Exercises.

(1) Given two complexes \( F \) and \( G \), we can define a new complex by

\[
(F \otimes G)_i := \bigoplus_{p+q=i} F_p \otimes G_q
\]

with boundary

\[
\partial(f \otimes g) = \partial f \otimes g + (-1)^p f \otimes \partial g.
\]

Fix a (commutative, Noetherian) ring \( R \) and \( x_1, x_2, x_3 \in R \). Confirm that, with this definition, there exists a natural isomorphism (up to signs):

\[
K(x_1, x_2) \otimes K(x_3) \cong K(x_1, x_2, x_3).
\]

(2) Let \( S = \mathbb{Q}[x, y] \). For the Koszul complex \( K(x^2, xy, y^2) \), compute all of the homology modules of this complex. What does that tell you about \( \text{depth}(x^2, xy, y^2, S) \)?

(3) Find an example of an ideal \( I \) in a Noetherian ring \( R \) where \( \text{depth}(I, R) > \text{codim}(I, R) \). Illustrate the correctness of your example in Macaulay2.

(4) Over a Noetherian ring \( R \), fix \( x_1, \ldots, x_n \in R \) such that \((x_1, \ldots, x_n)R = R\), i.e. such that \( 1 \in (x_1, \ldots, x_n) \). Prove that the Koszul complex \( K(x_1, \ldots, x_n) \) is exact, i.e. that all homology groups of this Koszul complex vanish. (Note: a more general fact is proven in Eisenbud §17.4. But there is a simpler proof in this case based on Corollary 17.10 from Eisenbud, where \( y_1 = 1 \).)

(5) For those who know some algebraic topology: The Koszul complex is closely related to the complex for computing the reduced homology of a simple. Namely, let \( R = \mathbb{Z} \) and let \( 1 = (1, \ldots, 1) \in \mathbb{Z}^{n+1} \). Show that (possibly up to signs) there is a natural way to identify \( K(1) \) with the complex for computing the reduced homology of the \( n \)-simplex \( \Delta_n \).