\mathfrak{g}_{x, r}} = X|_{\mathfrak{g}_{x, r}}\),

(ii) \(\text{Ad}(g)(X + C)|_{\mathfrak{r}'' \cap \mathfrak{g}_{x, r}} = 0 = X|_{\mathfrak{r}'' \cap \mathfrak{g}_{x, r}}\),

(iii) if \((\pi, V)\) is a representation of \(G(k)\) and \(V'\) is a subspace of \(V\) on which the group

\[G(k) \cap \langle T(E)_{r+}, U_{\alpha}(E)_{x,(d+\epsilon)+}, U_{\beta}(E)_{x,r+} \mid \alpha \in \Phi(G, T_{E}) - \Phi(G', T_{E}), \beta \in \Phi(G', T_{E}) \rangle\]

acts trivially and that is stable under the action of a subgroup \(H\) of \(G'^{\text{der}}(k) \cap G_{x, (2d-r+2\epsilon)+}\), then \(g^{-1}Hg\) preserves \(V'\) and \((g|_{H}, V') = (\pi|_{H}, V')\),

where \(\text{Ad}\) denotes the contragredient of the adjoint action.

Proof.

Let \(Y \in \mathfrak{r}''\). Recall that \([(g_{E})_{\alpha}, (g_{E})_{\beta}] \subset (g_{E})_{\alpha + \beta}\) for \(\alpha, \beta \in \Phi(G, T_{E})\), \(\alpha \neq -\beta\) (where \((g_{E})_{\alpha + \beta} = \{0\}\) if \(\alpha + \beta \notin \Phi(G, T_{E})\)). Hence, if \(Z \in \mathfrak{r}'\), then \([Y, Z] \in \mathfrak{r}'\), and \(X([Y, Z]) = 0\)
by Lemma 3.6. Similarly, if $Z \in t$, then $[Y, Z] \in t''$, and $X([Y, Z]) = 0$. Thus the image of the linear map $f_X$ is contained in $j^* \subset g^*$.

Choose a Chevalley system $\{x_\alpha : G_\alpha \to G_E \mid \alpha \in \Phi(G, T_E)\}$ for $G_E$ with corresponding Lie algebra elements $\{X_\alpha = dx_\alpha(1) \mid \alpha \in \Phi(G, T_E)\}$. Then for $\alpha \in \Phi(G, T_E) - \Phi(G', T_E)$, we have $[X_\alpha, X_{-\alpha}] = H_\alpha = d\alpha(1)$. Hence, extending $f_X$ linearly to $t'' \otimes_k E$, the element $f_X(X_\alpha)$ in $j^* \otimes_k E$ is a map that sends $X_\beta$ to $c_\alpha \cdot \beta$ for $\beta \in \Phi(G, T_E) - \Phi(G', T_E)$ for some constant $c \in E$ with $\text{val}(c) = \text{val}(X(H_\alpha)) = -r$, by Lemma 3.6. From this description, we see that $d(x, f_X(X_\alpha)) = -r - d(x, X_\alpha) = -r + \langle -\alpha(x) \rangle = \alpha(x) - r$ while $d(x, X_\alpha) = \alpha(x)$. Thus $f_X(t'' \otimes_k E \cap (g_E)_{x,r'}) = j^* \otimes_k E \cap (g^*)_{x,r' - r}$, and hence $f_X(t'' \cap g_{x,r'}) = j^* \cap (g^*)_{x,r' - r}$ because $E$ is tamely ramified over $k$. In particular, $f_X : t'' \to j^*$ is a vector space isomorphism.

(b) By (a), there exists $Y \in t''$ of depth $\geq r - d - \epsilon > 0$ such that $C = X([Y, \_])$. Let $\text{exp}$ denote a mock exponential function from $g_{x,r-d-\epsilon}$ to $G_{x,r-d-\epsilon}$ as defined in [Adl98, Section 1.5], i.e. if $Y = \sum_{\alpha \in \Phi(G,T_E) - \Phi(G',T_E)} x_\alpha X_\alpha$ for some $a_\alpha \in E$, then

$$\text{exp}(Y) \equiv \prod_{\alpha \in \Phi(G,T_E) - \Phi(G',T_E)} x_\alpha(a_\alpha) \mod G(E)_{x,2(r-d-\epsilon)}$$

(viewing $\text{exp}(Y) \in G(k)$ inside $G(E)$) for some fixed (arbitrarily chosen) order of the roots $\Phi(G, T_E) - \Phi(G', T_E)$. We set $g = (\text{exp}(-Y))^{-1}$. Note that $Y \in g_{x,r-d-\epsilon}$ implies that $x_\alpha(-a_\alpha) \in G(E)_{x,r-d-\epsilon}$.

Let $Z \in g_{x,r'}$ for some $r' \in \mathbb{R}$. Then by [Adl98, Proposition 1.6.3], we have

$$\text{Ad}(g^{-1})(Z) \equiv Z + [-Y, Z] \mod g_{x,r'+2(r-d-\epsilon)}.$$ 

Hence, using that $g \in G_{x,r-d-\epsilon}$, $X \in g_{x,-d-\epsilon}$, $C \in g_{x,-(d+\epsilon)}^{*}$, $Y \in g_{x,r-d-\epsilon}$ and $\epsilon < \frac{r-d}{2}$, we obtain

$$\text{Ad}(g)(X + C)(Z) = X(\text{Ad}(g^{-1})(Z)) + C(\text{Ad}(g^{-1})(Z)) = X(Z) \quad \text{for } Z \in g_{x,r},$$

$$\text{Ad}(g)(X + C)(Z) = X(\text{Ad}(g^{-1})(Z)) + C(\text{Ad}(g^{-1})(Z)) = X(Z) + [-Y, Z] + C(Z) = X(Z) \quad \text{for } Z \in t'' \cap g_{x,d+} \quad \text{(using } C = X([Y, \_])).}$$

To prove the remaining claim, observe that for $h \in H \subset G'(k) \cap G_{x,(2d-r+2\epsilon)+}$, $\alpha \in \Phi(G, T_E) - \Phi(G', T_E)$ and $x_\alpha(a_\alpha) \in G(E)_{x,r-d-\epsilon}$, we have $x_\alpha(a_\alpha)hx_\alpha(a_\alpha)^{-1} = h \cdot u$ for some $u$ in $\langle U_h(E)_{x,(d+\epsilon)+} \mid \alpha \in \Phi(G, T_E) - \Phi(G', T_E) \rangle$, and hence

$$ghg^{-1} \equiv h \cdot u' \mod G(E)_{x,r+}$$

for some $u'$ in $\langle U_h(E)_{x,(d+\epsilon)+} \mid \alpha \in \Phi(G, T_E) - \Phi(G', T_E) \rangle$. Thus $\pi(ghg^{-1})$ and $\pi(h)$ agree on $V'$.

\[\square\]

**Corollary 5.2.** Let $n$ be a positive integer and $(x, (X_i)_{1 \leq i \leq n})$ a truncated datum of length $n$ (with corresponding truncated extended datum $(x, (r_i)_{1 \leq i \leq n}, (X_i)_{1 \leq i \leq n}, (G_i)_{1 \leq i \leq n+1})$). Let $1 \leq j \leq n$, choose a maximal torus $T_j$ of $\text{Cent}_{H_j}(X_j) \subset H_j$ that splits over a tame extension
$E$ of $k$ and such that $x_j \in \mathcal{A}(T_j, E) \cap \mathcal{B}(H_j, k)$. Write $t_j = \text{Lie}(T_j)(k)$. Let $(\pi, V)$ be a representation of $G(k)$. Let $d, \epsilon \in \mathbb{R}$ such that $\frac{r_i}{2} \leq d < r_j$ and $\frac{r_i - d}{2} > \epsilon > 0$. Suppose that $V'$ is a nontrivial subspace of $V \cup_{1 \leq i \leq j} (H_i)_{x,r_i+}$ on which $(H_i)_{x,r_i, \frac{r_i}{2}+}/(H_i)_{x,r_i+} \simeq (h_i)_{x,r_i, \frac{r_i}{2}+} / (h_i)_{x,r_i+}$ acts via $\varphi \circ X_i$ for $1 \leq i \leq j - 1$ and $(H_j)_{x,r_j-\epsilon,d+}/(H_j)_{x,r_j+} \simeq (h_j)_{x,r_j-\epsilon,d+} / (h_j)_{x,r_j+}$ acts via $\varphi \circ (X_j + C)$ for some $C \in (h_j)^*_x, -(d+\epsilon)$ that is trivial on $t_j \oplus h_{j+1}$. Then there exists $g \in (H_j)_{x,r_j-d-\epsilon}$ such that

(i) $V'' := \pi(g)V' \subset V \cup_{1 \leq i \leq j} (H_i)_{x,r_i+}$

(ii) $(H_i)_{x,r_i, \frac{r_i}{2}+}/(H_i)_{x,r_i+} \simeq (h_i)_{x,r_i, \frac{r_i}{2}+} / (h_i)_{x,r_i+}$ acts on $V''$ via $\varphi \circ X_i$ for $1 \leq i \leq j - 1$

(iii) $(H_j)_{x,r_j,d+}/(H_j)_{x,r_j+} \simeq (h_j)_{x,r_j,d+} / (h_j)_{x,r_j+}$ acts on $V''$ via $\varphi \circ X_j$

(iv) any subgroup $H$ of $(H_{j+1})_{x,(2d-r_j+2\epsilon)}$ that stabilizes $V'$ also stabilizes $V''$ and $(\pi|_H, V'') \simeq (\pi|_H, V')$.

Proof.
Let $g \in (H_j)_{x,r_j-d-\epsilon}$ be as constructed in the proof of Lemma 5.1 applied to the group $H_j$ with almost stable and generic element $X_j$ of depth $r_j$. Hence $(\pi|_H, V'') \simeq (\pi|_H, V')$ by Lemma 5.1(iii). Note that $g^{-1}(H_i)_{x,r_i+}g = (H_i)_{x,r_i+}$ for $1 \leq i \leq j$. Thus $V'' \subset V \cup_{1 \leq i \leq j} (H_i)_{x,r_i+}$. To show (iii), recall that $g^{-1} = \prod_{(i) \in \Phi(H_j,(T_j)_E) - \Phi(\text{Cent}_{H_j}(X_j),(T_j)_E)} x_{\alpha}(a_{\alpha})g'$ with $g' \in H_j(E)_{x,2r_j-2d-2\epsilon}$ and $x_{\alpha}(-a_{\alpha}) \in H_j(E)_{x,r_j-d-\epsilon}$. Hence for $h \in (H_i)_{x,r_i, \frac{r_i}{2}+}$ for $1 \leq i < j$ we have $g^{-1}hg \equiv h \mod (H_i)_{x,r_i, \frac{r_i}{2}+}$ with $r_j - d - \epsilon > 0$, and therefore $(H_i)_{x,r_i, \frac{r_i}{2}+}/(H_i)_{x,r_i+} \simeq (h_i)_{x,r_i, \frac{r_i}{2}+}/(h_i)_{x,r_i+}$ acts on $V''$ via $\varphi \circ X_i$. In addition, we have $g^{-1}(H_j)_{x,r_j,d+}g \subset (H_j)_{x,r_j-d+}$. Since $(H_j)_{x,r_j-\epsilon,d+}/(H_j)_{x,r_j+} \simeq (h_j)_{x,r_j-\epsilon,d+} / (h_j)_{x,r_j+}$ acts via $\varphi \circ (X_j + C)$ on $V'$, we obtain from Lemma 5.1(iii) that $(H_j)_{x,r_j,d+}/(H_j)_{x,r_j+} \simeq (h_j)_{x,r_j,d+} / (h_j)_{x,r_j+}$ acts on $V''$ via $\varphi \circ X_j$.

