Complex Contagions on Configuration Model Graphs with a Power-Law Degree Distribution

Grant Schoenebeck, Fang-Yi Yu
Contagions, diffusion, cascade...

• Ideas, beliefs, behaviors, and technology adoption spread through network
• Why do we need to study this phenomena?
  – Better Understanding
  – Promoting good behaviors/beliefs
  – Stopping bad behavior
Outline

• Models
  – Complex contagions model
  – Power-Law and configuration model graph
• Main result
• Related work
• A happy proof sketch
Outline

• Models
  – Complex contagions model
  – Power-Law and configuration model graph

• Main result

• Related work

• A happy proof sketch
Model of Contagions

- K-Complex Contagions [GEG\textsubscript{13}; CLR 79]
  - Given an initial infected set $I = \{u, v\}$. 
Model of Contagions

- K-Complex Contagions [GEG13; CLR 79]
  - Given an initial infected set $I = \{u, v\}$.
  - Node becomes infected if it has at least $k$ infected neighbors.
• K-Complex Contagions [GEG13; CLR 79]
  – Given an initial infected set \( I = \{u, v\} \).
  – Node becomes infected if it has at least \( k \) infected neighbor
Model of Contagions

- K-Complex Contagions [GEG13; CLR 79]
  - Given an initial infected set \( I = \{u, v\} \).
  - Node becomes infected if it has at least \( k \) infected neighbor.
Model of Contagions

- K-Complex Contagions [GEG13; CLR 79]
  - Given an initial infected set $I = \{u, v\}$.
  - Node becomes infected if it has at least $k$ infected neighbors.
Why k complex contagions?

- One of most classical and simple contagions model
  - Threshold model [Gra 78]
  - Bootstrap percolation [CLR 79]
- Non-submodular
Motivating Question

• Do k complex contagions spread on social networks?
Question

• Do k complex contagions spread on Erdos-Renyi model $G_{n,p}$ where $p = O\left(\frac{1}{n}\right)$?
  – $n$ vertices
  – Each edge $(u, v)$ occurs with probability $p$
• Need $\Omega(n)$ (random) seeds to infect constant fraction of the graph[JLTV89]?
• Can we categorize all networks which spread slowly/quickly?
What is a social network?

• Qualitatively: special structure
  – Power law degree distribution
  – low-diameter/small-world...

• Quantitatively: generative model?
  – Configuration model graphs
  – Preferential attachment model
  – Kleinberg’s small world model
Motivating Question

• Do k complex contagions spread on social networks?
Motivating Question

- Do $k$ complex contagions spread on social networks?
  - What properties are shared by social networks?
  - Do these properties alone permit complex contagion spreads?
Outline

• Models
  – Complex contagions model
  – Power-Law and configuration model graph

• Main result
• Related work
• A happy proof sketch
Power-law distribution

• A power-law distribution with $\alpha$ if the $\Pr[X = x] \sim x^{-\alpha}$
Configuration Model

- Given a degree sequence \( \text{deg}(v_1), \text{deg}(v_2), \ldots, \text{deg}(v_n) \)
- The node \( v_i \) has \( \text{deg}(v_i) \) stubs

Nodes with stubs
Configuration Model

- Given a degree sequence $\text{deg}(v_1), \text{deg}(v_2), \ldots, \text{deg}(v_n)$
- The node $v_i$ has $\text{deg}(v_i)$ stubs
- Choose a uniformly random matching on the stubs
Outline

• Models
  – Complex contagions model
  – Power-Law and configuration model graph

• Main result

• Related work

• A happy proof sketch
Main result
• Configuration Model
  – power-law degree distribution $2 < \alpha < 3$
• Initial infected node
  – the highest degree node
• $k$-complex contagions spreads to $\Omega(1)$ fraction of nodes with high probability.

Corollary from [Amini 10]
• Configuration Model
  – power-law degree distribution $3 < \alpha$
• Initial infected nodes:
  – $o(1)$ fraction of highest degree node
• $k$-complex contagions spreads to $o(1)$ fraction of nodes with high probability.
The Bottom Line

Main result
• Configuration Model
  – power-law degree distribution $2 < \alpha < 3$
• $k$-complex contagions
  – Initial infected node: the highest degree node
• Contagions spreads to $\Omega(1)$ fraction of nodes with high probability.

Corollary from [Amini 10]
• Configuration Model
  – power-law degree distribution $3 < \alpha$
• $k$-complex contagions
  – Initial infected node: $o(1)$ fraction of highest degree node
• Contagions spreads to $o(1)$ fraction of nodes with high probability.

