§0: Some basics

Liminfs and Limsups

- **Def.** Let \((x_n) \subseteq \mathbb{R}\) be a sequence. The **limit inferior** of \((x_n)\) is defined by

\[
\liminf_{n \to \infty} x_n = \lim_{n \to \infty} \inf_{m \geq n} x_m
\]

and, similarly, the **limit superior** of \((x_n)\) is

\[
\limsup_{n \to \infty} x_n = \lim_{n \to \infty} \sup_{m \geq n} x_m.
\]

- **Remark.**

\[
\liminf_{n \to \infty} x_n = \sup_{n \in \mathbb{N}} \inf_{m \geq n} x_m,
\]

\[
\limsup_{n \to \infty} x_n = \inf_{n \in \mathbb{N}} \sup_{m \geq n} x_m.
\]

- **Def.** A number \(\xi \in \mathbb{R} \cup \{-\infty, +\infty\}\) is a **subsequential limit** of \((x_n)\) if there exists a subsequence of \(x_n\) that converges to \(\xi\). We denote by \(E\) the set of subsequential limits of \((x_n)\). **Lemma.**

\[
\liminf(x_n) = \inf E
\]

\[
\limsup(x_n) = \sup E.
\]

In fact, \(E\) is closed so we can replace the above with min and max.

- **Lemma.** For any sequence \((x_n)\), \(\liminf(x_n) \leq \limsup(x_n)\) and \((x_n)\) converges to \(L\) if and only if \(\liminf(x_n) = L = \limsup(x_n)\).
§1: Measure Theory [SS]

Preliminaries

• Thm.- Every open subset \( O \) of \( \mathbb{R} \) can be written uniquely as a disjoint union of countably many open intervals.

• Thm.- Every open subset \( O \) of \( \mathbb{R}^d \) can be written as a countable union of almost disjoint closed cubes.

The exterior measure

• Def.- If \( E \subseteq \mathbb{R}^d \) is any subset, the exterior measure of \( E \) is

\[
m_*(E) := \inf \left\{ \sum_{j=1}^{\infty} |Q_j| : \bigcup_{n=1}^{\infty} Q_j \supseteq E, Q_j \text{ cubes} \right\}.
\]

• Prop.- (Properties of the exterior measure)
  (i) If \( E_1 \subseteq E_2 \) then \( m_*(E_1) \leq m_*(E_2) \).
  (ii) If \( E = \bigcup_{j=1}^{\infty} E_j \) then \( m_*(E) \leq \sum_{j} m_*(E_j) \).
  (iii) If \( E \subseteq \mathbb{R}^d \) then \( m_*(E) = \inf m_*(O) \) where the infimum is taken over all open \( O \) that contain \( E \).
  (iv) If \( E = E_1 \cup E_2 \) and \( d(E_1, E_2) > 0 \) then \( m_*(E) = m_*(E_1) + m_*(E_2) \).
  (v) If \( E \) is an almost disjoint union of countably many cubes \( Q_j \) then \( m_*(E) = \sum_j m_*(Q_j) \).

Measurable sets and Lebesgue measure

• Def.- A subset \( E \) of \( \mathbb{R}^d \) is measurable if for all \( \epsilon > 0 \) there exists an open \( O \subseteq \mathbb{R}^d \) with \( E \subseteq O \) and \( m_*(O \setminus E) \leq \epsilon \).

• Prop.- (Properties of measurable sets)
  (i) Every open set of \( \mathbb{R}^d \) is measurable.
  (ii) If \( m_*(E) = 0 \) then \( E \) is measurable – thus if \( F \subseteq E \) and \( m(E) = 0 \) then \( F \) is measurable.
  (iii) A countable union of measurable sets is measurable.
  (iv) Closed sets are measurable.
  (v) The complement of a measurable set is measurable.
  (vi) A countable intersection of measurable sets is measurable.

• Thm.- If \( E = \bigcup_{j=1}^{\infty} E_j \) is a countable union of disjoint measurable sets then \( m(E) = \sum_j m(E_j) \). Cor.- Suppose \( E_j, E_{j+1}, \ldots \) are measurable. If \( E_j \not\supset E \) then \( \lim_{j \to \infty} m(E_j) = m(E) \). If \( E_j \not\subset E \) and \( m(E_j) < \infty \) for some \( j \) then \( \lim_{j \to \infty} m(E_j) = m(E) \).

• Thm.- Suppose \( E \subseteq \mathbb{R}^d \) is measurable. Then for every \( \epsilon > 0 \):
  (i) There exists an open \( O \) with \( E \subseteq O \) and \( m(O \setminus E) \leq \epsilon \).
  (ii) There exists a closed \( F \) with \( F \subseteq E \) and \( m(E - F) \leq \epsilon \).
  (iii) Furthermore, if \( m(E) < \infty \) then the \( F \) in (ii) can be taken to be compact.
  (iv) If \( m(E) < \infty \) then there exists a finite union \( F = \bigcup_{j=1}^{N} Q_j \) of closed cubes with \( m(E \triangle F) \leq \epsilon \).

Borel subsets

• Def.- The Borel \( \sigma \)-algebra of \( \mathbb{R}^d \), denoted \( B(\mathbb{R}^d) \), is the \( \sigma \)-algebra generated by the open subsets of \( \mathbb{R}^d \).
A \( G \)-delta (or \( G_{\delta} \)) set is a countable intersection of open sets. An \( F \)-sigma (or \( F_{\sigma} \)) set is a countable union of closed sets.
Thm.- Let $E \subseteq \mathbb{R}^d$ be any subset. The following are equivalent.
(i) $E$ is Lebesgue measurable.
(ii) $E$ differs from a $G_\delta$ by a set of measure 0.
(iii) $E$ differs from an $F_\sigma$ by a set of measure 0.

Cor.- The Lebesgue $\sigma$-algebra is the completion of the Borel $\sigma$-algebra.

Measurable functions

Remark.- We allow functions to take the values $-\infty$ and $\infty$.

Def.- A function $f$ on $\mathbb{R}^d$ is measurable if, for all $a \in \mathbb{R}$, $f^{-1}([-\infty, a))$ is measurable.

Lemma.- The following are equivalent for a function $f$.
(i) $f$ is measurable. (ii) $f^{-1}(O)$ is measurable for every open $O \subseteq \mathbb{R}$.
(iii) $f^{-1}(F)$ is measurable for every closed $F \subseteq \mathbb{R}$.

