Morita Equivalence

Eamon Quinlan

Given a (not necessarily commutative) ring, you can form its category of right modules. Take this category and replace the names of all the modules with dots. The resulting category is a bunch of dots with a bunch of arrows. The question is: what can you then say about the original ring from this category of dots? The study of this question leads to the notion of Morita equivalence, two rings being Morita equivalent if they have equivalent categories of right modules. The question then becomes: what properties are preserved under Morita equivalence? In this talk we will explore this and some other related questions, as well as explain a theorem of Morita that gives a very nice criterion for when two rings are Morita equivalent.

1 Introduction

Notation: rings always have 1, but need not be commutative. Given a ring A, we denote by $A - \mathrm{Mod}$ (resp. $\mathrm{Mod} - A$) the category of left (resp. right) A-modules. Two modules will be considered equal if they are isomorphic.

As in the abstract, suppose we are given a category $\mathcal{C}$ which we know is equivalent to the category $\mathrm{Mod} - A$ via a functor $F : \mathrm{Mod} - A \xrightarrow{\sim} \mathcal{C}$. What information about A can we obtain purely based on $\mathcal{C}$? Another way to think about this question is: suppose we start with our category $\mathrm{Mod} - A$, mod out by isomorphism, and erase all the names of the modules and replace them with a symbol like $\bullet$. Now all we have left is a bunch of $\bullet$'s together with a lot of arrows between them; call this category $\mathcal{C}$. What can we recover about our original category $\mathrm{Mod} - A$, and more generally our original ring A, from this new category?

Exercise 1.1.- (Easy if familiar with some homological algebra, hard otherwise)

(i) Show you can find which $\bullet$ corresponded to the zero module.

(ii) Show you can determine which arrows were injective – we can therefore make sense of what subdots are in our new category of dots.

(iii) Show you can determine which arrows were surjective – we can therefore make sense of what quotient dots are in our new category of dots.

(iv) Given a (two-ended) chain of subdots of a given dot, we can find the dot corresponding to the union.

(v) Given a (two-ended) chain of subdots of a given dot, we can find the dot corresponding to the intersection.

(vi) Given a dot, we can determine whether its corresponding module was right noetherian.

(vii) Given a dot, we can determine whether its corresponding module was right artinian.

Exercise 1.2.- Show that an A-module M is finitely generated if and only if given any chain $0 \subseteq N_0 \subseteq N_1 \subseteq \cdots \subseteq M$ with $\bigcup_{i\in\mathbb{N}} N_i = M$ we have $N_K = M$ for some $K \gg 0$. We can thus determine which dots corresponded to the finitely generated modules.

Exercise 1.3.- Show that the following properties of A can be deduced from $\mathcal{C}$.

(i) Whether A is right noetherian. [Recall: A is right noetherian if and only if all finitely generated A-modules are right noetherian; one can then use 1].

(ii) Whether A is right artinian. [Similar to previous].

(iii) The center of A, up to isomorphism. [Hint: the center of A will be isomorphic to the endomorphism ring of the identity functor on $\text{Mod}-A$]. Thus, if we knew A was commutative to start with, we would be able to recover all of A.

(iv) The characteristic of A.

Exercise 1.4.- The following example shows that not all information can be recovered from this category of dots. For this fix a ring R and let $M_n(R)$ be the ring of n by n matrices over R. Show:

(i) Given an right R-module M there is a natural right action of $M_n(R)$ on $M^{\oplus n}$ coming from the action of R.

(ii) This construction actually gives a functor

$$\text{Mod}-R \longrightarrow \text{Mod}-M_n(R)$$

that is actually an equivalence [Hint: show the functor is fully faithful and essentially surjective to prove it is an equivalence].

(iii) Observe that then after applying our "dot" construction to the rings R and $M_n(R)$ from before get the same category! For example R could be $\mathbb{C}$ and then we would not be able to distinguish whether we started from the ring $\mathbb{C}$ or from the ring $M_n(\mathbb{C})$. As a consequence, whether the ring we started with is commutative cannot be inferred from the category of dots.

(iv) Finally, observe that this example provides a proof that $Z(M_n(\mathbb{C})) \cong \mathbb{C}$.

2 Formalizing the problem

Let's now try to make the discussion more rigorous. As we have already mentioned or hinted at, whether two rings R and S do or don't have the same "category of dots" is in fact testing whether the categories $\text{Mod}-R$ and $\text{Mod}-S$ are equivalent. We thus make the following definition.

Definition 2.1.- Two rings R and S are Morita equivalent if the categories $\text{Mod}-R$ and $\text{Mod}-S$ are equivalent. We denote this by $R \sim S$ (not standard!). A property P is Morita-invariant if whenever R has P and $R \sim S$ we have that S has P.

What we have thus proven is that being right noetherian, right artinian, and having center isomorphic to Z (for a fixed commutative ring Z) are all Morita equivalent properties. We should also remark that in the case where R and S are commutative, $R \sim S$ if and only if $R \cong S$.

