Coverings of random ellipsoids, and invertibility of matrices with i.i.d. heavy-tailed entries

Liza Rebrova*
University of Michigan

Konstantin Tikhomirov
University of Alberta

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Invertibility problem

Theorem (Rudelson, Vershynin)

Let $n \geq n_0$ and let $A = (a_{ij})$ be an $n \times n$ random matrix with i.i.d mean zero subgaussian entries. Then for any $\varepsilon > 0$ we have

$$\mathbb{P}\left\{s_n(A) \leq \varepsilon n^{-1/2}\right\} \leq L\varepsilon + u^n,$$

where $L > 0$ and $u \in (0,1)$ depend only on the distribution of a_{ij}.

Definition

A random variable ξ is called subgaussian, if for any $t > 0$

$$\mathbb{P}\{|\xi| > t\} \leq C\exp(-ct^2).$$

Invertibility problem

Theorem (R,Tikhomirov)

Let $n \geq n_0$ and let $A = (a_{ij})$ be an $n \times n$ random matrix with i.i.d mean zero $\mathbb{E}a_{ij}^2 = 1$ entries. Then for any $\varepsilon > 0$ we have

$$\mathbb{P}\{s_n(A) \leq \varepsilon n^{-1/2}\} \leq L\varepsilon + u^n,$$

where $L > 0$ and $u \in (0,1)$ depend only on the distribution of a_{ij}.

Corollary: i.i.d. matrices with heavy-tailed entries (as well as subgaussian) are well-invertible, as

$$\|A^{-1}\| = s_{max}(A^{-1}) = 1/s_n(A) \sim \sqrt{n}.$$

Subgaussian case, idea of the proof

$$s_n(A) := s_{min}(A) = \min_{x \in S^{n-1}} \|Ax\|$$

Approximation by the ε-net $\mathcal{N} \subset S^{n-1}$.
For any $x \in S^{n-1}$ find the closest $y \in \mathcal{N}$:

$$\|Ax\| \geq \|Ay\| - \|A(x-y)\| \geq \|Ay\| - \|A\|\|x-y\| \geq \inf_{y \in \mathcal{N}} \|Ay\| - \sqrt{n} \cdot \varepsilon$$

Lemma (norm of a subgaussian matrix $\sim \sqrt{n}$)

For A be an $n \times n$ random matrix with i.i.d. subgaussian entries.

$$\mathbb{P}\{\|A\| \geq t\sqrt{n}\} \leq \exp(-c_0 t^2 n) \qquad \text{for } t > C_0.$$

Challenge: find an ε-net with the sufficiently low cardinality

$$|\mathcal{N}| \sim \left(\frac{c}{\varepsilon\sqrt{n}}\right)^n.$$

Heavy-tailed case

$$s_n(A) := s_{min}(A) = \min_{x \in S^{n-1}} \|Ax\|$$

Approximation by the ε-net $\mathcal{N} \subset S^{n-1}$.
For any $x \in S^{n-1}$ find the closest $y \in \mathcal{N}$:

$$\|Ax\| \geq \|Ay\| - \|A(x-y)\| \geq \|Ay\| - \|A\|\|x-y\| \geq ???$$

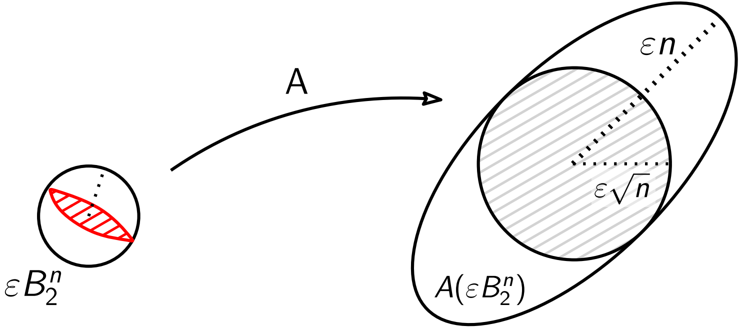
Norm is too large:

$$\|A\| \sim n >> \sqrt{n}$$

New challenge: obtain an estimate $\|A(x-y)\| \leq \sqrt{n}\varepsilon$, where x, y are in the same ε-net element.

So, for any x, y taken from one net element we would like to have

$$\|A(x-y)\| \leq \sqrt{n}\varepsilon$$



The net should be refined without blowing up of cardinality $|\mathcal{N}|$. It is possible as A cannot have too many large directions!

(*) Our model: A is an $n \times n$ random matrix with i.i.d $\mathbb{E}a_{ij} = 0$ and $\mathbb{E}a_{ij}^2 = 1$ entries.

Definition

We call a diagonal matrix $D = (d_{ij})_{i,j=1}^n$ with $d_{ii} \in (0,1)$ a (positive diagonal) *contraction*.

Key proposition

Let $\delta \in (0,1]$ *and* A *satisfies (*). Then with probability* $1 - 4\exp(-\delta n/8)$ *there exists a contraction* D*, such that (1)* $\det D \geq \exp(-\delta n)$ *and (2)* $\|AD\|_{\infty\to 2} \leq \frac{C}{\sqrt{\delta}} n$.