Prop.- (Properties of measurable functions)
(i) If $f$ is measurable on $\mathbb{R}^d$ and finite-valued and $\Phi : \mathbb{R} \to \mathbb{R}$ is continuous then $\Phi \circ f$ is measurable.
(ii) If $\{f_k\}$ is a sequence of measurable functions then the pointwise sup, inf, limsup, and liminf are all measurable. Then the pointwise limit, when it exists – at least a.e. – is also measurable.
(iii) If $f$ is measurable then $f^k$ ($k \geq 1$) is also measurable.
(iv) If $f$ and $g$ are measurable and finite-valued then $f + g$ and $fg$ are also measurable. (iv) If $f$ is measurable and $f(x) = g(x)$ for a.e. $x$ then $g$ is measurable.

Approximations by simple functions

Thm.- If $f$ is a non-negative measurable function then there exists an increasing sequence of non-negative simple functions $\{\phi_n\}$ that converges pointwise to $f$.

Thm.- If $f$ is any measurable function then there exists a sequence of simple functions $\{\phi_k\}_{k=1}^\infty$ with $|\phi_k(x)| \leq |\phi_{k+1}(x)|$ that converges pointwise to $f$.

Thm.- If $f$ is any measurable function then there exists a sequence of step functions (simple functions made with rectangles only) that converges pointwise to $f$ almost everywhere.

Littlewood’s three principles

Remark.- Littlewood’s three principles are:
(i) Every set is nearly a finite union of intervals. (c.f. some theorem from before)
(ii) Every function is nearly continuous. (c.f. Lusin’s theorem)
(iii) Every convergent sequence is nearly uniformly convergent. (c.f. Egorov’s theorem)

Thm.- (Egorov) Let $\{f_k\}$ be a sequence of measurable functions supported on $E$ where $m(E) < \infty$ such that $f_k(x) \to f(x)$ for a.e. $x$. Then for every $\epsilon > 0$ there exists a closed subset $A_\epsilon \subseteq E$ with $m(E - A_\epsilon) \leq \epsilon$ such that $f_k \to f$ uniformly on $A_\epsilon$. Example.- The convergence of $f_n(x) = x^n$ on $[0, 1]$.

Thm.- (Lusin) Suppose $f$ is a measurable, finite-valued function on $E$ where $E$ is of finite measure. Then for every $\epsilon > 0$ there exists a closed $F_\epsilon \subseteq E$ with $m(E - F_\epsilon) \leq \epsilon$ such that $f|_{F_\epsilon}$ is continuous. Remark.- This is different than saying $f$ is continuous on $F_\epsilon$, e.g. $\chi_{\mathbb{Q}}$ on $[0, 1]$. 3
§2: Integration Theory [SS]

The Lebesgue Integral

- **Def.** Given a simple function $\phi = \sum_k c_k \chi_{E_k}$ we define its integral to be
  \[ \int \phi := \sum_k c_k m(E_k). \]

  **Fact.** This is independent of the representation.

- **Def.** Now given a function $f$ that is (i) bounded and (ii) supported on a set $E$ of finite measure, and given a sequence $\{\phi_n\}$ of simple functions (i) bounded (uniformly) by some $M$ (ii) supported on the set $E$ with (iii) $\phi_n(x) \to f(x)$ for a.e. $x$ then we define the **Lebesgue integral** of $f$ by
  \[ \int f := \lim_{n \to \infty} \int \phi_n. \]

  **Fact.** The limit always exists, and it does not depend on the sequence $\phi_n$. If $f$ is measurable then such a sequence always exists.

- **Def.** If $f$ is a (i) measurable, (ii) non-negative (but we allow infinite values) then its **Lebesgue integral** is given by
  \[ \int f := \sup_g \int g \]
  where the sup is taken over all measurable bounded $g$ that are supported on a set of finite measure. We say such a function is **Lebesgue integrable** if the integral is finite.

- **Def.** Now given a measurable function $f$ such that (i) $|f|$ is integrable then we define its **Lebesgue integral** by
  \[ \int f := \int f_+ - \int f_. \]

  **Fact.** All the definitions above agree.

- **Prop.** The Lebesgue and Riemann integrals agree for Riemann-integrable functions defined on closed intervals.

- **Prop.** The integral of Lebesgue integrable functions is linear, additive, monotonic and satisfies the triangle inequality.

- **Prop.** Suppose $f$ is integrable on $\mathbb{R}^d$. Then for every $\epsilon > 0$
  (i) There exists a ball $B \subseteq \mathbb{R}^d$ such that $\int_{B} |f| < \epsilon$.
  (ii) There exists a $\delta > 0$ such that $\int_E |f| < \epsilon$ whenever $m(E) < \delta$.

- **Lemma.** (Fatou) Suppose $\{f_n\}$ is a sequence of non-negative measurable functions. If $\lim_{n \to \infty} f_n(x) = f(x)$ for a.e. $x$ then
  \[ \int f \leq \liminf_{n \to \infty} \int f_n. \]

- **Cor.** Suppose $f$ is a non-negative measurable function and $\{f_n\}$ is a sequence of non-negative measurable functions with $f_n(x) \leq f(x)$ and $f_n(x) \to f(x)$ for a.e. $x$. Then
  \[ \lim_{n \to \infty} \int f_n = \int f. \]

- **Cor.** (Monotone Convergence Theorem) Suppose $\{f_n\}$ is a sequence of non-negative measurable function with $f_n \uparrow f$. Then
  \[ \lim_{n \to \infty} \int f_n = \int f. \]
• **Thm.-** (Dominated Convergence Theorem) Suppose \( \{f_n\} \) is a sequence of measurable functions such that \( f_n(x) \to f(x) \) a.e. \( x \). If \( |f_n(x)| \leq g(x) \), where \( g(x) \) is integrable then
\[
\int |f_n - f| \to 0 \quad \text{as} \quad n \to \infty,
\]
and thus
\[
\int f_n \to \int f \quad \text{as} \quad n \to \infty.
\]

• **Counterexample.-** Consider the function \( f_n(x) = 1/n \). Then \( f_n \to f \) where \( f = 0 \). This provides a counterexample for Dominated Convergence when the \( f_n \) are not dominated by an \( L^1 \) function, and also a counterexample to monotone convergence when \( f_n \searrow f \) and thus \( -f_n \nearrow f \), i.e. negative functions.

• **Prop.-** (Tchebyshev’s inequality) Let \( f \) be integrable. Then for all \( \alpha > 0 \)
\[
m(\{x : |f(x)| > \alpha\}) \leq \frac{\|f\|_1}{\alpha}.
\]

The space \( L^1 \)

• **Def.-** The space \( L^1(\mathbb{R}^d) \) is the space of equivalence classes of Lebesgue integrable functions, where we regard two functions as equivalent if the are equal almost everywhere. **Remark.-** The integral is still defined as an operator in \( L^1 \) and \( \|f\| = \int |f| \) defines a norm on \( L^1 \), and thus \( d(f,g) = \int |f-g| \) defines a metric on \( L^1 \).