Corollary 5.3. Let $n$ be a positive integer and $(x, (X_i)_{1 \leq i \leq n})$ a truncated datum of length $n$ (with truncated extended datum $(x, (r_i)_{1 \leq i \leq n}, (X_i)_{1 \leq i \leq n}, (G_i)_{1 \leq i \leq n+1})$). Let $T$ be a maximal torus of $G_{n+1}$, set $t = \text{Lie}(T)(k)$, and let $(\pi, V)$ be a representation of $G(k)$. Let $0 < \epsilon < \frac{r_i}{4}$ such that $(H_i)_{x,r_i-\epsilon, \frac{r_i}{2}+} = (H_i)_{x,r_i, \frac{r_i}{2}+}$ for all $1 \leq i \leq n$. Suppose that $V'$ is a nontrivial subspace of $V \cup_{1 \leq i \leq n} (H_i)_{x,r_i+}$ on which the action of $(H_i)_{x,r_i, \frac{r_i}{2}+}/(H_i)_{x,r_i+} \simeq (h_i)_{x,r_i, \frac{r_i}{2}+}/(h_i)_{x,r_i+}$ via $\pi$ is given by $\varphi \circ (C_i + X_i)$ for some $C_i \in (h_i)_{x,r_i, \frac{r_i}{2}+}$ that is trivial on $(t \cap h_i) \oplus h_{i+1}$ for all $1 \leq i \leq n$. Then there exists $g \in G_{x, \frac{r_i}{2}+\epsilon} \subset G_{x,0+}$ such that

(i) $V'' := \pi(g)V' \subset V \cup_{1 \leq i \leq n} (H_i)_{x,r_i+}$

(ii) $(H_i)_{x,r_i, \frac{r_i}{2}+}/(H_i)_{x,r_i+} \simeq (h_i)_{x,r_i, \frac{r_i}{2}+}/(h_i)_{x,r_i+}$ acts on $V''$ via $\varphi \circ X_i$ for $1 \leq i \leq n$

(iii) any subgroup $H$ of $(H_{n+1})_{x,2\epsilon}$ that stabilizes $V'$ also stabilizes $V''$ and $(\pi|_H, V'') \simeq (\pi|_H, V')$.
Proof.
Since \((H_i)_{y, r_i - \epsilon, \frac{3}{2} +} = (H_i)_{y, r_i, \frac{3}{2} +}\) for all \(1 \leq i \leq n\), we can apply Corollary \ref{corollary:commutator} successively for \(j = 1, 2, \ldots, n\) with \(d = \frac{r_1}{2}, \frac{r_2}{2}, \ldots, \frac{r_n}{2}\), respectively. We obtain \(g = g_n \cdot \ldots \cdot g_1 \in (H_n)_{x, \frac{r_n}{2} - \epsilon} \cdot \ldots \cdot (H_1)_{x, \frac{r_1}{2} - \epsilon} \subset G_{x, \frac{r_n}{2} - \epsilon}\) such that \(\pi(g)V' \subset V_{1 \leq i \leq j - 1}(H_{y, r_i})\) and the action of \((H_i)_{y, r_i, \frac{3}{2} +} / (H_i)_{y, r_i +} \cong (h_i)_{y, r_i, \frac{3}{2} +} / (h_i)_{y, r_i +}\) on \(\pi(g)V'\) via \(\pi\) is given by \(\varphi \circ X_i\) for \(1 \leq i \leq n\), and the action of any subgroup \(H\) of \((H_{n+1})_{x, 2+}\) that stabilizes \(V'\) also stabilizes \(V''\) and \((\pi|_H, V'') \cong (\pi|_H, V')\).

\(\square\)

Corollary 5.4. Let \(n\) be a positive integer and \((x, (X_i)_{1 \leq i \leq n})\) a truncated datum of length \(n\) (with corresponding truncated extended datum \((x, (r_i)_{1 \leq i \leq n}, (X_i)_{1 \leq i \leq n}, (G_i)_{1 \leq i \leq n+1})\)). Let \((\pi, V)\) be a representation of \(G(k)\). Let \(0 < \epsilon < \frac{r_n}{4}\) such that \((H_n)_{x, r_n - 2\epsilon} = (H_n)_{x, r_n}\). Suppose that \(V'\) is a nontrivial subspace of \(V_{1 \leq i \leq n}(H_i)_{x, r_i +}\) on which the action of \((h_i)_{x, r_i, \frac{3}{2} +} / (h_i)_{x, r_i +}\) via \(\pi\) is given by \(\varphi \circ X_i\) for all \(1 \leq i \leq n - 1\), and the action of \((H_n)_{x, r_n} / (H_n)_{x, r_n +} \cong (h_n)_{x, r_n} / (h_n)_{x, r_n +}\) via \(\pi\) is given by \(\varphi \circ X_n\). Then there exists a subspace \(V'' \subset V_{1 \leq i \leq n}(H_i)_{x, r_i +}\) such that \((h_i)_{x, r_i, \frac{3}{2} +} / (h_i)_{x, r_i +}\) acts on \(V''\) via \(\varphi \circ X_i\) for \(1 \leq i \leq n\).

Proof.
Let \(T\) be a maximal torus of \(G_{n+1}\) with Lie algebra \(t = \text{Lie}(T)(k)\). Let \(d = \max(\frac{r_n}{2}, r_n - 3\epsilon)\). Note that the commutator \([\prod_{1 \leq i \leq n-1}(H_i)_{x, r_i, \frac{3}{2} +} / (H_n)_{x, r_n, d +}]\) acts trivially on \(V'\) and hence we can replace \(V'\) without loss of generality by \(\pi((H_n)_{x, r_n, d +})V'\). Since \((H_n)_{x, r_n - 2\epsilon} = (H_n)_{x, r_n}\), the action of \((H_n)_{x, r_n - \epsilon, d +}\) on \(V'\) factors through \((H_n)_{x, r_n - \epsilon, d +} / (H_n)_{x, r_n +}\) and, after replacing \(V'\) by a subspace if necessary, is given by \(\varphi \circ (X_n + C_3)\) for some \(C_3 \in (h_n)_{x, (r_n - 2\epsilon)}\) that is trivial on \(t \cap h_n \oplus h_{n+1}\). Applying Corollary \ref{corollary:commutator} for \(j = n\) and \(d = \max(\frac{r_n}{2}, r_n - 3\epsilon)\), we obtain \(g_3 \in (H_n)_{x, \epsilon}\) such that \(\pi(g_3)V' \subset V_{1 \leq i \leq n}(H_i)_{x, r_i +}\). The group \((H_i)_{x, r_i, \frac{3}{2} +} / (H_i)_{x, r_i +}\) acts on \((h_i)_{x, r_i, \frac{3}{2} +} / (h_i)_{x, r_i +}\) via \(\pi(g_3)\) for \(1 \leq i \leq n - 1\) and \((H_n)_{x, r_n, d +} / (H_n)_{x, r_n +} \cong (h_n)_{x, r_n, d +} / (h_n)_{x, r_n +}\) acts on \((h_n)_{x, r_n +}\) via \(\varphi \circ X_n\). Replacing \(V'\) by \((g_3)\) and using the same reasoning, we can apply Corollary \ref{corollary:commutator} for \(j = n\) repeatedly with \(d = r_n - 4\epsilon, r_n - 5\epsilon, r_n - 6\epsilon, \ldots, r_n - (N - 1) \cdot \epsilon\) (and replacing \(V'\) at each step if necessary), where \(N\) is the largest integer for which \(\bar{N} \cdot \epsilon < \frac{r_n}{2}\). After the final step we obtain a subspace \(V'' \subset V_{1 \leq i \leq n}(H_i)_{x, r_i +}\) on which \((h_i)_{x, r_i, \frac{3}{2} +} / (h_i)_{x, r_i +}\) acts via \(\varphi \circ X_i\) for \(1 \leq i \leq n\).

\(\square\)

6 Every irreducible representation contains a datum

Theorem 6.1. Let \((\pi, V_{\pi})\) be a smooth irreducible representation of \(G(k)\). Then \((\pi, V_{\pi})\) contains a datum.

Proof.
Let \(j\) be an integer such that \((\pi, V_{\pi})\) contains a truncated datum \((x_{j-1}, (X_i)_{1 \leq i \leq j-1})\) of \(G\) of length \(j - 1\), and write \(B_j := B(G, j) \subset B = B(G, k)\).
If \((\pi, V_\pi)\) does not contain a truncated datum, then we set \(j = 1\), \(G_j = G\), hence \(\mathcal{B}_j = \mathcal{B}\). Moreover, we let \(x_0 = x_{j-1}\) be an arbitrary point of \(\mathcal{B}\) and denote by \(r_0 = r_{j-1}\) the depth of \((\pi, V_\pi)\) at \(x_0\).

The strategy of this proof is to successively increase the length of the truncated datum until we reach its maximum and then to show how to turn it into a datum that is contained in \((\pi, V_\pi)\).

We define a function \(f : \mathcal{B}_j \to \mathbb{R}_{\geq 0} \cup \infty\) as follows: For \(y \in \mathcal{B}_j\), we set \(f(y)\) to be the smallest non-negative real number \(r_j\) such that

- there exists \(X_j \in (g_j)_{y, -r_j}\) almost stable, where \(g_j = \text{Lie}(G_j)(k)\), and
- there exists \(V_{j-1} \subset V_\pi^{\cup 1 \leq i \leq j-1}((H_i)_{y, r_i})\)

satisfying the following two properties

(a) for \(1 \leq i \leq j - 1\) the group

\[(H_i)_{y, r_i, r_j + } / (H_i)_{y, r_i, r_j} \cong (h_i)_{y, r_i, r_j + } / (h_i)_{y, r_i} \]

associated to \((y, (X_i)_{1 \leq i \leq j-1})\) acts on \(V_{j-1}\) via \(\varphi \circ X_i\) (this condition is automatically satisfied for \(j = 1\)), and

(b) \(V_{j-1}^{(H_j)_{y, r_j}}\) contains a nontrivial subspace \(V'_j\) such that if \(r_j > 0\), then \((H_j)_{y, r_j} / (H_j)_{y, r_j} \cong (h_j)_{y, r_j} / (h_j)_{y, r_j}\) acts on \(V'_j\) via \(\varphi \circ X_j\).

If such a real number \(r_j\) does not exist, then we set \(f(y) = \infty\).

Note that \(f\) is well defined, because the Moy–Prasad filtration is semi-continuous and for every \(r \in \mathbb{R}\) every \((g_j)^{y, r}+\)-coset contains an almost stable element (e.g. take an element dual to a semisimple element under the non-degenerate bilinear form \(B\) provided by [AR00] that we discussed on page 11 in Section 3). Moreover, by our assumption, \(f(x_{j-1}) \leq r_{j-1}\) (because if \(j > 1\), we could take \(X_j = 0\) for \(r_j = r_{j-1}\)).

**Lemma 6.1.1.**

(i) \(f(g.x) = f(x)\) for all \(x \in \mathcal{B}_j\) and \(g \in G_j(k)\)

(ii) The subset \(f^{-1}(\mathbb{R}_{\geq 0})\) of \(\mathcal{B}_j\) is open in \(\mathcal{B}_j\) and the function \(f : \mathcal{B}_j \to \mathbb{R}_{\geq 0} \cup \infty\) is continuous on \(f^{-1}(\mathbb{R}_{\geq 0})\).

(iii) The subset \(f^{-1}(\mathbb{R}_{\geq 0})\) of \(\mathcal{B}_j\) is closed in \(\mathcal{B}_j\), hence equal to \(\mathcal{B}_j\).
Proof of Lemma 6.1.1.

Proof of part (ii). Observe that $X_i$ (1 ≤ $i < j$) and $G_i$ (1 ≤ $i ≤ j$) are stabilized by $G_j(k)$, hence the $G_j(k)$-invariance of $f$ follows.

Proof of part (iii). If $j = 1$, then $f(x)$ is the depth of $\pi$ at $x$, and the claim is true. Hence we assume $j > 1$. Let $(x, (X_i)_{1 ≤ i ≤ j})$ be a truncated datum contained in $(\pi, V_\pi^j)$, $X_j \in (g_j)^*_{x, f(x)}$ almost stable and $V_j' \subset V_{j-1} \subset V_{\pi}^{1 ≤ i ≤ j-1}((H_i)_{x, r_i+})$ satisfying the conditions (a) and (b) above. If $f(x) > 0$, then set $r_j = f(x)$, otherwise let 0 < $r_j ≤ r_{j-1}$ be arbitrary. For $1 ≤ i ≤ j-1$, let $d_i < r_i$ be a positive real number such that $(G_i)_{x, r_i-d_i} = (G_i)_{x, r_i}$. Note that $d_i > 0$ exists for $1 ≤ i ≤ j-1$ by the semi-continuity of the Moy–Prasad filtration.

