Math 615, Projects

Broadly speaking, I would like student work on a project that emphasizes one of: learning mathematics, doing research, or coding. Any of these projects can be completed by an individual or by a small group of 2-3 students. I will be happy to help you design or refine your project as much as needed.

Key dates:

- February 20: a couple of ideas for your project.
- February 27: outline for project due (a paragraph or two that clearly outlines the project and that includes some possible references)
- March 13: progress report due (1-2 page writeup that demonstrates progress on both the expository and computational sides of the project)
- March 27: rough draft due (this should be a fairly complete rough draft).
- April 1: return comments on rough draft.
- April 17: final draft due.

1. Learning focused. Such a project would tackle a subject in commutative algebra that you are interested in learning about. Choose a topic that you are interested in learning about and a reasonable reference for the subject. The goal would be to produce the following:

- A short paper summarizing some of the main results about this subject;
- Code that illustrates how to compute interesting examples and functions related to the subject.

For instance, here is what one such topic might look like more concretely. Imagine that you are interested in monomial ideals and Stanley–Reisner theory. Then, you might:

- Study Chapter 1 of Miller–Sturmfels’ book *Combinatorial Commutative Algebra*.
- Explore the Macaulay2 package SimplicialComplexes for working with Stanley–Reisner ideals.
- Write code for doing some function not covered by that package (e.g. code for determining whether or not the graded Betti numbers depend on the characteristic of the ground field)
- Compute a broad range of Stanley–Reisner examples. These should give you a feel for the capabilities and limitations of the package. Even better would be if your examples are targeted at some phenomenon you want to understand better (e.g. how do you construct a simplicial complex whose Betti numbers are different in characteristic $p$ than in characteristic 0)?

A good starting point for such a project would be to choose 1-2 chapters from one of the following textbooks:

- Cox, Little, O’Shea’s *Using Algebraic Geometry*
- Bruns and Herzog’s *Cohen–Macaulay Rings*
- Eisenbud’s *The Geometry of Syzygies*
- Miller and Sturmfels’ *Combinatorial Commutative Algebra*
- Sturmfels’s *Algorithms in Invariant Theory*

Mel’s notes include lots of good topics, too. And if you have some other idea for such a project, I am entirely to open this.
2. **Experiment focused.** This would be focused on performing a collection of organized experiments with the hope of discovering interesting phenomenon. The best way to generate such a project would be in relation to something that you are already interested in, such as a thesis problem, leftovers from an REU, etc. Even if you only have a vague area in mind (e.g. Hilbert schemes of points, F-signature, Boij–Söderberg theory, etc) I am happy to help design an interesting experiment, or to contact someone who would be able to help design such a project.\(^1\)

The experiments do not have to consist of original research. For instance, lots is known about monomial curves, but it can still be fun and enlightening to perform experiments and try and rediscover some of those ideas for yourself.

One of these projects will likely require you to come and chat with me several times. In particular, it often happens that the original question will turn out to be too hard or uninteresting for some reason, and so we may need to refine the question a couple of times before finding something interesting. You may also need to learn some new mathematics along the way. Said another way, I expect that this type of project will probably require the most work.

Here are some ideas I have for these types of problems.

*Laurent polynomials and Eulerian numbers.* A short paper of mine discovered a connection between certain Laurent polynomials and the Eulerian numbers. There are some leftover questions listed in Section 4 of that paper. It would be interesting to do further computations and see if we can gain new insights into those questions. This might involve exploring the “NumericalAlgebraicGeometry” package in Macaulay2.

*Modular curves and Hilbert functions.* This is a question that somebody else asked me, and I just haven’t had the time to do some experiments with it. Fix some prime \(p\) Let \(S := \mathbb{Q}[s_0, \ldots, s_{p-1}, t_0, \ldots, t_{p-1}]\). Let \(I\) be the ideal

\[
I = \langle s_a + s_{-a}, t_a - t_{-a} | a \in \mathbb{Z}/p \rangle + \langle s_as_b + s bs_c + s cs_a + t_a + t_b + t_c | a + b + c = 0 \in \mathbb{Z}/p \rangle.
\]

What can you say about the Hilbert function and syzygies of \(S/I\)? This question is apparently related to the study of modular curves and other questions from arithmetic geometry.

*Random semigroup rings.* Use the new “MonomialAlgebras” package to study the properties of random semigroup rings. (See §7 of Miller–Sturmfels’ *Combinatorial Commutative Algebra* for background on semigroup rings.) For instance, here is one such question: let \(S_r\) be a random, simplicial semigroup in \(\mathbb{N}^r\) where the vectors have coordinate sum 3 and the ideal has codimension 2. As \(r \to \infty\), what is the probability that the semigroup ring \(\mathbb{Q}[S_r]\) is seminormal (or normal or Cohen–Macaulay or . . . )?\(^2\) Or you could try to compute the “average” value of some numerical invariant, such as the Hilbert function of \(\mathbb{Q}[S_r]\).

*Points on random curves of low genus over a finite field.* Fix a prime \(p\). Let \(C\) be a random curve of genus \(g\) (for \(g = 3, 4, 5,\) or 6) over the finite field \(\mathbb{F}_p\). How many \(\mathbb{F}_p\) points does \(C\) have? Do the experiment for small values of \(p\) and then see if you can work out an explicit formula or a good estimate.

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\(^1\)For instance, if you are interested in F-signature, then I would email Karl Schwede to try and help design such a project.

\(^2\)If you are interested in this one, be sure and come talk to me. The key is to find some question where the probability is not 0% or 100%, and then to explain why.
Random integer matrices. Fix $a < b$. Let $\Phi_N$ be a random $a \times b$ matrix of integers of absolute value at most $N$. $\Phi_N$ induces a map

$$\mathbb{Z}^a \xrightarrow{\Phi_N} \mathbb{Z}^b.$$  

As $N \to \infty$, what is the probability that $\Phi_N$ is surjective?\(^3\)

Eisenbud’s book also include includes some interesting possible projects on pages 379–384. My personal favorites are projects 3, 4, 5, and 6.

3. Coding based. This is a variant on the “learning based” projects. But there would be a much heavier focus on adding functionality to a Macaulay2 package, and less of a focus on writing an expository treatment of a subject. For instance, let’s say you are interested in D-modules. Then you could contact the authors of that package and ask about what type of functions it would be interesting to add. In the best case scenario, you would be adding functionality and writing documentation for that package. If you are particularly interested into getting a code-heavy project, start by thinking about a few areas that you would be interested (the list of packages on the M2 webpage is a good place to start).

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\(^3\)I believe that this phenomenon has been well studied. But it is connected to some really interesting stuff in arithmetic geometry such as the Cohen–Lenstra heuristics, and work of Ellenberg, Venkatesh, and Westmoreland.