• **Thm.-** (Riesz-Fischer) The vector space \( L^1 \) is complete in its metric. Moreover, any Cauchy sequence \( \{f_n\} \) in \( L^1 \) has a subsequence that converges pointwise almost-everywhere.

• **Thm.-** The following families are dense in \( L^1 \).
  (i) Simple functions.
  (ii) Step functions (characteristic functions of finite union of rectangles).
  (iii) Continuous functions with compact support.

• **Thm.-** Let \( f(x) \) be integrable, \( h \in \mathbb{R} \), \( \delta \in \mathbb{R}_{>0} \). Then \( f(x-h) \), \( f(\delta x) \), \( f(-x) \) are integrable and
  (i) \( \int f(x-h) = \int f(x) \).
  (ii) \( \int f(\delta x) = \delta^d \int f(x) \).
  (iii) \( \int f(-x) = \int f(x) \).

Fubini’s Theorem

• **Def.-** For this section we set \( \mathbb{R}^d = \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \), and thus a point on \( \mathbb{R}^d \) takes the form \( (x,y) \) where \( x \in \mathbb{R}^{d_1} \), \( y \in \mathbb{R}^{d_2} \). If \( f(x,y) \) is a function on \( \mathbb{R}^d \) we define the slice \( f^y(x) := f(x,y) \), which is then a function on \( \mathbb{R}^{d_1} \), and \( f_x(y) \) similarly. Given a set \( E \subseteq \mathbb{R}^d \) denote its slice by \( E^y \) by \( E_y = \{ x \in \mathbb{R}^{d_1} : (x,y) \in E \} \) and \( E_x = \{ y \in \mathbb{R}^{d_2} : (x,y) \in E \} \).

• **Thm.-** (Fubini) Suppose \( f(x,y) \) is integrable on \( \mathbb{R}^d \). Then, for almost every \( y \in \mathbb{R}^{d_2} \),
  (i) The slice \( f^y \) is integrable on \( \mathbb{R}^{d_1} \).
  (ii) The function \( \int_{\mathbb{R}^{d_1}} f^y(x) dx \) is integrable on \( \mathbb{R}^{d_2} \).
  (iii) Moreover,
  \[
  \int_{\mathbb{R}^{d_2}} \left( \int_{\mathbb{R}^{d_1}} f(x,y) dx \right) dy = \int_{\mathbb{R}^d} f(x,y) dx.
  \]
• Thm.- (Tonelli) Suppose \( f(x, y) \) is a non-negative measurable function on \( \mathbb{R}^d \). Then, for almost every \( y \in \mathbb{R}^{d_2} \),
  (i) The slice \( f^y \) is measurable on \( \mathbb{R}^{d_1} \).
  (ii) The function \( \int_{\mathbb{R}^{d_1}} f^y(x)dx \) is measurable on \( \mathbb{R}^{d_2} \).
  (iii) Moreover,
  \[
  \int_{\mathbb{R}^{d_2}} \left( \int_{\mathbb{R}^{d_1}} f(x, y)dx \right) dy = \int_{\mathbb{R}^{d}} f(x, y)d(x, y)
  \]
  (where now this can be an equality \( \infty = \infty \)).
• Cor.- If \( E \) is a measurable set in \( \mathbb{R}^d \) then, for almost all \( y \in \mathbb{R}^{d_2} \), the slice
  \[
  E^y = \{ x \in \mathbb{R}^{d_1} : (x, y) \in E \}
  \]
  is a measurable subset of \( \mathbb{R}^{d_1} \). Moreover, \( m(E^y) \) is a measurable function of \( \mathbb{R}^{d_2} \) and \( m(E) = \int_{\mathbb{R}^{d_2}} m(E^y)dy \).
• Prop.- If \( E = E_1 \times E_2 \) is a measurable subset of \( \mathbb{R}^d \) and \( m_*(E_2) > 0 \) then \( E_1 \) is measurable.
• Lemma.- If \( E_1 \subseteq \mathbb{R}^{d_1}, E_2 \subseteq \mathbb{R}^{d_2} \) are any sets, then
  \[
  m_*(E_1 \times E_2) \leq m_*(E_1)m_*(E_2).
  \]
  (with the understanding that \( 0 \cdot \infty = 0 \)).
• Prop.- If \( E_1, E_2 \) are measurable subsets of \( \mathbb{R}^{d_1}, \mathbb{R}^{d_2} \) resp. then \( E = E_1 \times E_2 \) is a measurable subset of \( \mathbb{R}^d \) and, moreover,
  \[
  m(E) = m(E_1)m(E_2).
  \]
  (where \( 0 \cdot \infty = 0 \)). Cor.- If \( f(x) \) is any function on \( \mathbb{R}^{d_1} \) then \( \tilde{f}(x, y) := f(x) \) is measurable in \( \mathbb{R}^d \). Cor.- Suppose \( f(x) \) is a non-negative function on \( \mathbb{R}^d \) and let
  \[
  A := \{ (x, y) \in \mathbb{R}^d \times \mathbb{R} : 0 \leq y \leq f(x) \}.
  \]
  Then (i) \( A \) is measurable in \( \mathbb{R}^{d+1} \) if and only if \( f \) is measurable on \( \mathbb{R}^d \) and, whenever these hold, (ii) \( m(A) = \int f f(x) \).

Convolutions

• If \( f, g \) are measurable functions on \( \mathbb{R}^d \) then \( f(x-y)g(y) \) is measurable in \( \mathbb{R}^{2d} \).
• If, furthermore, \( f, g \) are integrable on \( \mathbb{R}^d \) then \( f(x-y)g(y) \) is integrable in \( \mathbb{R}^{2d} \).
• The convolution of \( f \) and \( g \) is
  \[
  (f \ast g)(x) := \int_{\mathbb{R}^d} f(x-y)g(y)dy.
  \]
• The function \( (f \ast g)(x) \) is well-defined for almost all \( x \).
• \( f \ast g \) is integrable whenever \( f \) and \( g \) are, and
  \[
  \|f \ast g\|_{L^1} \leq \|f\|_{L^1}\|g\|_{L^1}
  \]
  with equality whenever \( f \) and \( g \) are non-negative.
§3: Differentiation and Integration [SS]

Differentiation of the integral.

• Notation.- Throughout this section, $B$ always denotes balls.