Let $\min\{\frac{r_j}{4} + \frac{d_i}{2} | 1 ≤ i < j-1\} > \epsilon > 0$ and let $y \in \mathcal{B}_j$ with $d(x, y) < \epsilon$. Let $T$ be a maximal torus of $G_j$ that splits over a tamely ramified extension $E$ of $k$ such that $x$ and $y$ are contained in $\mathcal{A}(T_E, E)$. Note that $(y, (X_i)_{1 ≤ i ≤ j-1})$ is a truncated datum (by Lemma 3.6 and Corollary 3.8). By the normalization of the distance $d$ on the building $\mathcal{B}$, we have $|\alpha(x - y)| ≤ d(x, y) < \epsilon$ for all $\alpha \in \Phi(G, T_E)$. Hence, since $X_i$ vanishes on $g_i \cap \bigoplus_{\alpha \in \Phi(G_i, T_E)} g(E)\alpha$ for $1 ≤ i ≤ j-1$, we have $V_j' \subset V_{\pi}^{1 ≤ i ≤ j-1}((H_i)_{y, r_i+})$ and the commutator $[\prod_{1 ≤ i ≤ j-1} (H_i)_{y, r_i+ - \frac{d_j}{4} + \frac{r_j}{4} + (H_j)_{y, r_j+}, \prod_{1 ≤ i ≤ j-1} (H_i)_{y, r_i+ - \frac{d_j}{4} + \frac{r_j}{4} + (H_j)_{y, r_j+}}]$ is contained in $\prod_{1 ≤ i ≤ j-1} \ker(\varphi \circ X_i) \setminus (H_i)_{y, r_i+ - \frac{d_j}{4} + \frac{r_j}{4} + (H_j)_{y, r_j+}}$. Hence, adjusting $V_j'$ if necessary (to a subspace of $\pi(\prod_{1 ≤ i ≤ j-1} (H_i)_{y, r_i+ - \frac{d_j}{4} + \frac{r_j}{4} + (H_i)_{y, r_i+}})$), the action of

$$(H_i)_{y, r_i+ - \frac{d_j}{4} + \frac{r_j}{4} + (H_i)_{y, r_i+}} \simeq (h_i)_{y, r_i+ - \frac{d_j}{4} + \frac{r_j}{4} + (h_i)_{y, r_i+}}$$

via $\pi$ on $V_j'$ is given by

$$\varphi \circ (C_i + X_i)$$

for some $C_i \in (h_i)^*_{y, -(\frac{r_j}{4} + \frac{d_j}{4})}$ being trivial on $(t \cap h_i) \oplus h_{i+1}$ for all $1 ≤ i ≤ j-1$. Moreover, $X_j \in (g_j)^*_{x, r_j} \subset (g_j)^*_{y, r_j - \epsilon}$, and the action of $(H_j)_{y, r_j+}$ on $V_{j-1}$ factors through $(H_j)_{y, r_j+} / (H_j)_{y, r_j+}$ which is given by $\varphi \circ X_j$ (which, as an aside, yields the trivial action). By Corollary 5.3 there exists $g \in G_{y, r_j+ - \epsilon} \subset G_{y, r_j+}$ such that $\pi(g)V_j' \subset V_{\pi}^{1 ≤ i ≤ j-1}((H_i)_{y, r_i+})$, the action of $(H_i)_{y, r_i+ - \frac{d_j}{4} + (H_i)_{y, r_i+}} \simeq (h_i)_{y, r_i+ - \frac{d_j}{4} + (h_i)_{y, r_i+}}$ on $\pi(g)V_j'$ via $\pi$ is given by $\varphi \circ X_i$ for $1 ≤ i ≤ j-1$, and the action of $(H_j)_{y, r_j+}$ on $\pi(g)V_j'$ factors through $(H_j)_{y, r_j+} / (H_j)_{y, r_j+}$ which is given by $\varphi \circ X_j$. Thus $f(y) ≤ r_j + \epsilon$. Hence the set $f^{-1}(\mathbb{R}_{≥ 0})$ is open in $\mathcal{B}_j$.

Moreover, if $f(x) = 0$, then this implies that $f$ is continuous on $f^{-1}(\mathbb{R}_{≥ 0})$, because $f(y) ≥ 0$ and $r_j > 0$ can be chosen arbitrarily small in this case.

It remains to prove continuity around $x$ in the case $f(x) = r_j > 0$. Suppose $f(y) < r_j - \epsilon$, and let $X_j \in (g_j)^*_{y, -(r_j - \epsilon) +}$ be almost stable satisfying condition (b) above. Note that $(G_i)_{y, r_i - \frac{d_j}{4} + \frac{r_j}{4} + (h_i)_{y, r_i+}}$ for $1 ≤ i ≤ j-1$. Hence, by the same reasoning as above (switching $x$ and $y$), we deduce that $f(x) < r_j$, a contradiction. Thus $f(y) ≥ r_j - \epsilon$ and $f$ is continuous on $f^{-1}(\mathbb{R}_{≥ 0})$.

Proof of part (iii). Suppose $y \in \mathcal{B}_j$ is in the closure of $f^{-1}(\mathbb{R}_{≥ 0})$, and let $d > 0$ be sufficiently small such that for all $r \in \mathbb{R}_{≥ d}$ with $G_{y, r} \neq G_{y, r+}$ we have $G_{y, r-d} = G_{y, r}$. Let $\frac{d}{8} > \epsilon > 0$ and
\( x \in f^{-1}(\mathbb{R}_{\geq 0}) \) with \( d(x,y) < \epsilon \). Then \( G_{x,r} \neq G_{x,r+} \) implies \( G_{x,r-d+2\epsilon} = G_{x,r} \) (if \( r \in \mathbb{R}_{\geq d-2\epsilon} \)), hence \( G_{x,r-\frac{d}{2}} = G_{x,r} \), and \( G_{x,0+} = G_{x,\frac{d}{2}} \). Thus we can apply the proof of part (ii) to deduce that \( f(y) \) is finite.

\[ \square \text{Lemma 6.1.1} \]

Since \( f \) is \( G_j(k) \)-equivariant, continuous, bounded below by zero, and the fundamental domain for the action of \( G_j(k) \) on \( B_j \) is bounded, there exists a point \( x_j \in B_j \) such that \( f(x_j) \leq f(x) \) for all \( x \in B_j \). Define \( r_j = f(x_j) \) and note that \( r_j \leq f(x_{j-1}) \leq r_{j-1} \).

We distinguish two cases.

**Case 1:** \( r_j > 0 \).

Let \( (x_j, (X_i)_{1 \leq i \leq j-1}) \) be a truncated datum contained in \( (\pi, V_\pi) \), \( X_j \in (g_j)^{x,r} \) almost stable and \( V_j' \subset V_{j-1} \subset V_\pi^{1 \leq i \leq j-1}((H_i)_{x,r_{i+}}) \) satisfying the conditions \([a]\) and \([b]\) above.

**Lemma 6.1.2.** The element \( X_j \) of \( g_j^* \) is almost strongly stable at \( x_j \in B_j \).

**Proof of Lemma 6.1.2.** Suppose \( X_j \) is not almost strongly stable. Since \( X_j \) is almost stable, this implies that \( X_j \in (g_j)^{x,r}_{x,r_{i+}} \) is unstable. Thus, by [Kem78, Corollary 4.3] there exists a non-trivial one parameter subgroup \( \lambda : \mathbb{G}_m \to (G_j)_{x_j} \) in the reductive quotient \( (G_j)_{x_j} \) of \( G_j \) at \( x_j \) such that \( \lim_{t \to 0} \lambda(t)X_j = 0 \). This means \( X_j \) is trivial on the root spaces corresponding to roots \( \alpha \) with \( \langle \alpha, \lambda \rangle < 0 \). Let \( \mathcal{A} \) be a split torus of the parahoric group scheme \( (\mathcal{P}_j)_{x_j} \) of \( G_j \) such that \( \mathcal{A}_j \) is a maximal split torus of \( (G_j)_{x_j} \) containing \( \lambda(\mathbb{G}_m) \) and such that \( \mathcal{A}_k \) is contained in a maximal torus \( T_j \subset G_j \) which splits over a tame extension \( E \) of \( k \) and whose apartment \( \mathcal{A}(T_E, E) \cap B_j \) contains \( x_j \). Let \( \lambda : \mathbb{G}_m \to \mathcal{A}_k \) be the one parameter subgroup corresponding to \( \lambda \). Then for \( \epsilon > 0 \) small enough, we have \( (H_j)_{x,j+\epsilon \lambda, r_j} \subset (H_j)_{x,j,r_j} \) and \( X_j \in (g_j)^{x,j+\epsilon \lambda, r_j}_{x,j+\epsilon \lambda, r_j} \), where \( x_j + \epsilon \lambda \in \mathcal{A}(T_E, E) \cap B_j \). Moreover, the image of \( X_j \) in \( (g_j)^{x,j+\epsilon \lambda, r_j}_{x,j+\epsilon \lambda, r_{j+}} \) is trivial. Let \( r_j > \delta > 0 \) such that the subgroup \( (H_j)_{x,j+\epsilon \lambda, r_j-\delta} \) equals \( (H_j)_{x,j+\epsilon \lambda, r_j} \) and therefore acts trivially on \( V_j' \). Analogously to the first part of the proof of Lemma 6.1.1, for \( \epsilon \) sufficiently small, there exist \( d_i > 0 \) for \( 1 \leq i \leq j-1 \) such that we have \( V_j' \subset V_\pi^{1 \leq i \leq j-1}((H_i)_{x,j+\epsilon \lambda, r_{i+}}) \) and (after potentially adjusting \( V_j' \subset V_\pi^{1 \leq i \leq j-1}((H_i)_{x,j+\epsilon \lambda, r_{i+}}) \)) the action of \( (H_i)_{x,j+\epsilon \lambda, r_{i+}}/\pi \) on \( V_j' \) is given by \( \varphi \circ (C_i + X_i) \) for some \( C_i \in (h_i)^{x,j+\epsilon \lambda, r_{i+}} \) being trivial on \( (t \in h_i) \oplus h_{i+1} \) for all \( 1 \leq i < j \), the group \( (H_j)_{x,j+\epsilon \lambda, r_{j-\delta}} \) acts trivially on \( V_j' \) and \( (x_j + \epsilon \lambda, (X_i)_{1 \leq i \leq j-1}) \) is a truncated datum. Assuming \( \epsilon \) is sufficiently small and applying Corollary 5.3 (or if \( j = 1 \), set \( g = 1 \)), we obtain \( g \in G_{x,j+\epsilon \lambda, 0+} \) such that \( \pi(g)V_j' \subset V_\pi^{1 \leq i \leq j-1}((H_i)_{x,j+\epsilon \lambda, r_{i+}}) \) the action of \( (H_j)_{x,j+\epsilon \lambda, r_{i+}}/\pi \) on \( \pi(g)V_j' \) is given by \( \varphi_X \circ X_j \) for \( 1 \leq i \leq j-1 \), and the action of \( (H_j)_{x,j+\epsilon \lambda, r_{j-\delta}} \) on \( \pi(g)V_j' \) is trivial. Hence \( f(x_j + \epsilon \lambda) \leq r_j - \delta < r_j = f(x_j) \), which contradicts the choice of \( x_j \). Thus \( X_j \) is almost strongly stable. \( \square \text{Lemma 6.1.2} \)

Now we can show that, after changing \( x_j \) and \( X_j \) if necessary, we obtain a truncated datum of \( G \) of length \( j \) that is contained in \( (\pi, V_\pi) \).
Lemma 6.1.3. There exists a choice of $x_j$ and $X_j$ as above such that $(x_j,(X_i)_{1\leq i \leq j})$ is a truncated datum contained in $(\pi, V_\pi)$.

Proof of Lemma 6.1.3. Let $x_j$ and $X_j$ be as in Lemma 6.1.2. Let $\epsilon > 0$ be sufficiently small (as specified later). By Proposition 3.10 (applied to $G_j$) there exists $y \in B_j \subset B$ such that $d(x_j, y) < \epsilon$, the element $\tilde{X}$ is almost strongly stable and generic of depth $-r_j$ at $y$, and $x_j$ and $y$ are contained in $B(Cent_{G_j}(\tilde{X}), k) \subset B_j$. Note that for $\epsilon$ sufficiently small, we have $(H_j)_{y,r_j} \subset (H_j)_{x_j,r_j}$ and the action of $(H_j)_{y,r_j}$ on $V'_j$ factors through $(H_j)_{y,r_j}/(H_j)_{x_j,r_j+}$ on which it is given by $X_j$. Since $X_j - \tilde{X} \in (h_j)_{y,(-r_j+r_j)+}$, this difference is trivial on $(h_j)_{y,r_j}$. Therefore the action of $(H_j)_{y,r_j}/(H_j)_{x_j,r_j+}$ on $V'_j$ is also given by $\tilde{X}$, and, in particular, it factors through $(H_j)_{y,r_j}/(H_j)_{x_j,r_j+}$. Moreover, the tuple $(y,(X_i)_{1\leq i \leq j-1})$ is a truncated datum (by Lemma 3.6 and Corollary 3.8 as in the proof of Lemma 6.1.1(ii)). Substituting $X_j$ by $\tilde{X}$ and applying Corollary 5.3 (if $j > 1$) as in the proofs of Lemma 6.1.1 and Lemma 6.1.2 and possibly substituting $V'_j$ by $\pi(g)V'_j$ for some $g \in G_{y,0+}$, we can achieve that (a) and (b) above are satisfied at the point $y$. This implies that $f(y) = r_j$.