Then the shifts of parallelepiped $D(n^{-\frac{1}{2}} B_\infty^n)$ can serve as "dilated net elements":

$$\|AD(n^{-\frac{1}{2}} B_\infty^n)\| \leq \frac{1}{\sqrt{n}} \|AD\|_{\infty\to 2} \leq \frac{C}{\sqrt{\delta}} \sqrt{n},$$

Consequences of the key proposition

1) This solves an invertibility problem for heavy-tailed matrices: there is a net refinement of cardinality $|\mathcal{N}'| \leq C^n \left(\frac{c}{\varepsilon\sqrt{n}}\right)^n$ such that $\|A(x-y)\| \sim \sqrt{n}\varepsilon$ for every A with high probability.

2) Also, it implies the following geometric theorem about the existence of a good covering for a random ellipsoid:

Theorem A

For any $\delta \in (0, 1/4]$ there is a subset $\mathcal{N} \subset B_2^n$ of cardinality at most $\exp(C_\delta n)$, such that for any random matrix A satisfying (*) with probability at least $1 - 4\exp(-n\delta/32)$ we have

$$A(B_2^n) \subset \bigcup_{y \in A(\mathcal{N})} \left(y + c_\delta \sqrt{n} B_2^n\right).$$

Proof of key proposition

Key proposition

Let $\delta \in (0,1]$ and A satisfies (*). Then with probability $1 - 4\exp(-\delta n/8)$ there exists a contraction D, such that (1) $\det D \geq \exp(-\delta n)$ and (2) $\|AD\|_{\infty\to 2} \leq \frac{C}{\sqrt{\delta}} n$.

Construction of D:

1. Step 1: construct D with the following properties:
   - Euclidean norms of all the rows of AD are bounded by $C_\delta\sqrt{n}$
   - $\det D \geq \exp(-\delta n)$
2. Step 2: show that the norms of the rows condition implies that $\|AD\|_{\infty\to 2} \leq C'_\delta n$ with high probability

Step 1: Bernoulli case, for one row

Suppose a vector $X := (a_{i1}^2, \dots, a_{in}^2)$ has Bernoulli coordinates.
We define D_i for X:

$$\begin{cases} d_{ii} := \frac{Cn}{\|X\|_1} \cdot \mathbb{I}_{\{x_i \neq 0\}} \text{ if } \|X\|_1 > Cn \\ d_{ii} := 1 \text{ otherwise} \end{cases}$$

Then $\|\langle X, D_i \rangle\|_1 \leq Cn$ and $\det D_i \geq \exp(-\delta)$.

Example 1 (making l_1 norm a bad row to be at most 2):

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & d & 0 & 0 & 0 \\ 0 & 0 & d & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & d \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & d & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & d & d & 0 & d \\ 1 & 0 & 0 & 0 & d \end{bmatrix}$$

Step 1: Bernoulli case, for a matrix

Define

$$D := \prod_{i=1}^{n} D_i$$

Then

- $\|\langle X, D\rangle\|_1 \leq Cn$ for every row $X = (a_{i1}^2, \ldots, a_{in}^2)$
- $\det D = \prod(\det D_i) \geq \exp(-\delta n)$.

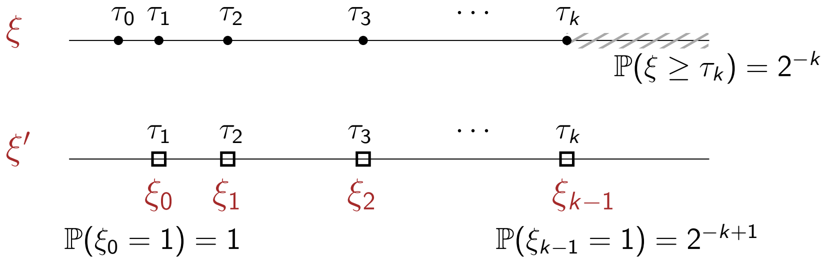
Example 2 (making l_1 norm of every row be at most 2):

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} d_1 \cdot 1 & 0 & 0 & 0 & 0 \\ 0 & d_1 d_2 & 0 & 0 & 0 \\ 0 & 0 & 1 \cdot d_2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & d_1 d_2 \end{bmatrix} = \begin{bmatrix} d_1 & 0 & 0 & 1 & 0 \\ 0 & d_1 d_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & d_1 d_2 & d_2 & 0 & d_1 d_2 \\ d_1 & d_1 d_2 & 0 & 0 & d_1 d_2 \end{bmatrix}$$

Step 1: General distribution case

Main idea: any random variable ξ (for us $\xi = a_{ij}^2$) can be almost surely approximated above by the sum of Bernoulli random variables ξ_i, such that $\mathbb{P}(\xi_i = 1) = 2^{-i}$,

$$\xi' := \sum_{i=1}^{\infty} \tau_i \xi_i \geq \xi, \qquad \text{and} \qquad \mathbb{E}\xi' \leq 2\mathbb{E}\xi.$$



Step 3: from bounded rows to bounded $\|.\|_{\infty\to 2}$ norm

Lemma

Let A has symmetrically distributed entries and satisfies (*). If there exists $D \in \mathcal{F}$ (any countable subset of the set of contractions), such that Euclidean norms of the rows AD are at most $K\sqrt{n}$, then

$$\|AD\|_{\infty\to 2} \le CKn$$

with exponentially high probability.

Main idea: as

$$\|AD\|^2_{\infty\to 2} = \sup_{v\in Vert(B^n_\infty)} \|ADv\|^2 = \sum \langle ADv, e_i\rangle^2,$$

after conditioning on $|A|$ and on D (only signs of the entries matter) we can use Bernstein's inequality to bound the norm.

References and related work

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Thanks for your attention!

Please share your questions and comments with me.

My contact email is erebrova@umich.edu