• Remark.- For a continuous function $f$ on $\mathbb{R}^d$ we have
  \[ \lim_{m(B) \to 0} \frac{1}{m(B)} \int_B f(y)dy = f(x). \]

• Def.- Given an integrable function $f$ on $\mathbb{R}^d$ we define its Hardy-Littlewood maximal function $f^*$ by
  \[ f^*(x) = \sup_{B \ni x} \frac{1}{m(B)} \int_B |f(y)|dy. \]

• Thm.- (Maximal theorem) Suppose $f$ is integrable. Then:
  (i) $f^*$ is measurable.
  (ii) $f^*(x) < \infty$ for a.e. $x$.
  (iii) $f^*(x)$ satisfies
  \[ m(\{x \in \mathbb{R}^d : f^*(x) > \alpha\}) \leq \frac{A}{\alpha} \|f\|_{L^1} \]
  for all $\alpha > 0$, where $A = 3^d$. Proof idea.- (i) is easy, (ii) follows from (iii). (iii) is hard and uses a version of Vitali covering argument. Remark.- (iii) is a weak-type inequality, i.e. weaker than inequality on $L^1$-norms, by Tchebyshev’s inequality. Observe we could have defined $f^*(x)$ using balls centered at $x$.

• Thm.- (Lebesgue differentiation theorem) If $f$ is integrable then
  \[ \lim_{m(B) \to 0} \frac{1}{m(B)} \int_B f(y)dy = f(x) \quad \text{for a.e. } x. \]

Cor.- $f^*(x) \geq |f(x)|$ for a.e. $x$.

• Def.- A measurable function $f$ is locally integrable if, for all balls $B$, $f\chi_B$ is integrable. We denote by $L^1_{loc}(\mathbb{R}^d)$ the space of locally integrable functions. Remark.- The Lebesgue differentiation theorem holds for locally integrable functions.

• Def.- If $E$ is a measurable set and $x \in \mathbb{R}^d$ we say $x$ is a point of Lebesgue density of $E$ if
  \[ \lim_{m(B) \to 0} \frac{\frac{m(B \cap E)}{m(B)}}{m(B)} = 1. \]

• Cor.- (Lebesgue’s density theorem) Suppose $E$ is a measurable subset of $\mathbb{R}^d$. Then:
  (i) Almost every $x \in E$ is a point of Lebesgue density of $E$.
  (ii) Almost every $x \notin E$ is not a point of Lebesgue density of $E$ – and, in fact, the limit above is 0 for almost all $x \notin E$.

• Def.- If $f$ is locally integrable on $\mathbb{R}^d$ the Lebesgue set of $f$ consists of all points $x \in \mathbb{R}^d$ for which $f(x) < \infty$ and
  \[ \lim_{m(B) \to 0} \frac{1}{m(B)} \int_B |f(y) - f(x)|dy = 0. \]
Remark.- If \( f \) is continuous at \( x \) then \( x \) is in the Lebesgue set of \( x \). If \( x \) is in the Lebesgue set of \( x \) then
\[
\lim_{m(B) \to 0} \frac{1}{m(B)} \int_B f(y) dy = f(x).
\]

- **Remark.-** The Lebesgue set of \( f \) depends on the choice of representative.
- **Cor.-** If \( f \) is locally integrable on \( \mathbb{R}^d \) then almost every point belongs to the Lebesgue set of \( f \).
- **Def.-** A collection of sets \( \{U_\alpha\} \) is said to shrink regularly to \( x \), or to have bounded eccentricity at \( x \), if there is a constant \( c > 0 \) such that for each \( U_\alpha \) there is a ball \( B \) with \( x \in B \), \( U_\alpha \subseteq B \) and \( m(U_\alpha) \geq cm(B) \). (Perhaps we also need that \( x \) is contained in arbitrarily small \( U_\alpha \)'s? Folland has a more clear discussion).
- **Cor.-** Suppose \( f \) is locally integrable on \( \mathbb{R}^d \). If \( \{U_\alpha\} \) shrinks regularly to \( x \) then
\[
\lim_{m(U_\alpha) \to 0} \frac{1}{m(U_\alpha)} \int_{U_\alpha} f(y) dy = f(x)
\]
for all \( x \) in the Lebesgue set of \( f \) – and thus for almost every \( x \).

### Approximations to the identity

**Def.-** A family \( \{K_\delta\}_{\delta > 0} \) of integrable functions on \( \mathbb{R}^d \) are an approximation to the identity if:

1. \( \int K_\delta(x) dx = 1 \).
2. \( |K_\delta(x)| \leq A\delta^{-d} \).
3. \( |K_\delta(x)| \leq A\delta/|x|^{d+1} \).

for all \( \delta > 0 \) and \( x \in \mathbb{R}^d \), where \( A \) is a constant independent of \( \delta \).

**Thm.-** If \( \{K_\delta\} \) is an approximation to the identity and \( f \) is integrable on \( \mathbb{R}^d \) then
\[
(f * K_\delta)(x) \to f(x) \quad \text{as} \quad \delta \to 0
\]
whenever \( x \) is in the Lebesgue set of \( f \) – and thus for a.e. \( x \).

**Thm.-** With the hypotheses of the previous theorem, we also have
\[
\|(f * K_\delta) - f\|_{L^1} \to 0 \quad \text{as} \quad \delta \to 0.
\]

**Remark.-** Recall \( f * K_\delta \) are integrable.

### Differentiability of functions

- **Def.-** Let \( \gamma \) be a parametrized curve in the plane given by \( z(t) = (x(t), y(t)) \) where \( a \leq t \leq b \) and \( x(t), y(t) \) are continuous real valued functions on \( [a, b] \). Then \( \gamma \) is **rectifiable** if there exists some \( M > 0 \) such that, for any partition \( a = t_0 < t_1 < \cdots < t_N = b \) of \( [a, b] \),
\[
\sum_{j=1}^N |z(t_j) - z(t_{j-1})| \leq M.
\]

The length \( L(\gamma) \) of \( \gamma \) is the supremum over all partitions of the left-hand side – or, equivalently, the infimum of all \( M \) that satisfy the above.
• **Def.-** Similarly, if $F : [a, b] \to \mathbb{C}$ is continuous and $a = t_0 < t_1 < \cdots < t_N = b$ then the variation with respect to this partition is given by

$$\sum_{j=1}^{N} |F(t_j) - F(t_{j-1})|.$$ 

The function $F$ is said to be of **bounded variation** if there exists some uniform bound for all variations.

• **Thm.-** A real-valued function $F$ on $[a, b]$ is of bounded variation if and only if $F$ is the difference of two increasing (not necessarily strictly) bounded functions.