Note that $(y,(X_i)_{1\leq i \leq j})$ is a truncated datum. (If $j > 2$, then $G_j \neq Cent_{G_j}(X_j)$, because otherwise $X_j(h_j) = 0$ and hence $f(x_j)$ would not be minimal.) By Corollary 5.4 this truncated datum is contained in $(\pi, V_\pi)$.

Since $G_j \subset G_{j-1}$ for $j > 2$, after repeating this construction finitely many times we obtain an integer $n$ and a truncated datum $(x_n,(X_i)_{1\leq i \leq n})$ contained in $(\pi, V_\pi)$ with $r_{n+1} = 0$, i.e. we move to the second case.

Case 2: $r_j = 0$. Let $V'_j$ be the maximal subspace of $V_\pi^{l_{1\leq i \leq j}}((H_i)_{x_j,r_i+})$ satisfying (a) and (b) above. Note that $(H_j)_{x_j,0}$ stabilizes $V'_j$, because $(H_j)_{x_j,0}$ centralizes $X_1$ and stabilizes $(H_i)_{x_j,r_i+}$ and $(H_i)_{x_j,r_i+}$ for $1 \leq i < j$. Let $(\rho_0, V_{\rho_0})$ be an irreducible $(H_j)_{x_j,0}/(H_j)_{x_j,0+}$-subrepresentation of $V'_j$ viewed as a representation of $(H_j)_{x_j,0}/(H_j)_{x_j,0+}$. Then $(x_j,(X_i)_{1\leq i \leq j-1}, (\rho_0, V_{\rho_0}))$ is a datum contained in $(\pi, V_\pi)$.

Remark 6.2. In the next section we will use the existence of a datum for a given representation $(\pi, V)$ to deduce the existence of a type for $(\pi, V)$. Note however that a datum itself might not determine the Bernstein component, i.e. a given datum might be a datum for representations in different Bernstein components. If one is interested in determining the Bernstein component uniquely, one has to enhance the datum slightly (to a representation of $(M_{n+1})_x$, where $M_{n+1}$ is a Levi subgroup of $G_{n+1}$ that we are going to attached to $x$ and $G_{n+1}$ in Section 7 page 25). Such an enhancement determines the Bernstein component uniquely by [KY17, 10.3 Theorem], which is based on the work of Hakim–Murnaghan [HM08 for supercuspidal representations. The assumption required in Hakim–Murnaghan was removed by Kaletha in [Kal17 Corollary 3.5.5].
7 From a datum to types

Let \((\pi, V)\) be a smooth irreducible representation of \(G(k)\), and let \((x, (X_i)_{1 \leq i \leq n}, (\rho_0, V_{\rho_0}))\) be a datum for \((\pi, V)\), which exists by Theorem 6.1. In this section we show how to use this datum in order to exhibit a type contained in \((\pi, V)\). In order to do so we will define characters \(\phi_i : G_{i+1}(k) \to \mathbb{C}\) of depth \(r_i\) for \(1 \leq i \leq n\) and a depth-zero representation of a compact open subgroup \(K_{G_{n+1}}\) of \(G_{n+1}(k)\) that contains \((G_{n+1})_{x,0}\). We will prove that these objects satisfy all necessary conditions imposed by Kim and Yu ([KY17]) so that Yu’s construction \(([Yu01])\) yields a type. In Theorem 7.12 we will conclude that the resulting type is contained in \((\pi, V)\).

Recall that Moy and Prasad ([MP96, 6.3 and 6.4]) attach to \(x\) and \(G_{n+1}\) a Levi subgroup \(M_{n+1}\) of \(G_{n+1}\) such that \(x \in \mathcal{B}(M_{n+1}, k) \subset \mathcal{B}(G_{n+1}, k)\) and \((M_{n+1})_{x,0}\) is a maximal parahoric subgroup of \(M_{n+1}\) with \((M_{n+1})_{x,0}/(M_{n+1})_{x,0+} \cong (G_{n+1})_{x,0}/(G_{n+1})_{x,0+}\). We denote by \((M_{n+1})_x\) the stabilizer of \(x \in \mathcal{B}(M_{n+1}, k)\) in \(M_{n+1}(k)\). Then we define following Kim and Yu ([KY17, 7.1 and 7.3]) the group \(K_{G_{n+1}}\) to be the group generated by \((M_{n+1})_x\) and \((G_{n+1})_{x,0}\).

Let \(V'\) be a subspace of \(V\) as provided by Definition 4.3 for the datum \((x, (X_i)_{1 \leq i \leq n}, (\rho_0, V_{\rho_0}))\) contained in \((\pi, V)\), and let \(\tilde{V}\) be the irreducible \(K_{G_{n+1}}\)-subrepresentation of \(V\) containing \(V'\). Note that any \(g \in K_{G_{n+1}} \subset (G_{n+1})_x\) centralizes \(X_i\) for \(1 \leq i \leq n\) and hence stabilizes \((H_i)_{x,r_i+}\) (\(1 \leq i \leq n+1\)). Thus \(\tilde{V}\) is contained in \(V^{\cup_{1 \leq i \leq n+1}((H_i)_{x,r_i+})}\).

Moreover, let \(T\) be a maximal torus of \(M_{n+1} \subset G_{n+1}\) whose apartment contains \(x\). Then, for \(t \in T(k)_{0+}\) and \(g \in K_{G_{n+1}}\), we have \(tg^{-1}1g^{-1} \in (H_{n+1})_{x,0+}\). Hence, if \(v \in \tilde{V}\) is an element such that \(T(k)_{0+}\) preserves \(\mathbb{C} \cdot v\), then \(T(k)_{0+}\) also preserves \(\mathbb{C} \cdot gv\) and acts on both spaces via the same character. Since \(\tilde{V}\) is an irreducible \(K_{G_{n+1}}\)-representation, we deduce that \(T(k)_{0+}\) acts on \(\tilde{V}\) via some character \(\phi_T\) (times identity).

Before using \(\phi_T\) to define the characters \(\phi_i\), we recall Lemma 3.1.1 of [Kal17].

**Lemma 7.1 ([Kal17]).** If \(r \in \mathbb{R}_{>0}\) and \(1 \to A \to B \to C \to 1\) is an exact sequence of tori that are defined over \(k\) and split over a tamely ramified extension of \(k\), then

\[1 \to A(k)_r \to B(k)_r \to C(k)_r \to 1\]

is an exact sequence.

**Corollary 7.2.** Let \(r \in \mathbb{R}_{>0}\) and \(1 \leq j \leq n+1\). Then \((G_j)_{x,r}\) is generated by \(T(k)_r\) and \((H_j)_{x,r}\).

**Proof.**

Note that \(T \cap H_{n+1}\) is a maximal torus of \(H_{n+1}\) ([Con14, Example 2.2.6]). Then by Lemma 7.1 the map \(T(k)_r \to (T/T \cap H_{n+1})(k)_r = (G_{n+1}/H_{n+1})(k)_r\) is surjective, and hence \(T(k)_r\) also surjects onto \((G_{n+1})_{x,r}/H_{n+1}(k)_{x,r} \subset (G_{n+1}/H_{n+1})(k)_r\). (That \((G_{n+1})_{x,r}\) maps to \((G_{n+1}/H_{n+1})(k)_r\) can be seen by considering a tame extension over which \(T\) splits.) \(\square\)
Now we define $\phi_i$, recursively, first for $i = n$, then $i = n-1, n-2, \ldots, 1$. Suppose we have already defined $\phi_n, \ldots, \phi_{j+1}$ of depth $r_n, \ldots, r_{j+1}$ for some $1 \leq j \leq n$ ($j = n$ meaning no character has been defined yet) such that

$$
\pi |T(k) \cap (H_{j+1})_{x,0+} = \phi_n |T(k) \cap (H_{j+1})_{x,0+} \cdot \ldots \cdot \phi_{j+1} |T(k) \cap (H_{j+1})_{x,0+} \cdot \Id \text{ on } \widetilde{V}.
$$

Then we let $\phi'_j = \phi_T \cdot \phi_n |T(k)_{x,0+} \cdot \ldots \cdot \phi_{j+1} |T(k)_{x,0+}$, which is trivial on $T(k) \cap (H_{j+1})_{x,0+} = (T \cap H_{j+1})_{x,0+}$ and on $T(k) \cap (H_j)_{x,r_j} = (T \cap H_j)_{r_j+}$. By Lemma 7.1 we have $(G_{j+1}/H_{j+1})_{x,0+} \simeq T/(T \cap H_{j+1})_{x,0+} \simeq T(k)_{0+}/(T \cap H_{j+1})_{0+}$. Hence $\phi'_j$ defines a character of $(G_{j+1}/H_{j+1})_{0+}$ that extends via Pontryagin duality to some character $\tilde{\phi}'_j$ of $(G_{j+1}/H_{j+1})(k)$. Restricting $\tilde{\phi}'_j$ to the image of $G_{j+1}(k)$, we obtain a character of $G_{j+1}(k)$ that we also denote by $\tilde{\phi}'_j$. Note that $\tilde{\phi}'_j$ is trivial on $H_{j+1}(k)$ and $\tilde{\phi}'_j |T(k)_{x,0+}$ coincides with $\phi'_j$. Similarly, the character $\phi'_j$ gives rise to a character $\hat{\phi}'_j$ of $G_{j+1}(k)$ that can be extended and composed to yield a character (also denoted by $\hat{\phi}'_j$) of $G_j(k)$ that is trivial on $H_j(k)$ and coincides with $\hat{\tilde{\phi}}_j$ on $T(k)_{r_j+}$. If $j = 1$, we may and do choose $\hat{\phi}'_j$ to be the trivial character. We define $\phi_j = \hat{\tilde{\phi}}_j \cdot (\hat{\phi}'_j)^{-1}|_{G_{j+1}(k)}$. Then $\phi_j$ has depth $r_j$ (because by considering a tame extension that splits the torus $T$, we see that $G_{j+1}(k)_{x,r_j}$ maps to $(G_{j+1}/H_{j+1})(k)_{r_j+} \simeq T(k)_{r_j+}/(T \cap H_{j+1})(k)_{r_j+}$ or use Corollary 7.2) and

$$
\pi |T(k) \cap (H_j)_{x,0+} = \phi_n |T(k) \cap (H_j)_{x,0+} \cdot \ldots \cdot \phi_{j+1} |T(k) \cap (H_j)_{x,0+} \cdot \Id \text{ on } \widetilde{V}.
$$

Lemma 7.3. For $1 \leq j \leq n$ the character $\phi_j : G_j(k) \to \mathbb{C}^*$ satisfies the following properties:

(i) $\phi_j$ is trivial on $(G_{j+1})_{x,r_j+}$ and on $H_{j+1}(k)$,

(ii) $\phi_j|((H_j)_{x,r_j} \cap G_{j+1}(k))$ factors through

$$
((H_j)_{x,r_j} \cap G_{j+1}(k))/((H_j)_{x,r_j} \cap G_{j+1}(k)) \simeq ((h_j)_{x,r_j} \cap g_{j+1})/((h_j)_{x,r_j} \cap g_{j+1})
$$

and is given by $\varphi \circ X_j|((h_j)_{x,r_j} \cap g_{j+1})$,

(iii) $\phi_j$ is $G_j$-generic of depth $r_j$ (in the sense of [Yu01] § 9) relative to $x$, and

(iv) the group $(G_{n+1})_{x,0+}$ acts on $\widetilde{V}$ via $\prod_{1 \leq i \leq n} \phi_i|((g_{n+1})_{x,0+})$.

Proof.
Part (i) follows immediately from the above construction.

For Part (ii), note that using Corollary 7.2 we see that $(H_j)_{x,r_j} \cap G_{j+1}(k) = H_j(k) \cap (G_{j+1})_{x,r_j}$ is generated by $T(k)_{r_j} \cap H_j(k)$ and $(H_{j+1})_{x,r_j}$. Since $\phi_j$ is trivial on $H_{j+1}(k)$ and coincides with $\phi_T$ on $T(k)_{r_j} \cap H_j(k)$, the claim follows from the properties of $V'$ in Definition 4.3.

For Part (iii), note that by Part (ii) there exists $Y \in (g_{j+1})_{x,r_j}$ such that $Y$ is trivial on $h_j$ and $\phi_j|((g_{j+1})_{x,r_j})$ is given by the character of $(G_{j+1})_{x,r_j}/(G_{j+1})_{x,r_j+} \simeq (g_{j+1})_{x,r_j}/(g_{j+1})_{x,r_j+}$
arising from $\varphi \circ (Y + X_j)$. Since $Y$ is trivial on $h_j$, the element $Y$ is fixed under the dual of the adjoint action of $G_{j+1}$ on $g_{j+1}$. Hence by the definition of $G_{j+1}$, the group $G_{j+1}$ centralizes $Y + X_j$. Moreover, if $T$ is a maximal torus of $G_{j+1}$, and $\alpha \in \Phi(G_j, T) - \Phi(G_{j+1}, T)$, then $H_\alpha \in \langle h_j \rangle_\mathbb{F}$ and hence

$$\text{val}((Y + X_j)(H_\alpha)) = \text{val}(X_j(H_\alpha)) = -r_j,$$

where the last equality follows from Lemma 3.6. Since $p$ is not a torsion prime for the dual root datum of $G_j$ by Lemma 2.2(a), (b) and (c) (applied to the dual root datum of $G_j$), the character $\phi_j$ is $G_j$-generic of depth $r_j$ by [Yu01, Lemma 8.1].