Another important remark is that at this point the relation $\sim$ should really be called something like "right"-Morita equivalence, to reflect the fact that we are testing whether the categories of "right"-modules are equivalent. If you don't mind spoilers, see Theorem 5.1 to address this concern.

3 Bimodules associated to an equivalence

From now on, most of the discussion is extracted from [AF].

We would now like to work towards a criterion due to Morita for when two rings are Morita equivalent. First observe that one of the most fundamental problems of not being able to reconstruct A from $\mathrm{Mod}-A$ comes from the fact that when we forget the "names" of the modules we can no longer recover where A (as a right A-module) was (c.f. 1.4). In other words, given an equivalence $F : \mathrm{Mod}-A \xrightarrow{\sim} \mathrm{Mod}-B$ we need not have that $F(A) \cong B$.

However, the right A-module A has some very special properties that do get transferred to the B-module $F(A)$.

Definition 3.1.- An A-module G is a generator if one of the following equivalent conditions hold:

- (i) Given two morphisms $f \neq g : M \to N$ of right A-modules there exists a morphism $h : A \to M$ with the property that $f \circ h \neq g \circ h$.
- (ii) The functor $\mathrm{Mod}-A \to Sets$, $(M \mapsto \mathrm{Hom}_A(G, M))$ is faithful.
- (iii) Every A-module M is a quotient of a (possibly infinite) direct sum of G's.

An A-module Q is projective if the following equivalent conditions hold:

- (i) The A-module Q is a direct summand of a free A-module.
- (ii) Given any surjection $N \to M \to 0$ of A-modules and a map $f : Q \to M$ there is an extension of f to N.

We say a right A-module P is a progenerator if it is a finitely generated projective generator.

Exercise 3.2.- (i) Prove the equivalence of the conditions in the above definition.

- (ii) Show that A is a progenerator in $\mathrm{Mod}-A$. (c.f. 1.2 for finitely generated).
- (iii) Show that if $F : \mathrm{Mod}-A \to \mathrm{Mod}-B$ is an equivalence then $F(A)$ is a progenerator in $\mathrm{Mod}-B$.
- (iv) Observe that
$$A \cong \mathrm{End}_A A \cong \mathrm{End}_B F(A)$$
where all the isomorphisms are isomorphisms of rings.

The above exercise thus proves:

Proposition 3.3.- *Suppose $A \sim B$. Then there is a progenerator P in $\mathrm{Mod}-B$ such that $A \cong \mathrm{End}_B P$ as rings, and a progenerator Q in $\mathrm{Mod}-A$ with $B \cong \mathrm{End}_A Q$.*

Finally, observe that because $\mathrm{End}_B P$ (resp. $\mathrm{End}_A Q$) act on P (resp. Q) on the *left*, the isomorphisms above actually give P (resp. Q) the structures of (A, B)- (resp. (B, A)-) bimodules.

Remark 3.4.- Something that is a bit confusing is why $\mathrm{End}_A M$ always acts on M on the left, regardless of whether M is a left or a right module. The reason is that we always write functions as $f(x)$ as opposed to $(x)f$. If M is a left-module it becomes a ring-module under $(\mathrm{End}_A M)^{op}$, the opposite ring, which would be the endomorphism ring if we wrote functions as $(x)f$. One also observes this symmetry-breaking by the fact that given a ring A we have $A \cong \mathrm{End}(A_A)$ and $A^{op} \cong \mathrm{End}({}_A A)$.

4 Equivalence from the bimodules

We have associated to an equivalence $\mathrm{Mod}-A \cong \mathrm{Mod}-B$ and (A,B)-bimodule P with the property that it is a progenerator as a right B-module, and with $A \cong \mathrm{End}_B P$. As it turns out from the following proposition, P will also be a progenerator as an A-module. This allows us to start building the symmetry that will culminate in Morita's theorem.

Proposition 4.1.- *Suppose P is an (A,B)-bimodule, progenerator as a right B-module. Then P is a progenerator as a left A-module.*

Proof. Pick some k so that $B^{\oplus k} \cong P \oplus P'$ as B-modules. Then:

$$\begin{aligned} {}_AP^{\oplus k} &\cong \mathrm{Hom}_B(B^{\oplus k},\ {}_AP_B) \\ &\cong \mathrm{Hom}_B(P_B \oplus P'_B,\ {}_AP_B) \\ &\cong A \oplus A'. \end{aligned}$$

This shows that P is a generator.

Also observe that for some m we have $P^{\oplus m} \cong B \oplus B'$ as B-modules. Then

$$\begin{aligned} A^{\oplus m} &\cong \mathrm{Hom}_B(P,\ {}_AP)^{\oplus m} \\ &\cong \mathrm{Hom}_B(P^{\oplus m},\ {}_AP) \\ &\cong \mathrm{Hom}_B(B \oplus B',\ {}_AP) \\ &\cong P \oplus P'. \end{aligned}$$

which shows that P is projective and finitely generated as a left A-module. $\square$

Once we have this result, we can go the other way. That is, we can prove that starting with one of these special bimodules we can get an equivalence.