• **Thm.-** If $F$ is of bounded variation on $[a, b]$ then $F$ is differentiable almost everywhere. **Cor.-** If $F$ is increasing and continuous then $F'$ exists almost everywhere. Moreover, $F'$ is measurable, non-negative and

$$\int_{a}^{b} F'(x)dx \leq F(b) - F(a).$$

In particular, if $F$ is bounded then $F'$ is integrable. **Remark.-** There is a continuous function for which the left-hand side is 0 and the right-hand side is 1, called the Cantor function.

• **Def.-** A function $F$ on $[a, b]$ is **absolutely continuous** of, for any $\epsilon > 0$, there exists some $\delta > 0$ such that

$$\sum_{k=1}^{N} |F(b_k) - F(a_k)| < \epsilon \quad \text{whenever} \quad \sum_{k=1}^{N} (b_k - a_k) < \delta$$

where the $(a_k, b_k)$, $k = 1, \ldots, N$, are disjoint intervals. **Remark.-** Absolute continuity implies uniform (and thus plain-old) continuity. It also implies bounded variation. The total variation is then also absolutely continuous and thus $F$ is the difference of two continuous monotonic functions. If $F(x) = \int_{a}^{x} f(y)dy$, where $f$ is integrable, then $F$ is absolutely continuous.

• **Thm.-** If $F$ is absolutely continuous on $[a, b]$ then $F'$ exists almost everywhere and it is integrable. Moreover,

$$F(b) - F(a) = \int_{a}^{b} F'(y)dy.$$ 

Conversely, if $f$ is integrable on $[a, b]$ there exists an absolutely continuous function $F$ such that $F' = f$ almost everywhere and, in fact, we may take $F(x) = \int_{a}^{x} f(y)dy$.

• **Thm.-** If $F$ is a bounded increasing function on $[a, b]$ then $F'$ exists almost everywhere.
§4: Abstract Measure and Integration Theory [SS]

Abstract measure spaces

• Def.- Let $X$ be a non-empty set. A $\sigma$-algebra $\mathcal{M}$ is a non-empty collection of subsets of $X$ that is closed under complements and countable unions. Remark.- A $\sigma$-algebra $\mathcal{M}$ is then closed under countable intersection as well. Moreover, $X, \phi \in \mathcal{M}$.

• Def.- Let $\mathcal{M}$ be a $\sigma$-algebra. A measure on $\mathcal{M}$ is a function $\mu : \mathcal{M} \to [0, \infty]$ such that whenever $E_1, E_2, \ldots$ is a countable disjoint family of sets in $\mathcal{M}$ then

$$\mu(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \mu(E_n).$$

Remark.- Observe then that $\mu(\phi) = 0$ and thus the above formula holds for finite unions too.

• Def.- A measure space is a triple $(X, \mathcal{M}, \mu)$ where $X$ is a set, $\mathcal{M}$ is a $\sigma$-algebra on $X$ and $\mu$ is a measure on $\mathcal{M}$. It is said to be complete if whenever $F \in \mathcal{M}$ is such that $\mu(F) = 0$ and $E \subseteq F$ then $E \in \mathcal{M}$. It is said to be $\sigma$-finite whenever $X$ is a countable union of sets of finite measure.

Exterior measures, Carathéodory’s theorem

• Def.- If $X$ is a non-empty set, an exterior measure or outer measure $\mu_*$ on $X$ is a function from all subsets of $X$ to $[0, \infty]$ that satisfies:

(i) $\mu_*(\phi) = 0$.
(ii) If $E_1 \subseteq E_2$ then $\mu_*(E_1) \leq \mu_*(E_2)$.
(iii) If $E_1, E_2, \ldots$ is a countable family of sets then

$$\mu_*(\bigcup_{j=1}^{\infty} E_j) \leq \sum_{j=1}^{\infty} \mu_*(E_j).$$

A subset $E \subseteq X$ is then called Carathéodory measurable, or measurable if for every $A \subseteq X$ we have

$$\mu_*(A) = \mu_*(E \cap A) + \mu_*(E^c \cap A).$$

• Thm.- Given an exterior measure $\mu_*$ on $X$ the collection $\mathcal{M}$ of measurable subsets is a $\sigma$-algebra, and $\mu_*$ restricted to $\mathcal{M}$ is a measure. Moreover, the resulting measure space is complete.

Metric exterior measures

• Def.- If $(X, d)$ is a metric space the Borel $\sigma$-algebra $\mathcal{B}_X = \mathcal{B}$ on $X$ is the smallest $\sigma$-algebra that contains all open sets of $X$. An exterior measure $\mu_*$ on $X$ is a metric exterior measure if

$$\mu_*(A \cap B) = \mu_*(A) + \mu_*(B) \quad \text{whenever} \quad d(A, B) > 0.$$ 

• Thm.- If $\mu_*$ is a metric exterior measure on $X$ then the Borel sets in $X$ are measurable – thus, $\mu_*$ restricted to $\mathcal{B}_X$ is a measure.

• Def.- Given a metric space $X$, a measure on the Borel sets is called a Borel measure.

• Prop.- Suppose the Borel measure $\mu$ is finite on all balls in $X$ of finite radius. Then for any Borel set $E$ an any $\epsilon > 0$ there is an open set $\mathcal{O}$ and a closed set $F$ with $F \subseteq E \subseteq \mathcal{O}$ such that $\mu(F - E) < \epsilon$, $\mu(\mathcal{O} - E) < \epsilon$. 

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The extension theorem

• **Def.-** If $X$ is a non-empty set, an algebra on $X$ is a non-empty collection $\mathcal{A}$ of subsets closed under complements and finite unions – and thus under finite intersection. A pre-measure is a function $\mu_0 : \mathcal{A} \to [0, \infty]$ with:
  (i) $\mu_0(\emptyset) = 0$,
  (ii) If $E_1, E_2, \ldots$ is a countable disjoint collection of sets in $\mathcal{A}$ with $\bigcup_{n=1}^{\infty} E_n \in \mathcal{A}$ – e.g. finite union – then

\[ \mu_0\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu_0(E_n). \]

• **Thm.-** (Carathéodory’s Extension Theorem) Suppose $\mathcal{A}$ is an algebra on $X$ and $\mu_0$ is a premeasure on $\mathcal{A}$. Let $\mathcal{M}$ be the $\sigma$-algebra generated by $\mathcal{A}$. Then there exists a measure $\mu$ on $\mathcal{M}$ that extends $\mu_0$. This extension is unique whenever $(X, \mu_0)$ is $\sigma$-finite.

Integration on a measure space

• Fix throughout this section a $\sigma$-finite measure space $(X, \mathcal{M}, \mu)$.