Part (iv) follows from the observation above that $\phi_j|_{T(k)_{0^+}} = \phi_{T} \cdot \phi_{n}|_{T(k)_{0^+}}^{-1} \cdot \ldots \cdot \phi_{2}|_{T(k)_{0^+}}^{-1}$ and that $(H_{n+1})_{x_0^+}$ acts trivially on $\tilde{V}$ together with Corollary 7.2.

**Corollary 7.4.** The irreducible representation $(\prod_{1 \leq i \leq n} \phi_i^{-1}|_{K_{G_{n+1}}} \cdot \pi|_{K_{G_{n+1}'}, \tilde{V}})$ of $K_{G_{n+1}}$ is trivial on $(G_{n+1})_{x_0^+}$ and its restriction to $(H_{n+1})_{x_0}$ contains $(\rho_0, V_0)$ as an irreducible sub-representation.

**Proof.**
This is an immediate consequence of Lemma 7.3 (i) and (iv) and the definition of $\tilde{V}$. □

Recall our convention that by “type” we mean an $s$-type for some inertial equivalence class $s \in \mathcal{I}$. In order to obtain a type for our representation $(\pi, V)$ of $G(k)$ using the construction of Kim and Yu in [KY17] we denote by $r_\pi$ the depth of the representation $(\pi, V)$, i.e. $r_\pi = r_1$ if $n \geq 1$ and $r_\pi = 0$ if $n = 0$, and we make the following definitions:

$$\begin{align*}
\vec{G} &= \left\{ \begin{array}{ll}
(G_{n+1}, G_n, \ldots, G_2, G_1 = G) & \text{if } G_2 \neq G_1 \text{ or } n = 0 \\
(G_{n+1}, G_n, \ldots, G_3, G_2 = G) & \text{if } G_2 = G_1
\end{array} \right. \\
\vec{r} &= \left\{ \begin{array}{ll}
(r_n, r_{n-1}, \ldots, r_2, r_1) & \text{if } G_2 \neq G_1 \text{ or } n = 0 \\
(r_n, r_{n-1}, \ldots, r_2, r_1) & \text{if } G_2 = G_1
\end{array} \right. \\
\vec{\phi} &= \left\{ \begin{array}{ll}
(\phi_n, \phi_{n-1}, \ldots, \phi_2, \phi_1, 1) & \text{if } G_2 \neq G_1 \text{ or } n = 0 \\
(\phi_n, \phi_{n-1}, \ldots, \phi_2, \phi_1) & \text{if } G_2 = G_1
\end{array} \right. \\
K &= K_{G_{n+1}}(G_n)_{x_{\frac{r_1}{2}}} \cdots (G_1)_{x_{\frac{1}{2}}} \\
K_{0^+} &= (G_{n+1})_{x_0^+} (G_n)_{x_{\frac{r_1}{2}}} \cdots (G_1)_{x_{\frac{1}{2}}} \\
K_+ &= (G_{n+1})_{x_0^+} (G_n)_{x_{\frac{r_1}{2}}} \cdots (G_1)_{x_{\frac{1}{2}}} \\
K_{0^+}^H &= (H_{n+1})_{x_0^+} (H_n)_{x_{\frac{r_1}{2}}} \cdots (H_1)_{x_{\frac{1}{2}}} \\
K_+^H &= (H_{n+1})_{x_0^+} (H_n)_{x_{\frac{r_1}{2}}} \cdots (H_1)_{x_{\frac{1}{2}}}
\end{align*}$$

**Lemma 7.5.** We have the following identities:

$$\begin{align*}
K_+ &= (G_{n+1})_{x_0^+} (H_n)_{x_{\frac{r_1}{2}}} \cdots (H_1)_{x_{\frac{1}{2}}} = (G_{n+1})_{x_0^+} K_+^H \\
K_{0^+} &= (G_{n+1})_{x_0^+} (H_n)_{x_{\frac{r_1}{2}}} \cdots (H_1)_{x_{\frac{1}{2}}} = (G_{n+1})_{x_0^+} K_{0^+}^H \\
K_{0^+}^H &= (H_{n+1})_{x_0^+} (H_n)_{x_{\frac{r_1}{2}}} \cdots (H_1)_{x_{\frac{1}{2}}} \\
K^H_+ &= (H_{n+1})_{x_0^+} (H_n)_{x_{\frac{r_1}{2}}} \cdots (H_1)_{x_{\frac{1}{2}}} \cdots
\end{align*}$$
Proof.
The first two lines follow from Corollary §7.2. It is clear that
\[ K_{0+}^H \supset (H_{n+1})_{x,0+}(H_n)_{x,r_n,\frac{r_n}{2}} \cdots (H_1)_{x,r_1,\frac{r_1}{2}}. \]
In order to prove the fourth identity, it remains to show that
\[ (H_i)_{x,r_i,\frac{r_i}{2}} \subset (H_{n+1})_{x,0+}(H_n)_{x,r_n,\frac{r_n}{2}} \cdots (H_1)_{x,r_1,\frac{r_1}{2}} \]
for all \( n + 1 \geq i \geq 1 \) (where \( r_{n+1} = 0 \)). We show this by induction. For \( i = n + 1 \) the statement is obvious, so assume \( n \geq i \geq 1 \) and that the statement holds for \( i + 1 \). Then \( (H_{i+1})_{x,r_{i+1},\frac{r_{i+1}}{2}} \subset (H_i)_{x,r_i,\frac{r_i}{2}} \subset (H_{i+1})_{x,r_{i+1},\frac{r_{i+1}}{2}} \cdots (H_1)_{x,r_1,\frac{r_1}{2}} \), and it suffices to prove that \( (H_i)_{x,r_i,\frac{r_i}{2}} = (H_{i+1})_{x,r_{i+1},\frac{r_{i+1}}{2}} \), or, equivalently,
\[ (\mathfrak{h}_i)_{x,r_i,\frac{r_i}{2}}/(\mathfrak{h}_i)_{x,r_i} = ((\mathfrak{h}_i)_{x,r_i,\frac{r_i}{2}} + (\mathfrak{h}_{i+1})_{x,r_{i+1,\frac{r_{i+1}}{2}}})/(\mathfrak{h}_i)_{x,r_i} \]
This follows by taking \( \text{Gal}(E/k) \)-invariants of the following equality (of abelian groups with \( \text{Gal}(E/k) \)-action)
\[ (\mathfrak{h}_i(E))_{x,r_i,\frac{r_i}{2}}/(\mathfrak{h}_i(E))_{x,r_i} = (\mathfrak{h}_i(E))_{x,r_i,\frac{r_i}{2}}/(\mathfrak{h}_i(E))_{x,r_i} \oplus (\mathfrak{h}_{i+1}(E))_{x,r_{i+1,\frac{r_{i+1}}{2}}}/(\mathfrak{h}_{i+1}(E))_{x,r_i}, \]
where \( E \) is a tamely ramified extension of \( k \) over which \( G_{i+1} \) and \( G_i \) split. The third identity is proved analogously. \( \square \)

Let \( \rho_{Yu} \) be an irreducible representation of \( K_{G_{n+1}} \) such that \( \rho_{Yu}|_{(G_{n+1})_{x,0}} \) factors through \( (G_{n+1})_{x,0}/(G_{n+1})_{x,0+} \) and contains a cuspidal representation of \((G_{n+1})_{x,0}/(G_{n+1})_{x,0+} \). By Lemma 7.3 and [KY17, 7.3 Remark] the tuple \((\tilde{G}, x, \tilde{F}, \rho_{Yu}, \tilde{\phi})\) satisfies Conditions D1, D3, D4 and D5 in [KY17, 7.2]. Using this tuple we can carry out Yu’s construction \( \Sigma \) as explained in [KY17, 7.4] to obtain a representation of \( K \) that we denote by \((\pi_K, \pi_K)\). By construction, the representation \( \pi_K \) is of the form \( \rho_{Yu} \otimes \kappa_{\tilde{\phi}} \), where \( \rho_{Yu} \) also denotes the extension of \( \rho_{Yu} \) from \( K_{G_{n+1}} \) to \( K \) that is trivial on \((G_n)_{x,\frac{r_n}{2}} \cdots (G_1)_{x,\frac{r_1}{2}} \), and \((\kappa_{\tilde{\phi}}, \pi_K)\) is a representation of \( K \) that depends only on \((\tilde{G}, \tilde{F}, \tilde{\phi})\), i.e. not on the choice of \( \rho_{Yu} \) ([Yu01][4] or [Kim07][12.4]). In particular, \((\pi_K|_{K+}, \pi_K)\) does not depend on \( \rho_{Yu} \).

We denote by \( \tilde{\phi}_i \) \((1 \leq i \leq n)\) the character of \( K_{G_{n+1}}(G_{i+1})_{x,0}G_{x,\frac{r_i}{2}} \) defined in [Yu01][4], i.e. the unique character of \( K_{G_{n+1}}(G_{i+1})_{x,0}G_{x,\frac{r_i}{2}} \) satisfying

\(^1\text{Remark 7.3. in [KY17] explains how to get the 5-tuple } \Sigma \text{ (using the notation from [KY17]) from our 5-tuple. The authors mention in this remark that as a last step one “can then extend/modify } \iota \text{ to a family } \{ \iota \} \text{ which is } \tilde{\iota}-\text{generic”. However, by doing so one might have to change our point } x \text{ (which is denoted by } y \text{ in [KY17]} \text{) to a nearby point in the building. In order to keep working with } x \text{ we will not perform this last modification. As a consequence the requirement of } \{ \iota \} \text{ being } \tilde{\iota}-\text{generic in Condition D2 of [KY17, 7.2] might not be satisfied. However, we can still carry out Yu’s construction with our tuple.} \)

\(^2\text{Ju-Lee Kim confirmed that ”relative to } x \text{ for all } x \in \mathcal{B}(G') \text{” in Condition D5 in [KY17, 7.2] should be “relative to } y \text{ (using the notation of [KY17]).} \)

\(^3\text{Yu assumes that } p \text{ is odd. Our condition } p \nmid |W| \text{ implies that } p \text{ is odd unless } G \text{ is a torus, but Yu’s construction works for tori even if } p = 2 \text{ (but is rather trivial in that case).} \)
• $\hat{\phi}_i|_{K_{G_n+1}(G_{i+1})_{x,R}} = \phi_i|_{K_{G_n+1}(G_{i+1})_{x,R}}$, and
• $\hat{\phi}_i|_G$ factors through

\[
G_{x,R}/G_{x,R_i} \cong \frac{\mathfrak{g}_{x,R}/\mathfrak{g}_{x,R_i} = \left(\mathfrak{g}_{i+1} \oplus \mathfrak{r}^\prime\right)}{\left(\mathfrak{g}_{i+1} \oplus \mathfrak{r}^\prime\right)}_{x,R_i} \cong \left(\mathfrak{g}_{i+1} \oplus \mathfrak{r}^\prime\right)_{x,R_i}.
\]

on which it is induced by $\hat{\phi}_i$. Here $\mathfrak{r}^\prime$ is as defined in Lemma 5.1, i.e.