Complex contagions spread on most of graphs with power-law degree distribution $2 < \alpha < 3$. 
Outline

• Models
  – Complex contagions model
  – Power-Law and configuration model graph
• Main result
• Related work
• A happy proof sketch
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>What has been done?</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>---------------------------------------------------------</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Lattice[98]</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Random regular[07]</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Configuration model</strong> $\alpha &gt; 3$ [10]</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Configuration model</strong> $2 &lt; \alpha &lt; 3$ [16]</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Watts-Strogatz[13]</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Kleinberg[14]</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Tree[79,06]</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Erdos-Renyi model[12]</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Chung-Lu model[12]</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Preferential Attachment[14]</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-------------</td>
<td>--------------------</td>
<td>----------------------------------------</td>
<td>----------------------------------------</td>
</tr>
</tbody>
</table>
### k-complex contagions don’t spread

**Physics**

|-------------|--------------------|----------------------------------------|-------------------|---------------|
## k-complex contagions spread

<table>
<thead>
<tr>
<th>Lattice[98]</th>
<th>Random regular[07]</th>
<th>Configuration model $\alpha&gt;3$ [10]</th>
<th>Configuration model $2&lt;\alpha&lt;3$[16]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Preferential Attachment[14]</td>
</tr>
</tbody>
</table>

**Network Science**
Things get complicated

Physics

Lattice [98]
Tree [79, 06]

Random regular [07]
Erdos-Renyi model [12]

Network Science

Configuration model $\alpha > 3$ [10]
Configuration model $2 < \alpha < 3$ [16]

Chung-Lu model [12]

Watts-Strogatz [13]
Kleinberg [14]
Preferential Attachment [14]
What have we learned?

Physics

- Lattice[98]
- Tree[79,06]

Network Science

- Watts-Strogatz[13]
- Kleinberg[14]

Random regular[07]

Configuration model \( \alpha > 3 \) [10]
Configuration model \( 2 < \alpha < 3 \) [16]

Erdos-Renyi model[12]
Chung-Lu model[12]

Preferential Attachment[14]
Why do we want to solve it?

Lattice
Random regular
Erdos-Renyi model
Chung-Lu model
Configuration model 2<\alpha<3
Configuration model \alpha>3
Tree
Watts-Strogatz
Kleinberg
Preferential Attachment
Outline

• Models
  – Complex contagions model
  – Power-Law and configuration model graph
• Main result
• Related work
• A happy proof sketch
Idea

- Restrict contagions from high degree node to low degree nodes
- Reveal the edges when needed
Observations

- The highest degree nodes forms clique
Observations

• The highest degree nodes forms clique
• $k$ degree node has many edges to $l$ degree nodes where $l > k$
Observations

- The highest degree nodes forms clique
- $k$ degree node has many edges to $l$ degree nodes where $l > k$
- Inductive structure
Observations

- The highest degree nodes forms clique
- $k$ degree node has many edges to $l$ degree nodes where $l > k$
- Inductive structure
<table>
<thead>
<tr>
<th></th>
<th>Physics</th>
<th>Network Science</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td><em>Configuration model $\alpha&gt;3$ [10]</em></td>
<td></td>
</tr>
<tr>
<td></td>
<td><em>Configuration model $2&lt;\alpha&lt;3$[16]</em></td>
<td></td>
</tr>
</tbody>
</table>
Inductive Structure

• Partition nodes into buckets ordered by degree of nodes $B_1, B_2, ..., B_l$

• Induction: if infection spreads on previous buckets $B_i$ where $i < k$, the infection also spread on bucket $B_k$. 
Inductive Structure

• If infection spreads on previous buckets $B_i$, where $i < k$, the infection also spread on bucket $B_k$. 
Inductive Structure

• If infection spreads on previous buckets $B_i$ where $i < k$, the infection also spread on bucket $B_k$. 
Inductive Structure

• If infection spreads on previous buckets $B_i$ where $i < k$, the infection also spread on bucket $B_k$. 

```
Time 2
```

\[ B_1 \rightarrow B_2 \rightarrow B_3 \]
Inductive Structure

• If infection spreads on previous buckets $B_i$ where $i < k$, the infection also spread on bucket $B_k$. 
Inductive Structure

- Induction: if infection spreads on previous buckets $B_i$ where $i < k$, the infection also spread on bucket $B_k$.
  - Well connection between buckets
  - Infection spread in buckets
Inductive Structure

• Induction: if infection spreads on previous buckets $B_i$ where $i < k$, the infection also spread on bucket $B_k$.
  – Well connection between buckets: Chernoff bound
  – Infection spread in buckets: Chebyshev’s inequality
Thanks for your listening
How do we solve it?

• Chebyshev’s inequality

\[
\Pr[Z > E[Z] + t] \leq \frac{Var[Z]}{t^2}
\]

• Chernoff-type bound

\[
\Pr[Z_n > Z_0 + t] \leq \exp\left(\frac{-t^2}{c}\right)
\]