• **Def.-** A function $f$ on $X$ (with values on $\mathbb{R} \cup \{\pm \infty\}$) is **measurable** if for all $a \in \mathbb{R}$ $f^{-1}((-\infty, a])$ is measurable. **Remark.-** If $\{f_n\}$ is a sequence of measurable functions then the pointwise sup, inf, limsup and liminf and lim – when it exists – are measurable. If $f, g$ are measurable and of finite value then $f + g$ and $fg$ are measurable.

• **Def.-** A **simple function** on $X$ is a function of the form $\phi(x) = \sum_{k=1}^{N} a_k \chi_{E_k}$ where $a_k \in \mathbb{R}$ and the $E_k$ are measurable.

• **Thm.-** A measurable function $f$ is the pointwise limit of a sequence $\{\phi_k\}$ of simple functions. Moreover, the $\phi_k$ may be taken such that $|\phi_k(x)| \leq |\phi_{k+1}(x)|$ for all $k$. **Remark.-** We use $\sigma$-finiteness here, but the following results don’t (I think?).

• **Thm.-** (Egorov’s) If $\{f_k\}$ is a sequence of measurable functions defined on a measurable set $E$ of finite measure and $f_k(x) \to f(x)$ almost everywhere then for each $\epsilon > 0$ there is a measurable set $A_\epsilon \subseteq E$ with $\mu(E - A_\epsilon) \leq \epsilon$ such that $f_k \to f$ uniformly on $A_\epsilon$.

• **Def.-** Given a simple function $\phi = \sum_{k=1}^{N} a_k \chi_{E_k}$ on $X$, $\int_X \phi d\mu = \sum_{k=1}^{N} a_k \mu(E_k)$. Given a non-negative function $f$ on $X$ we define

\[ \int_X f d\mu := \sup \left\{ \int_X \phi d\mu : 0 \leq \phi \leq f, \phi \text{ simple} \right\}. \]

Finally, given any function $f$, $\int_X f d\mu := \int_X f_+ d\mu - \int_X f_- d\mu$.

• **Def.-** A measurable function $f$ on $X$ is **integrable** if $\int |f| d\mu < \infty$.

• **Lemma.-** (Fatou) If $\{f_n\}$ is a sequence of non-negative measurable functions on $X$ then

\[ \int \liminf_{n \to \infty} f_n d\mu \leq \liminf_{n \to \infty} \int f_n d\mu. \]

• **Thm.-** (Monotone convergence) If $\{f_n\}$ is a sequence of non-negative measurable functions on $X$ with $f_n \nearrow f$ then

\[ \int f d\mu = \lim_{n \to \infty} \int f_n d\mu. \]
• **Thm.-** (Dominated convergence) If \( \{f_n\} \) is a sequence of measurable functions with \( f_n(x) \to f(x) \) a.e. and such that \( |f_n(x)| < g(x) \) for an integrable function \( g \) then

\[
\int |f - f_n| d\mu \to 0 \quad \text{as} \quad n \to \infty
\]

and thus

\[
\int f d\mu = \lim_{n \to \infty} \int f_n d\mu.
\]

**The spaces \( L^1 \) and \( L^2 \)**

• **Def.-** The space \( L^1(X, \mu) \) is the space of integrable functions modulo functions that vanish everywhere. The space \( L^2(X, \mu) \) is the space of square-integrable (usually \( \mathbb{C} \)-valued) functions modulo functions that vanish everywhere.

• **Thm.-** The space \( L^1(X, \mu) \) is a **complete** normed vector space. The space \( L^2(X, \mu) \) is a (possible non-separable) Hilbert space.

**Product measures and a general Fubini theorem**

• In this section, we fix two **complete** and \( \sigma \)-**finite** measure spaces \((X_1, \mathcal{M}_1, \mu_1)\) and \((X_2, \mathcal{M}_2, \mu_2)\).

• **Def.-** A **measurable rectangle**, or rectangle for short, is a subset of \( X_1 \times X_2 \) of the form \( A \times B \) where \( A \subseteq X_1 \) and \( B \subseteq X_2 \) are measurable. **Remark.-** The collection \( \mathcal{A} \) of sets in \( X \) that are finite unions of disjoint rectangles is an algebra of subsets of \( X \).

• **Prop.-** There is a unique pre-measure \( \mu_0 \) on \( \mathcal{A} \) such that \( \mu_0(A \times B) = \mu_1(A)\mu_2(B) \) for all rectangles \( A \times B \).

• **Def.-** Let \( \mathcal{M} \) be the \( \sigma \)-algebra generated by \( \mathcal{A} \). Then \( \mu_0 \) extends to a measure \( \mu_1 \times \mu_2 \) on \( \mathcal{M} \). Given \( E \in \mathcal{M} \), \( x_1 \in X_1 \) and \( x_2 \in X_2 \) the **slices** are defined by \( E_{x_1} := \{x_2 \in X_2 : (x_1, x_2) \in E\} \) and \( E^{x_2} := \{x_1 \in X_1 : (x_1, x_2) \in E\} \).

• **Prop.-** If \( E \) is measurable in \( X_1 \times X_2 \) then \( E^{x_2} \) is \( \mu_1 \)-measurable for a.e. \( x_2 \in X_2 \). The function \( \mu_1(E^{x_2}) \) is \( \mu_2 \)-measurable and

\[
\int_{X_2} \mu_1(E^{x_2}) d\mu_2 = (\mu_1 \times \mu_2)(E).
\]

**Remark.-** Of course, a similar statement holds after replacing \( X_1 \) with \( X_2 \).

• **Thm.-** (Generalized Fubini) In the above setting, suppose \( f(x_1, x_2) \) is **integrable** on \((X_1 \times X_2, \mu_1 \times \mu_2)\).

Then:
(i) For a.e. \( x_2 \in X_2 \) the function \( f(x_1, x_2) \) is \( \mu_1 \)-integrable (in particular, measurable).
(ii) The function \( \int_{X_1} f(x_1, x_2) d\mu_1 \) is \( \mu_2 \)-integrable.
(iii)

\[
\int_{X_2} \left( \int_{X_1} f(x_1, x_2) d\mu_1 \right) d\mu_2 = \int_{X_1 \times X_2} f(x_1, x_2) d(\mu_1 \times \mu_2).
\]

• **Thm.-** (Generalized Tonelli) Again in the above setting, if \( f(x_1, x_2) \) is **non-negative** and measurable on \((X_1 \times X_2, \mu_1 \times \mu_2)\) then:
(i) For a.e. \( x_2 \in X_2 \) the function \( f(x_1, x_2) \) is \( \mu_1 \)-measurable.
(ii) The function \( \int_{X_1} f(x_1, x_2) d\mu_1 \) is \( \mu_2 \)-measurable.
(iii)

\[
\int_{X_2} \left( \int_{X_1} f(x_1, x_2) d\mu_1 \right) d\mu_2 = \int_{X_1 \times X_2} f(x_1, x_2) d(\mu_1 \times \mu_2).
\]
§5: $L^p$ spaces [F]

Basic Theory

- Fix a measure space $(X, \mathcal{M}, \mu)$. On this section we consider complex-valued functions.
- **Def.** If $f$ is a measurable function on $X$ and $0 < p < \infty$ then define its $p$-norm to be
  \[ \|f\|_p := \left( \int_X |f|^p d\mu \right)^{1/p}. \]
  Define the space $L^p(X, \mathcal{M}, \mu)$ to be the set of measurable functions $f$ with $\|f\|_p < \infty$ – modulo almost everywhere equality. **Remark.** $L^p$ is indeed a vector space.