Then Yu proves in [Yu01, Proposition 11.4] that if $p \neq 2$, then $(G_i)_{x,R,R_i,R_i}/\left((G_i)_{x,R,R_i,R_i} \cap \text{ker}(\hat{\phi}_i)\right)$ is a Heisenberg $p$-group with center $(G_i)_{x,R,R_i,R_i}/\left((G_i)_{x,R,R_i,R_i} \cap \text{ker}(\hat{\phi}_i)\right)$. Let $(\omega_i, V_{\omega_i})$ denote the Heisenberg representation of this Heisenberg $p$-group with central character $\hat{\phi}_i|_{(G_i)_{x,R,R_i,R_i}}$. Then we observe from the construction of $(\kappa_\phi, V_{\kappa})$ and [Yu01, Theorem 11.5] that $(\kappa_\phi|_{K_{G_n+1}, V_{\kappa}})$ is irreducible and that the underlying vector space $V_{\kappa}$ is isomorphic to any one dimensional Heisenberg vector space. In that case $(\kappa_\phi|_{K_{G_n+1}, V_{\kappa}})$ is the trivial one dimensional representation. If $p = 2$, then $G$ is a torus and $(\kappa_\phi|_{K_{G_n+1}, V_{\kappa}})$ is a one dimensional representation and we take $V_{\omega_i} = \mathbb{C}$. (This is the space underlying a Heisenberg representation whose symplectic vector space is of dimension zero, which aligns with the observation that $(G_i)_{x,R,R_i,R_i}/\left((G_i)_{x,R,R_i,R_i} \cap \text{ker}(\hat{\phi}_i)\right) = (G_i)_{x,R,R_i,R_i}/\left((G_i)_{x,R,R_i,R_i} \cap \text{ker}(\hat{\phi}_i)\right)$ for $p = 2$.) The restriction of $(\kappa_\phi, V_{\kappa})$ to $(G_i)_{x,R,R_i,R_i}/\left((G_i)_{x,R,R_i,R_i} \cap \text{ker}(\hat{\phi}_i)\right)$ for $1 \leq i \leq n$ is given by letting $(G_i)_{x,R,R_i,R_i}$ act via the Heisenberg representation $\omega_i$ of $(G_i)_{x,R,R_i,R_i}/\left((G_i)_{x,R,R_i,R_i} \cap \text{ker}(\hat{\phi}_i)\right)$ with central character $\hat{\phi}_i|_{(G_i)_{x,R,R_i,R_i}}$ on $V_{\omega_i}$ and via $\hat{\phi}_j|_{(G_i)_{x,R,R_i,R_i}}$ on $V_{\omega_j}$ for $j \neq i$.

**Lemma 7.6.** There exists an irreducible $K_+$-subrepresentation of $(\pi|_{K_+}, V)$ that is isomorphic to any one dimensional $K_+$-subrepresentation of $(\pi_{K_{K_+}}|_{K_+}, V_{\kappa})$.

**Proof.** By [Yu01, Proposition 4.4], the representation $(\pi_{K_{K_+}}|_{K_+}, V_{\kappa})$ is $\theta := \prod_{1 \leq i \leq n} \hat{\phi}_i|_{K_+}$-isotypic. Let $(\pi|_{K_+}, V''')$ be an irreducible $K_+$-subrepresentation of $(\pi|_{K_+}, V') \subset (\pi|_{K_+}, \tilde{V})$. By Lemma 7.3, the group $(G_{n+1})_{x,0+}$ acts on $V''$ via $\theta$. Moreover, by Lemma 7.3 and (i) the restriction $\theta|_{(H_i)_{x,0+}}$ for $1 \leq i \leq n$ factors through $(H_i)_{x,0+}/(H_i)_{x,0+} \cong (h_i)_{x,0+}/(h_i)_{x,0+}$, where it is given by $\varphi \circ X_i$ (by the last line of Lemma 3.6). Hence the group $(H_i)_{x,0+}$ acts on $V''$ via $\theta$ for $1 \leq i \leq n$. Since $(G_{n+1})_{x,0+}$ together with $(H_i)_{x,0+}$, $1 \leq i \leq n$, generate $K_+$ by Lemma 7.5, we are done.

We denote by $N^H$ the kernel in $K_+^H$ of $\theta|_{K_+^H} = \prod_{1 \leq i \leq n} \hat{\phi}_i|_{K_+^H}$.

**Lemma 7.7.** If $n > 0$ and $p > 2$, then $K_{G_n+1}/N^H$ is a Heisenberg $p$-group with center $K_+^H/N^H$. If $n = 0$ or $p = 2$, then $K_{G_n+1}/N^H = K_+^H/N^H$. 29
Proof.
Note that $[K^H_+, K^H_0] \subset K^H_+$ and $[K^H_0, K^H_+] \subset (H_{n+1})_{x,0+}(H_n)_{x,r_0+}, \ldots \in (H_1)_{x,r_1+} \subset N^H$. Thus the center of $K^H_0/N^H$ contains $K^H_+ / N^H$ and we have a pairing $(a, b) = \theta(ab^{-1}b^{-1})$ on $K^H_0 / K^H_+ \times K^H_0 / K^H_+$. Note that

$$K^H_0 / K^H_+ \simeq (H_1)_{x,r_1}, \ldots / (H_1)_{x,r_1}, \ldots \oplus \oplus (H_n)_{x,r_n}, \ldots / (H_n)_{x,r_n}, \ldots$$

and it is easy to check (as done in the proof of [Kim07 Proposition 18.1]) that $(\cdot, \cdot)$ is the sum of the pairings $(\cdot, \cdot)_i$ on $(H_1)_{x,r_1}, \ldots / (H_1)_{x,r_1}, \ldots$ defined by $(a, b)_i = \hat{\phi}_i(ab^{-1}b^{-1})$. By [Yu01 Lemma 11.1], the pairing $(\cdot, \cdot)_i$ is non-degenerate $1 \leq i \leq n$, and hence the pairing $(\cdot, \cdot)$ is non-degenerate. Thus the center of $K^H_0/N^H$ is contained in $K^H_+ / N^H$, and therefore equals $K^H_+ / N^H$. Moreover, the image of $\theta|_{K^H_+}$ is $\{ c \in \mathbb{C} | c^n = 1 \}$, which implies that $K^H_+ / N^H$ has order $p$. The remainder of the proof works completely analogous to Yu’s proof ([Yu01 Proposition 11.4]) that the group $(H_1)_{x,r_1}, \ldots / (H_1)_{x,r_1}, \ldots \oplus \oplus (H_n)_{x,r_n}, \ldots / (H_n)_{x,r_n}, \ldots$ contains an irreducible $K$-representation $(\pi|_{K^H_+}, \hat{V})$ is a Heisenberg $p$-group with center $(H_1)_{x,r_1}, \ldots / (H_1)_{x,r_1}, \ldots \oplus \oplus (H_n)_{x,r_n}, \ldots / (H_n)_{x,r_n}, \ldots$ for $1 \leq i \leq n$. We outline the proof as a convenience for the reader and refer to [Yu01 Proposition 11.4] for details: We first prove the statement over a tame extension $E$ over which $G_{n+1}$ is split, and denote by $K^H_0(E), K^H_+(E)$ and $N(E)$ the corresponding groups constructed over $E$. By [Yu01 Lemma 10.1] and the above observations (over $E$), it suffices to exhibit subgroups $W_1$ and $W_2$ of $K^H_0(E)/N^H(E)$ that have trivial intersection with the center and whose image in $K^H_0(E)/K^H_0(E)$ is a complete polarization. This can be achieved by using positive and negative root groups, respectively. To conclude that $K^H_0/N^H$ is a Heisenberg $p$-group, we then embed $K^H_0/N^H$ into $K^H_+(E)/N^H(E)$, observe that by above its image $K^H_0/K^H_+$ in $K^H_+(E)/K^H_+(E)$ is a non-degenerate subspace, and apply [Yu01 Lemma 10.3].

The second half of the lemma follows immediately from the definition of $K^H_0$ and $K^H_+$ and the observation that if $p = 2$, then $G$ is a torus.

Let $(\pi|_{K^H_+}, \hat{V})$ be the irreducible $K$-subrepresentation of $(\pi, V)$ that contains $\hat{V}$.

Lemma 7.8. There exists an irreducible representation $(\rho, V_\rho)$ of $K$ that is trivial on $K_{0+}$ such that $(\rho \otimes \kappa_{\hat{\phi}}, V_\rho \otimes V_\kappa) \simeq (\pi|_{K^H_+}, \hat{V})$.

Proof.
Since $K_{0+} = G_{x,0+}K^H_0$ (Lemma 7.5) and $G_{x,0+} \subset K_+$ acts on $V_{\pi K}$ via $\theta|_{G_{x,0+}}$ (times identity) by [Yu01 Proposition 4.4], we deduce from the irreducibility of $(\kappa_{\hat{\phi}}, V_\kappa)$ mentioned above that also its restriction $(\kappa_{\hat{\phi}}|_{K^H_0}, V_\kappa)$ to $K^H_0$ is irreducible. Recall that $(\kappa_{\hat{\phi}}|_{K^H_0}, V_\kappa)$ factors through $K^H_0 / N^H$ and $K^H_+ / N^H$. Moreover, $\theta|_{K^H_+}$ acts via the character $\theta|_{K^H_+}$ (times identity). By Lemma [7.7] and the theory of Heisenberg representations there exists a unique irreducible representation of $K^H_0$, factoring through $K^H_0 / N^H$ and having $K^H_+ / N^H$ act via the character $\theta|_{K^H_+}$ (times identity). On the other hand, Lemma [7.6] and the observation that $[K^H_+, K_+] \subset N^H$ imply that $(\pi|_{K^H_+}, \hat{V})$ contains an irreducible $K^H_+$-subrepresentation on which $K_+$ acts via the character $\theta|_{K_+}$ (times identity), and which therefore is isomorphic to $(\kappa_{\hat{\phi}}|_{K^H_0}, V_\kappa)$ as a
$K^H_0$-representation. Moreover, since $K_{0+} = K_+K^H_0$, we deduce from the $K_+$-action that $(\pi|_{K_{0+}}, \hat{\pi})$ contains an irreducible $K_{0+}$-subrepresentation isomorphic to $(\kappa_{\tilde{\phi}}|_{K_{0+}}, V_\kappa)$. Hence, by [Kim07, Proposition 18.5] (or rather the analogous statement in our setting that is proved in the same way), the irreducible representation $(\pi|_K, \hat{\pi})$ of $K$ that extends $(\kappa_{\tilde{\phi}}|_{K_{0+}}, V_\kappa)$ is of the form $(\rho \otimes \kappa_{\tilde{\phi}}, V_\rho \otimes V_\kappa)$ for some irreducible representation $(\rho, V_\rho)$ of $K$ that is trivial on $K_{0+}$.

\begin{corollary}
The subspace $\hat{\pi}$ is contained in $V_{\cup_{i=1}^{n+1}\pi_i}$ and the action of the group $(H_i)_{x,r_i,\frac{q}{2}+}/(H_i)_{x,r_i+} \simeq (h_i)_{x,r_i,\frac{q}{2}+}/(h_i)_{x,r_i+}$ on $\hat{\pi}$ via $\pi$ is given by the character $\varphi \circ X_i$ for $1 \leq i \leq n$.
\end{corollary}

\begin{proof}
Let $1 \leq i \leq n$. We have $(H_i)_{x,r_i,\frac{q}{2}+} \subset K_+$ and $(H_{n+1})_{x,0+} \subset K_+$ and by Lemma 7.8 the representation $(\pi|_{K_i}, \hat{\pi})$ is $\theta$-isotypic. As we saw in the proof of Lemma 7.6, the character $\theta|(H_i)_{x,r_i,\frac{q}{2}+}$ factors through $(H_i)_{x,r_i,\frac{q}{2}+}/(H_i)_{x,r_i+} \simeq (h_i)_{x,r_i,\frac{q}{2}+}/(h_i)_{x,r_i+}$, on which it is given by $\varphi \circ X_i$, and $\theta|(H_{n+1})_{x,0+}$ is trivial by Lemma 7.3.
\end{proof}

\begin{lemma}
The irreducible components of the representation $(\rho|_{(G_{n+1})_{x,0}}, V_\rho)$ provided by Lemma 7.8 are cuspidal representations of $(G_{n+1})_{x,0}/(G_{n+1})_{x,0+}$.
\end{lemma}

\begin{remark}
Readers familiar with Kim’s work may expect that we could mainly cite [Kim07] for the proof of Lemma 7.10. However, contrary to the claim in [Kim07, Proposition 17.2.(2)], the representation $\rho|(G_{n+1})_{x,0} \otimes \kappa_{\tilde{\phi}}|(G_{n+1})_{x,0} \otimes \prod_{1 \leq i \leq n} \phi^{-1}_i|(G_{n+1})_{x,0}$ might not necessarily be cuspidal when viewed as a representation of $(G_{n+1})_{x,0}/(G_{n+1})_{x,0+}$. Since the proof of the above mentioned proposition in [Kim07] is not correct, we provide a different and independent proof of Lemma 7.10.
\end{remark}