Proposition 4.2.- *Suppose that P is an (A,B)-bimodule, progenerator on both sides with $A \cong \mathrm{End}_B P$. Then $\mathrm{Mod}\, A \cong \mathrm{Mod}\, B$ via $M \mapsto M \otimes_A P$ and $N \mapsto \mathrm{Hom}_B(P, M)$.*

This proposition is the key to the whole story. Before we can prove it we need a key lemma.

Lemma.- (KEY) Suppose P is a right B-module, progenerator as such, with $A \cong \mathrm{End}_B P$ – in particular, P is an (A,B)-bimodule. Then $B \cong \mathrm{Hom}_A(P,P)$.

Observe that there is certainly a map $B \to \mathrm{Hom}_A(P,P)$ given by right multiplication. We prove this is an isomorphism again as a hinted exercise.

Exercise 4.3.- (i) Recall that for some $n \gg 0$ we have $P^{\oplus n} \cong B \oplus B'$ as right B-modules.

(ii) Observe that $\mathrm{Hom}_A(P,P)$ consists of those $\mathbb{Z}$-linear maps $P \to P$ that commute with every B-linear map.

(iii) Show that if $\phi \in \mathrm{Hom}_A(P,P)$ then $\phi^{\oplus n}(B) \subseteq B$, and that $\phi^{\oplus n}$ restricts to a *left* B-linear map $B \to B$. Conclude that the map $B \to \mathrm{Hom}_A(P,P)$ is injective.

(iv) Show that given any $x \in P^{\oplus n}$ there exists an S-linear map $f_x : P^{\oplus n} \to P^{\oplus n}$ and $s_x \in S$ such that $f_x(s_x) = x$.

(v) Show that if $\phi^{\oplus n}$ restricts to multiplication by $s \in S$ on the right in S then $\phi^{\oplus n}$ is multiplication by s on the right on the whole of $P^{\oplus n}$. Conclude that the map $B \to \mathrm{Hom}_A(P,P)$ is surjective.

We are now ready to prove the proposition.

Proof of Proposition 4.2. . Set $F : \mathrm{Mod}-A \to \mathrm{Mod}-B$ by $F(M) = M \otimes_A P$ and $G : \mathrm{Mod}-B \to \mathrm{Mod}-A$ by $G(N) = \mathrm{Hom}_B(P, N)$. Using 20.10 and 20.11 from the book one then has

$$\begin{aligned} \mathrm{Hom}_B(P, M \otimes_A P) &\cong M \otimes \mathrm{Hom}_B(P, P) \\ &\cong M \otimes_A A \cong M \end{aligned}$$

and

$$\begin{aligned} \mathrm{Hom}_B(P, M) \otimes_A P &\cong \mathrm{Hom}_A(\mathrm{Hom}_A(P, P), N) \\ &\cong \mathrm{Hom}_A(B, N) \cong N. \end{aligned}$$

Therefore, F and G are inverse equivalences. □

Observe we have use the fact that P is a progenerator as an A-module as well, to get the first isomorphism in the second chain of isomorphisms.

5 Morita's Theorem

We have thus shown that equivalences $\mathrm{Mod}-A \cong \mathrm{Mod}-B$ are in some kind of correspondence with special (A, B)-bimodules P. The facts that make these bimodules special is that $A \cong \mathrm{End}_B P$ and that P is a progenerator as a B-module. But in the previous section we have proven that whenever we have such a bimodule we also get $B \cong \mathrm{End}_A P$ and that P is a progenerator as an A-module.

Proposition 4.2 tells us that we also have $A - \mathrm{Mod} \cong B - \mathrm{Mod}$. We thus arrive at the theorem.

Theorem 5.1.- *[Morita] Let A and B be rings. The following are equivalent:*

(i) $\mathrm{Mod}-A \cong \mathrm{Mod}-B$.

(ii) There exists a (B, A)-bimodule P, with $A \cong \mathrm{End}_B P$, which is a progenerator on both sides.

(iii) $B - \mathrm{Mod} \cong A - \mathrm{Mod}$.

Example 5.2.- I would like to give one last example to illustrate why this is such a surprising result. Consider the ring

$$A = \begin{bmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{bmatrix} \subseteq M_2(\mathbb{Q}).$$

Then A is right noetherian but not left noetherian [H]. Now suppose that I give you the "category of dots" that we gave in the introduction associated to the right modules for A. At the beginning we showed how once you get this category you can tell that the ring is right noetherian. What Morita's theorem tells you is that this category *has enough information for you to conclude that the ring is not left noetherian.* In particular, there is no noetherian ring B with $A \sim B$.

6 References

[H]: Hernstein – Noncommutative Rings.

[AF]: Anderson, Fuller – Rings and Categories of Modules.