- **Lemma.** (Hölder’s inequality) Suppose $1 < p < \infty$ and $p^{-1} + q^{-1} = 1$ – we say $p$ and $q$ are Hölder conjugates. If $f$ and $g$ are measurable functions on $X$ then
  \[ \|fg\|_1 \leq \|f\|_p \|g\|_q. \]
  In particular, if $f \in L^p$ and $g \in L^q$ then $fg \in L^1$. Moreover, equality holds precisely when $\alpha|f|^p = \beta|g|^q$ for some $\alpha, \beta$ not both zero.

- **Thm.** (Minkowsky’s Inequality) If $1 \leq p < \infty$ and $f, g \in L^p$ then $\|f + g\|_p \leq \|f\|_p + \|g\|_p$. **Cor.** For $1 \leq p < \infty$, $L^p$ is a normed vector space.

- **Thm.** For $1 \leq p < \infty$, $L^p$ is a Banach space – i.e. it is complete.

- **Prop.** For $1 \leq p < \infty$, the set of simple functions with support of finite-measure is dense in $L^p$.

The case $p = \infty$

- **Def.** If $f$ is measurable on $X$ we define its $L^\infty$-norm by
  \[ \|f\|_\infty := \inf \{a \geq 0 : L\mu(\{x : |f(x)| > a\} = 0) \} \]
  (with the convention $\inf = \infty$). We define $L^\infty(X, \mathcal{M}, \mu)$ to be the space of measurable functions $f : X \to \mathbb{C}$ with $\|f\|_\infty < \infty$ – modulo everywhere equivalence. **Remark.** $f$ is in $L^\infty$ if and only if there is a bounded measurable function $g$ with $f = g$ a.e.

- **Thm.**
  (i) If $f$ and $g$ are measurable functions on $X$ then $\|fg\|_1 \leq \|f\|_1 \|g\|_\infty$ – extension of Hölder’s inequality.
  (ii) $\|\cdot\|_\infty$ is a norm on $L^\infty$.
  (iii) $\|f_n\|_\infty \to 0$ if and only $f_n \to f$ uniformly outside a set of measure zero.
  (iv) $L^\infty$ is a Banach space. (v) The simple functions are dense in $L^\infty$.

Relations between $L^p$-spaces

- **Prop.** If $0 < p < q < r \leq \infty$ then $L^q \subseteq L^p + L^r$; that is, each $f \in L^q$ is the sum of a function in $L^p$ and a function in $L^r$.

- **Prop.** If $0 < p < q < r \leq \infty$ then $L^p \cap L^r \subseteq L^q$, with
  \[ \|f\|_q \leq \|f\|^\lambda_\|f\|^{1-\lambda} \]
  where $\lambda \in (0, 1)$ is such that $q^{-1} = \lambda p^{-1} + (1 - \lambda)r^{-1}$. 

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COMPLEX ANALYSIS

§2: Cauchy’s Theorem and Applications [SS]

Goursat’s theorem
- Thm.- (Goursat) If Ω is an open set in \( \mathbb{C} \), and \( T \subseteq \Omega \) is a triangle whose interior is also contained in \( \Omega \) then
  \[
  \int_T f(z)dz = 0
  \]
whenever \( f(z) \) is holomorphic in \( \Omega \). Remark.- In fact, the proof only requires that \( f'(z) \) exists on \( \Omega \) – i.e. no continuity required.
- Cor.- Same for any contour that can be bisected into triangles – e.g. rectangle.

Local existence of primitives and Cauchy’s theorem on a disk
- Thm.- A holomorphic function on an open disk has a primitive on the disk.
- Thm.- (Cauchy, on a disk) If \( f \) is holomorphic in a disk then
  \[
  \int_C f(z)dz = 0
  \]
for any closed curve \( \gamma \) on the disk.

Cauchy’s integral formulas
- Thm.- Suppose \( f \) is holomorphic in an open set containing a disk \( D \) and its boundary \( C \), where \( C \) has positive (i.e. counterclockwise) orientation. Then
  \[
  f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta.
  \]
Proof idea.- Keyhole contour.
- Remark.- The same proof applies to any contour that admits a “keyhole”-ification. Observe the integral is zero for any \( z \) outside of the contour.
- Cor.- If \( f(z) \) is holomorphic in \( \Omega \) then it has infinitely many derivatives in \( \Omega \). Moreover, if \( \Omega \) contains a disk \( D \) and its boundary \( C \) then for all \( z \) in the interior of \( D \)
  \[
  f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta.
  \]
• Cor.-(Cauchy inequalities) If \( f \) is holomorphic in a neighbourhood of the closure of a disk \( D \) with boundary \( C \) centered at \( z_0 \) with radius \( R \) then
\[
|f^{(n)}(z_0)| \leq \frac{n!\|f\|_C}{R^n}
\]
where \( \|f\|_C \) denotes the supremum of \( f \) on the circle \( C \).

• Cor.-(Liouville's theorem) If \( f \) is entire and bounded then \( f \) is constant. **Proof idea.** Show \( f' = 0 \).

• Cor.-(Fundamental Theorem of Algebra) Every non-constant polynomial has a zero in \( \mathbb{C} \).

• Thm. Suppose \( f \) is holomorphic in a neighbourhood of the closure of a disk \( D \) centered at \( z_0 \). Then \( f \) admits a power series expansion
\[
f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n
\]
for all \( z \in D \) and the coefficients are given by
\[
a_n = \frac{1}{n!} f^{(n)}(z_0).
\]

• Cor. The zeros of a non-constant holomorphic function \( f(z) \) on a domain are isolated.

• Cor. If \( f \) is holomorphic on a domain \( \Omega \) and its zeros accumulate in \( \Omega \) then \( f = 0 \). If \( f(z), g(z) \) are holomorphic on \( \Omega \) and the points where they agree accumulate in \( \Omega \) then \( f = g \).