\begin{proof}[Proof of Lemma 7.10] If $G$ is a torus, then the claim is obvious. Hence we assume $p \neq 2$ for the remainder of the proof. Suppose $(\rho', V_{\rho'})$ is an irreducible subrepresentation of $(\rho|_{(G_{n+1})_{x,0}}, V_\rho)$ that is not cuspidal (viewed as a representation of $(G_{n+1})_{x,0}/(G_{n+1})_{x,0+}$). Then there exists (the $j$-points of) a unipotent radical $U_j$ of a (proper) parabolic subgroup of the reductive group (with $j$-points) $(G_{n+1})_{x,0}/(G_{n+1})_{x,0+} = (G_{n+1})_{x}(f)$ such that $\rho'|_{U_j}$ contains the trivial representation of $U_j$. Denote by $V_{\rho'}''$ a subspace of $V_{\rho'}$ on which $U_j$ acts trivially. By [CGP15, Corollary 2.2.5 and Proposition 2.2.9] there exists a one parameter subgroup $\lambda : \mathbb{G}_m \to (G_{n+1})_{x}$ such that $U_j = \{g \in (G_{n+1})_{x}(f) | \lim_{t \to 0} \lambda(t).g = 1 \}$. Let $\lambda : \mathbb{G}_m \to G_{n+1}$ denote a lift of $\lambda$ that factors through a maximally split maximal torus $T$ of $G_{n+1}$ such that the apartment $\mathcal{A}(T)$ contains $x$ (see the proof of Lemma 6.1.2 for more details about such a lift). Let $\kappa' = \kappa_{\tilde{\phi}}|(G_{n+1})_{x,0} \otimes \prod_{1 \leq i \leq n} \phi^{-1}_i|(G_{n+1})_{x,0}$, which is trivial on $(G_{n+1})_{x,0+}$ (by either combining Corollary 7.4 with Lemma 7.6 or by using the proof of Lemma 7.6) and therefore can also be regarded as a representation of $(G_{n+1})_{x,0}/(G_{n+1})_{x,0+}$ and hence of $U_j$. Recall that $(\omega_i, V_{\omega_i})$ denotes the Heisenberg representation of $(G_i)_{x,r_i,\frac{q}{2}+}/((G_i)_{x,r_i,\frac{q}{2}+} \cap \ker(\tilde{\phi}_i))$ with central character (the restriction of) $\tilde{\phi}_i$, and that the vector space $V_\kappa$ underlying the
representation of $\kappa'$ is $\bigotimes_{i=1}^{n} V_{\omega_i}$. By the construction of Yu ([Yu01][§4, p. 592 and Theorem 11.5]), the representation $\kappa'$ is defined by letting $(G_{n+1})_{x,0}/(G_{n+1})_{x,0+}$ act on each of the tensor product factors $V_{\omega_i}$ in $\bigotimes_{i=1}^{n} V_{\omega_i}$ by mapping $(G_{n+1})_{x,0}/(G_{n+1})_{x,0+}$ to the symplectic group $\text{Sp}(V_i)$ of the corresponding symplectic space $V_i := (G_i)_{x,r_i,\frac{m}{2}}/(G_i)_{x,r_i,\frac{m}{2}+}$, with pairing defined by $(a, b)_i = \hat{\omega}_i(aba^{-1}b^{-1})$ and composing with a Weil representation. The map from $(G_{n+1})_{x,0}/(G_{n+1})_{x,0+}$ to $\text{Sp}(V_i)$ is induced by the conjugation action of $(G_{n+1})_{x,0}$ on $(G_i)_{x,r_i,\frac{m}{2}}$.

Let $E$ be a tamely ramified extension of $k$ over which $T$ splits, and define for $1 \leq i \leq n$ the space $V_i^+$ to be the image of $G(k) \cap \left\langle U_{\alpha}(E)_{x, \frac{m}{2}} \mid \alpha \in \Phi(G_i, T) - \Phi(G_{i+1}, T), \lambda(\alpha) > 0 \right\rangle$ in $V_i$, the space $V_i^0$ to be the image of $G(k) \cap \left\langle U_{\alpha}(E)_{x, \frac{m}{2}} \mid \alpha \in \Phi(G_i, T) - \Phi(G_{i+1}, T), \lambda(\alpha) = 0 \right\rangle$ in $V_i$, and $V_i^-$ to be the image of $G(k) \cap \left\langle U_{\alpha}(E)_{x, \frac{m}{2}} \mid \alpha \in \Phi(G_i, T) - \Phi(G_{i+1}, T), \lambda(\alpha) < 0 \right\rangle$ in $V_i$. Then $V_i = V_i^+ \oplus V_i^0 \oplus V_i^-$, the subspaces $V_i^+$ and $V_i^-$ are both totally isotropic, the orthogonal complement of $V_i^+$ is $V_i^+ \oplus V_i^0$, and $V_i^0$ is a non-degenerate subspace of $V_i$. Let $P_i \subset \text{Sp}(V_i)$ be the (maximal) parabolic subgroup of $\text{Sp}(V_i)$ that preserves the subspace $V_i^+$. Note that the image of $U_i$ in $\text{Sp}(V_i)$ is contained in $P_i$. Let $U_{i,j}$ be the image of

$$U_i := G(k) \cap \left\langle U_{\alpha}(E)_{x, \frac{m}{2}} \mid \alpha \in \Phi(G_i, T) - \Phi(G_{i+1}, T), \lambda(\alpha) > 0 \right\rangle$$

in the Heisenberg group $(G_i)_{x,r_i,\frac{m}{2}}/((G_i)_{x,r_i,\frac{m}{2}+} \cap \ker(\hat{\phi}_i))$. Then by Yu’s construction of the special isomorphism

$$j_i : (G_i)_{x,r_i,\frac{m}{2}}/((G_i)_{x,r_i,\frac{m}{2}+} \cap \ker(\hat{\phi}_i)) \to V_i^g$$

in [Yu01], Proposition 11.4], where $V_i^g$ is the group $V_i \ltimes \mathbb{F}_p$ with group law $(v, a), (v', a') = (v + v', a + a' + \frac{1}{2}(v, v'))$, and since $\lambda(\mathbb{G}_m) \subset T$, we have $j_i(U_{i,j}) = V_i^+ \ltimes 0$. By [Gér77] Theorem 2.4.(b) the restriction of the Weil–Heisenberg representation $V_{\omega_i}$ (via $j_i$) to $P_i \ltimes U_{i,j}$ contains a subrepresentation $V_{\omega_i}$ on which $U_{i,j}$ acts trivially and on which the action of $P_i$ is as follows: By [Gér77] Lemma 2.3.(c) there exist surjections $p_1^i : P_i \to \text{GL}(V_i^+)$ and $p_2^i : P_i \to \text{Sp}(V_i^0)$. Then the action of $P_i$ on $V_{\omega_i}^+$ is the tensor product of $p_1^i$ composed with a (quadratic) character $\chi$ of $\text{GL}(V_i^+)$ and $p_2^i$ composed with a Weil representation of $\text{Sp}(V_i^0)$. Note that the image of $U_i$ in $\text{GL}(V_i^+)$ (by composing $U_i \to P_i$ with $p_1^i : P_i \to \text{GL}(V_i^+)$) is unipotent and hence contained in the commutator subgroup of $\text{GL}(V_i^+)$. Thus $\chi \circ p_1^i$ trivial on the image of $U_i$. Moreover, the image of $U_i$ in $\text{Sp}(V_i^0)$ (by composing $U_i \to P_i$ with the surjection $p_2^i : P_i \to \text{Sp}(V_i^0)$) is contained in a minimal parabolic subgroup of $\text{Sp}(V_i^0)$ ([BT71], 3.7. Corollaire) and hence also in a parabolic subgroup $P_i^0$ of $\text{Sp}(V_i^0)$ that fixes a maximal totally isotropic subspace of $V_i^0$. By [Gér77] Theorem 2.4.(b) the Weil

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4 As Loren Spice pointed out, the statement of [Gér77] Theorem 2.4.(b) contains a typo. From the proof provided by [Gér77], one can deduce that the stated representation of $P(E_{+}, j)H(E_{+}, j)$ (i.e. the pull-back to $P(E_{+}, j)H(E_{+}, j)$ of a representation of $SH(\text{End}_{0}, j)$ as in part (a')) should be tensored with $\chi^{\pm 1} \times 1$ before inducing it to $P(E_{+}, j)H(E, j)$ in order to define $\pi_+$ (using the notation of [Gér77]).
representation $V'_{\omega_i}$ restricted to $P^0_i$ contains a one dimensional subrepresentation $V''_{\omega_i}$ on which the action of $P^0_i$ factors through a character of $P^0_i/U(P^0_i)$ where $U(P^0_i)$ denotes the unipotent radical of $P^0_i$. Since the image of $U_i$ is unipotent and hence its image in $P^0_i/U(P^0_i)$ (which is isomorphic to a general linear group) is contained in the commutator subgroup of $P^0_i/U(P^0_i)$, the group $U_i$ acts trivially on $V''_{\omega_i}$.

Let $V''_n$ denote the subspace $\otimes_{1\leq i\leq n} V''_{\omega_i}$ of $\otimes_{1\leq i\leq n} V_{\omega_i} = V_\kappa$. Let $U^{H}_{n+1}$ be the preimage of $U_i$ in $(H_{n+1})_{x,0}$ under the surjection $(H_{n+1})_{x,0} \twoheadrightarrow (H_{n+1})_{x,0}/(H_{n+1})_{x,0+}$. Since $\phi_i$ is trivial on $H_{n+1}(k)$ for all $1 \leq i \leq n$ (Lemma 7.3(3)), the action of the group $U^{H}_{n+1}$ via $\rho \otimes \kappa_{\phi_i}$ on the subspace $V''_n\otimes V''_n$ of $V_\rho \otimes V_\kappa$ is the trivial action. Moreover, recall that the restriction of $(\kappa_{\phi_i},V_\kappa)$ to $(G_i)_{x,r_i,\frac{\omega}{2}}$ for $1 \leq i \leq n$ is given by letting $(G_i)_{x,r_i,\frac{\omega}{2}}$ act via the Heisenberg representation $\omega_i$ on $V_{\omega_i}$ and via $\hat{\phi}_j|(G_i)_{x,r_i,\frac{\omega}{2}}$ on $V_{\omega_j}$ for $j \neq i$. By Lemma 7.3(3) and the definition of $\hat{\phi}_j$, the character $\hat{\phi}_j$ is trivial on $(H_i)_{x,r_i,\frac{\omega}{2}}$ for $j \neq i$. Hence $U_i$ (which is contained in $(H_i)_{x,r_i,\frac{\omega}{2}}$) acts trivially via $\rho \otimes \kappa_{\phi_i}$ on $V''_n \otimes V''_n$.

If $\epsilon > 0$ is sufficiently small, then we have

$$
(H_{n+1})_{x+\epsilon\lambda,0+} \subset \langle (H_{n+1})_{x,0+},U^{H}_{n+1} \rangle \quad \text{and} \quad (H_i)_{x+\epsilon\lambda,r_i,\frac{\omega}{2}+} \subset \langle (H_i)_{x,r_i,\frac{\omega}{2}+},U_i \rangle
$$

for $1 \leq i \leq n$, where $x + \epsilon\lambda$ arises from the action of $\epsilon\lambda$ on $x \in \mathcal{A}(T)$. Since $\rho \otimes \kappa_{\phi_i}$ is by the definition of $\rho$ isomorphic to a $K$-subrepresentation of $(\pi|_K,V)$, we obtain a non-trivial subspace $V''_n$ of $V$ on which $(H_{n+1})_{x+\epsilon\lambda,r_i,\frac{\omega}{2}+}/(H_i)_{x+\epsilon\lambda,r_i+} \cong (H_{n+1})_{x+\epsilon\lambda,r_i+}/(H_i)_{x+\epsilon\lambda,r_i+}$ acts via $\varphi \circ X_i$ for $1 \leq i \leq n$ and that is fixed by $(H_{n+1})_{x+\epsilon\lambda,0+}$. Since $x + \epsilon\lambda \in \mathcal{A}(T)$, the tuple $(x + \epsilon\lambda,(X_i)_{1\leq i\leq n})$ is a truncated datum (by Lemma 3.6 and Corollary 3.8), and by the same arguments as in Case 2 of the proof of Theorem 6.1, we can extend it to a datum $(x + \epsilon\lambda,(X_i)_{1\leq i\leq n},(\rho_0,V_{\rho_0}))$ contained in $(\pi,V)$. However, since $U_i$ was non-trivial (and $\epsilon > 0$ sufficiently small), the dimension of the facet of $\mathcal{B}(G_{n+1},k)$ that contains $x + \epsilon\lambda$ is larger than the dimension of the facet of $\mathcal{B}(G_{n+1},k)$ that contains $x$. This is a contradiction to the choice of $(x,(X_i)_{1\leq i\leq n},(\rho_0,V_{\rho_0}))$, i.e. to the assumption that $(x,(X_i)_{1\leq i\leq n},(\rho_0,V_{\rho_0}))$ is a datum for $(\pi,V)$ (as in Definition 4.4).

In order to prove that $(\pi,V)$ contains a type as constructed by Kim and Yu in [KY17], we introduce some additional notation following [KY17, 2.4]. We denote by $Z_s(M_{n+1})$ the maximal split torus in the center of $M_{n+1}$ and by $M_i$ the centralizer of $Z_s(M_{n+1})$ in $G_i$ for $1 \leq i \leq n$. We say (compare [KY17, 3.5. Definition]) that the resulting commutative diagram of embeddings (where the embeddings are chosen as explained above)

$$
\begin{array}{c}
\mathcal{B}(M_{n+1},k) \hookrightarrow \mathcal{B}(M_n,k) \hookrightarrow \cdots \hookrightarrow \mathcal{B}(M_1,k) \\
\downarrow \quad \downarrow \quad \quad \quad \quad \quad \quad \downarrow \\
\mathcal{B}(G_{n+1},k) \hookrightarrow \mathcal{B}(G_n,k) \hookrightarrow \cdots \hookrightarrow \mathcal{B}(G_1,k)
\end{array}
$$

is $\left(\frac{r_{n+1}}{2}, \frac{r_2}{2}, \ldots, \frac{r_1}{2}\right)$-generic relative to $x$ if

$$
\sum_{j=1}^{n} \left( \dim((G_i)_{x,r_j}/(G_i)_{x,r_j+}) - \dim((M_i)_{x,r_j}/(M_i)_{x,r_j+}) \right) = 0.
$$

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Note that this property is independent of the choice of embeddings in Diagram $[3][6]$

**Theorem 7.12.** Let $(\pi, V)$ be a smooth irreducible representation of $G(k)$. Then $(\pi, V)$ contains one of the types constructed by Kim–Yu in $[KY17]$.

**Proof.**

By Theorem 6.1, the representation $(\pi, V)$ contains a datum. Let $(x, (X_i)_{1 \leq i \leq n}, (\rho_0, V_{\rho_0}))$ be a datum for $(\pi, V)$ such that the non-negative number $\sum_{j=1}^{n} \left( \dim((G_i)_{x, \frac{\epsilon}{2}}/(G_i)_{x, \frac{\epsilon}{2}+}) - \dim((M_i)_{x, \frac{\epsilon}{2}}/(M_i)_{x, \frac{\epsilon}{2}+}) \right)$ is minimal among all possible choices of data for $(\pi, V)$. Performing the constructions above (page 27 and Lemma 7.8) we obtain a tuple $(\tilde{G}, x, \tilde{\tau}, \rho|_{K_{G_n+1}}, \tilde{\phi})$ and an associated representation $(\pi_K, V_{\pi_K}) = (\rho \otimes \kappa_{\phi}, V_{\rho} \otimes V_{\kappa})$ as constructed by Kim and Yu that is contained in $(\pi, V)$. It remains to show that $(K, \pi_K)$ is a type, i.e. that all the requirements that Kim and Yu impose on the tuple $(\tilde{G}, x, \tilde{\tau}, \rho|_{K_{G_n+1}}, \tilde{\phi})$ for the construction of types are satisfied. By Lemma 7.3(iii) and Lemma 7.10 it therefore remains to show that Diagram $[3]$ is $(\frac{r_0+1}{2}, \frac{r_0}{2}, \ldots, \frac{r_0}{2})$-generic relative to $x$. Suppose that this is not the case. Then, by $[KY17]$, 3.6 Lemma (b)], there exists $\lambda \in X_+((\mathbb{Z}_s(M_{n+1}))$ such that if $\epsilon > 0$ is sufficiently small, then Diagram $[3]$ is $(\frac{r_0+1}{2}, \frac{r_0}{2}, \ldots, \frac{r_0}{2})$-generic relative to $x + \epsilon \lambda$ and $(G_i)_{x+\epsilon \lambda, \frac{\epsilon}{2}} \subset (G_i)_{x, \frac{\epsilon}{2}}$ for $1 \leq i \leq n$. Note that using the notation of the proof of Lemma 7.10 the image of $(G_i)_{x+\epsilon \lambda, \frac{\epsilon}{2}}/((G_i)_{x+\epsilon \lambda, \frac{\epsilon}{2}+} \cap \ker(\hat{\phi})) = j_i^{-1}(V_i \ltimes \mathbb{F}_p)$ is $j_i^{-1}(V_i^+ \ltimes \mathbb{F}_p)$, where $V_i^+$ is the totally isotropic subspace $(G_i)_{x+\epsilon \lambda, \frac{\epsilon}{2}+}/(G_i)_{x, \frac{\epsilon}{2}+}$ of $V_i = (G_i)_{x, \frac{\epsilon}{2}}/(G_i)_{x, \frac{\epsilon}{2}+}$. For $1 \leq i \leq n$, let $V_{\omega_i}$ be a subpace of the Heisenberg representation $V_{\omega_i}$ on which $V_i^+$ acts trivially, and denote by $V_{\kappa}$ the subspace $\otimes_{1 \leq j \leq n} V_{\omega_j}$ of $\otimes_{1 \leq j \leq n} V_{\omega_j} = V_{\kappa}$. Then the action of $(H_i)_{x+\epsilon \lambda, \frac{\epsilon}{2}+}$ on $V_{\rho} \otimes V_{\kappa}$ factors through $(H_i)_{x+\epsilon \lambda, \frac{\epsilon}{2}+}/(H_i)_{x+\epsilon \lambda, \frac{\epsilon}{2}+} \simeq (\mathfrak{h}_i)_{x+\epsilon \lambda, \frac{\epsilon}{2}+}/(\mathfrak{h}_i)_{x+\epsilon \lambda, \frac{\epsilon}{2}+}$ on which it is given by the character $\varphi \circ X_i$ for $1 \leq i \leq n$. Moreover, $H_{n+1})_{x+\epsilon \lambda, 0+} = (H_{n+1})_{x, 0+}$ acts trivially on $V_{\rho} \otimes V_{\kappa}$. Hence (by Lemma 3.6 Corollary 3.8 and the same arguments as those in Case 2 of the proof of Theorem 6.1) we obtain a datum $(x + \epsilon \lambda, (X_i)_{1 \leq i \leq n}, (\rho_0, V_{\rho_0}))$ for $(\pi, V)$ with $\sum_{j=1}^{n} \left( \dim((G_i)_{x+\epsilon \lambda, \frac{\epsilon}{2}}/(G_i)_{x+\epsilon \lambda, \frac{\epsilon}{2}+}) - \dim((M_i)_{x+\epsilon \lambda, \frac{\epsilon}{2}}/(M_i)_{x+\epsilon \lambda, \frac{\epsilon}{2}+}) \right) = 0$. This contradicts that $0 < \sum_{j=1}^{n} \left( \dim((G_i)_{x, \frac{\epsilon}{2}}/(G_i)_{x, \frac{\epsilon}{2}+}) - \dim((M_i)_{x, \frac{\epsilon}{2}}/(M_i)_{x, \frac{\epsilon}{2}+}) \right)$ was minimal among all possible choices of data for $(\pi, V)$. \hfill \Box

**Remark 7.13.** Theorem 7.12 has been derived by Kim and Yu ($[KY17]$ 9.1 Theorem) from the result about exhaustion of Yu’s supercuspidal representations by Kim ($[Kim07]$) under much more restrictive assumptions than our Assumption 2.1. First of all they require the local field $k$ to have characteristic zero, and secondly their assumption on the residual characteristic $p$ is much stronger than ours, i.e. far from optimal, see $[Kim07]$ § 3.4.\footnote{While our point $x$ is a point of $\mathcal{B}(G, k)$ that is viewed as a point of $\mathcal{B}(M_i, k)$ and $\mathcal{B}(G_i, k)$ via the above embeddings, Kim and Yu ($[KY17]$) fix a point in $\mathcal{B}(M_{n+1}, k)$ and consider its image in $\mathcal{B}(M_i, k)$ and $\mathcal{B}(G_i, k)$. Hence the genericity property in $[KY17]$ 3.5. Definition] does depend on the embeddings.}
8 Exhaustion of supercuspidal representations

Recall that we assume throughout the paper that $G$ splits over a tame extension and $p \nmid |W|$. Under these assumptions, we obtain the following corollary of Section 7.

**Theorem 8.1.** Every smooth irreducible supercuspidal representation of $G(k)$ arises from the construction of Yu ([Yu01]).

**Proof.**
Let $(\pi, V)$ be a smooth irreducible supercuspidal representation of $G(k)$. By Section 7, in particular Theorem 7.12, we can associate to $(\pi, V)$ a tuple $(\vec{G}, x, \vec{r}, \rho|_{K_{n+1}}, \vec{\phi})$ such that $(\pi, V)$ contains the type $(K, \pi_K)$ associated to it by Kim–Yu following Yu’s construction. Let $M_{n+1}$ be the Levi subgroup of $G_{n+1}$ attached to $x$ and $G_{n+1}$ as in Section 7, page 25.

Following Kim–Yu, we let $Z_S(M_{n+1})$ denote a maximal split torus of the center $Z(M_{n+1})$ of $M_{n+1}$, and we define the Levi subgroup $M$ of $G$ to be the centralizer $\text{Cent}_G(Z_S(M_{n+1}))$ of $Z_S(M_{n+1})$ in $G$. Kim and Yu ([KY17, 7.5 Theorem]) show that the type $(K, \pi_K)$ is a cover of a type for the group $M$. Hence, since $(\pi, V)$ is supercuspidal, we have $M = G$. This implies that $Z_S(M_{n+1})$ is contained in the center of $G$. Hence $Z(G_{n+1})/Z(G)$ is anisotropic, where $Z(G_{n+1})$ and $Z(G)$ denote the centers of $G_{n+1}$ and $G$, respectively, and $M_{n+1} = G_{n+1}$. Instead of working with $K_{G_{n+1}} = (G_{n+1})_x$ in Section 7, we could have equally well performed all constructions for the stabilizer $(G_{n+1})_x$ of the image $[x]$ of $x$ in the reduced Bruhat–Tits building of $G_{n+1}$ (by replacing $(M_{n+1})_x$ by $(M_{n+1})_x$ everywhere) to obtain a representation $(\vec{\rho}, V_\rho)$ of $\vec{K} = (G_{n+1})_x[G_{n+1}]_x \cdot \cdots \cdot (G_1)_x \cdot \cdots$ such that the representation $(\pi^k_\vec{K}, \pi^k_\vec{K})$ of $\vec{K}$ associated to $(\vec{G}, x, \vec{r}, \vec{\rho}|_{(G_{n+1})_x}, \vec{\phi})$ by Yu is contained in $(\pi^k_\vec{K}, V)$. Since $M_{n+1} = \text{Cent}_{G_{n+1}}(Z_S(M_{n+1})) = G_{n+1}$, the compactly induced representation $\text{ind}_{G_{n+1}}^{G(k)}(\vec{\rho}|_{(G_{n+1})_x}, \vec{\phi})$ is irreducible supercuspidal (by [MP96, Proposition 6.6]). Hence $(\vec{G}, x, \vec{r}, \vec{\rho}|_{(G_{n+1})_x}, \vec{\phi})$ satisfies all the conditions that Yu requires for his construction of supercuspidal representations ([Yu01, § 3]), and $\text{ind}_{G_{n+1}}^{G(k)}(\pi_K)$ is the corresponding irreducible supercuspidal representations ([Yu01, Proposition 4.6]). By Frobenius reciprocity, we obtain a non-trivial morphism from $(\text{ind}_{G_{n+1}}^{G(k)}(\pi_K), \text{ind}_{G_{n+1}}^{G(k)}(V_{\pi_K}))$ to $(\pi, V)$, and hence these two irreducible representations are isomorphic.

**Remark 8.2.** The exhaustion of supercuspidal representations by Yu’s construction has been known under the assumption that $k$ has characteristic zero and $p$ is a sufficiently large prime number thanks to Kim ([Kim07]). We refer the reader to [Kim07, § 3.4] for the precise conditions for $p$ being “sufficiently large”. These assumptions are much stronger than $p \nmid |W|$.

The proof of Theorem 8.1 also shows how to recognize if a representation is supercuspidal by only considering a datum for this representation.

**Corollary 8.3.** Let $(\pi, V)$ be a smooth irreducible representation of $G(k)$, and let $(x, (X_i)_{1 \leq i \leq n}, (\rho_0, V_{\rho_0}))$ be a datum for $(\pi, V)$. Then $(\pi, V)$ is supercuspidal if and only if $x$ is a facet of minimal dimension in $\mathcal{B}(G_{n+1}, k)$ and $Z(G_{n+1})/Z(G)$ is anisotropic, where $G_{n+1} = \text{Cent}_G(\sum_{i=1}^n X_i)$.
Proof.
The point $x$ is a facet of minimal dimension in $\mathcal{B}(G_{n+1}, k)$ if and only if $M_{n+1} = G_{n+1}$. Hence we have seen in the proof of Theorem 8.1 that $(\pi, V)$ being supercuspidal implies the other two conditions in the corollary. The proof of Theorem 8.1 also shows that the other two conditions are sufficient to prove that $(\pi, V)$ is supercuspidal. 

References


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Types for tame $p$-adic groups

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