Further applications

• Thm.-(Morera) Suppose \( f \) is a continuous function in the open disk \( D \) such that for all triangles \( T \) contained in \( D \)
\[
\int_T f(z)dz = 0.
\]
Then \( f \) is holomorphic. **Proof idea.** The function \( f \) has a holomorphic primitive.

• Cor. If \( \{f_n\}_{n=1}^{\infty} \) is a sequence of holomorphic functions on \( \Omega \) that converge uniformly to a function \( f \) on compacts then \( f \) is holomorphic.

• Thm. Under the hypothesis of the previous corollary, \( \{f'_n\} \) converges to \( f' \) uniformly on compacts.

• Thm. Let \( F(z,s) \) be a continuous function on \( \Omega \times [0,1] \) where \( \Omega \subseteq \mathbb{C} \) is open, and suppose that \( F(z,s_0) \) is holomorphic for every \( s_0 \in [0,1] \). Then
\[
f(z) := \int_0^1 F(z,s)ds
\]
is holomorphic.

• Thm.-(Symmetry principle) Let \( \Omega \) be an open subset of \( \mathbb{C} \) that is symmetric with respect to the real line, let \( \Omega^+ \) be the part of \( \Omega \) lying (strictly) in the upper half plane, \( \Omega^- \) be the part lying (strictly) in the lower half plane and \( I = \Omega \cap \mathbb{R} \). Suppose \( f^+ \) (resp. \( f^- \)) is holomorphic in \( \Omega^+ \) (resp. \( \Omega^- \)) and that it extends continuously to \( I \). Suppose \( f^+ \) and \( f^- \) agree on \( I \). Then the function
\[
f(z) = \begin{cases} f^+(z) & \text{if } z \in \Omega^+ \\ f^+(z) = f^-(z) & \text{if } z \in I \\ f^-(z) & \text{if } z \in \Omega^- \end{cases}
\]
is holomorphic on \( \Omega \).

• Cor.-(Schwarz's reflection principle) Suppose \( f \) is holomorphic in \( \Omega^+ \) and that it extends continuously onto \( I \), on which it is real valued. Then \( f \) can be extended to a holomorphic function \( F \) on \( \Omega \), where \( F(z) = \overline{f(z)} \) for \( z \in \Omega^- \).
• **Thm.** (Runge’s approximation theorem) Any function holomorphic on a neighbourhood of a compact set \( K \) can be approximated uniformly on \( K \) by rational functions whose singularities are in \( K^c \). If \( K^c \) is connected, any function holomorphic in a neighbourhood of \( K \) can be approximated uniformly on \( K \) by polynomials.

§3: Meromorphic Functions and the Logarithm [SS]

### Zeros and poles

- **Def.** A point singularity of a function \( f \) is a point \( z_0 \) such that \( f \) is defined on a deleted neighbourhood of \( z_0 \), but not at \( z_0 \). A point \( z_0 \) is called a zero of \( f \) if \( f(z_0) = 0 \).

- **Thm.** Suppose \( f \) is a holomorphic function on \( \Omega \), and that \( z_0 \in \Omega \) is a zero of \( f \). Then there exists a unique integer \( n \) and a holomorphic function \( g \) on \( \Omega - \{z_0\} \) such that \( f(z) = (z - z_0)^n g(z) \).

- **Def.** In the theorem above, \( n \) is called the multiplicity of \( f \) at \( z_0 \).

- **Def.** We say \( f \) has a pole at \( z_0 \) if it is defined in a deleted neighbourhood of \( z_0 \) and \( 1/f \), defined to be zero at \( z_0 \), is holomorphic on a full neighborhood of \( z_0 \).

- **Thm.** If \( z_0 \) is a pole of \( f \) then there is a unique integer \( n \) and a holomorphic function \( h \) defined on a neighbourhood of \( z_0 \), with \( h(z_0) \neq 0 \), such that \( f(z) = (z - z_0)^{-n} h(z) \) on a neighbourhood of \( z_0 \).

- **Def.** From the above theorem, \( n \) is called the order of the pole \( z_0 \).

- **Thm.** If \( f(z) \) has a pole of order \( n \) at \( z_0 \) then, on a neighbourhood of \( z_0 \),

\[
f(z) = a_{-n}(z - z_0)^{-n} + \cdots + a_{-1}(z - z_0)^{-1} + G(z)
\]

where \( G(z) \) is holomorphic on a neighbourhood of \( z_0 \).

- **Def.** In the above theorem, \( a_{-n}(z - z_0)^{-n} + \cdots + a_{-1}(z - z_0)^{-1} \) is called the principal part of \( f(z) \) at \( z_0 \). The coefficient \( a_{-1} \) is called the residue of \( f \) at \( z_0 \), denoted \( \text{res}_{z_0} f = a_{-1} \).

- **Thm.** If \( f \) has a pole of order \( n \) at \( z_0 \) then

\[
\text{res}_{z_0} f = \frac{1}{(n-1)!} \lim_{z \to z_0} \left( \frac{d}{dz} \right)^{n-1} (z - z_0)^n f(z).
\]

### The residue formula

- **Thm.** Suppose \( f \) is holomorphic in an open set containing a circle \( C \) and its interior, except for a pole \( z_0 \) inside of \( C \). Then

\[
\int_C f(z)dz = 2\pi i \text{res}_{z_0} f.
\]

- **Cor.** (Residue formula) Suppose \( f \) is holomorphic in an open set containing a toy contour \( \gamma \) and its interior, except for poles at \( z_1, \ldots, z_N \) inside \( \gamma \). Then

\[
\int_\gamma f(z)dz = 2\pi i \sum_{k=1}^N \text{res}_{z_k} f.
\]
Singularities and meromorphic functions

- **Thm.-** (Riemann’s theorem on removable singularities) Suppose $f$ is holomorphic on $\Omega$ except at a point $z_0 \in \Omega$. If $f$ is bounded in $\Omega \setminus \{z_0\}$ then $z_0$ is a removable singularity of $f$. **Proof idea.-** By using a keyhole, Cauchy’s formula still works, and this extends holomorphically onto $z_0$. **Cor.-** Suppose $f$ has an isolated singularity at the point $z_0$. Then $z_0$ is a pole of $f$ if and only if $|f(z)| \to 0$ as $z \to z_0$.

- **Thm.-** (Casorati-Weierstrass) Suppose $f$ is holomorphic in the punctured disc $D \setminus \{z_0\}$ and that $f$ has an essential singularity at $z_0$. Then $f(D \setminus \{z_0\})$ is dense on the complex plane. (c.f. Picard’s theorem for a stronger result).

- **Def.-** A function $f$ is **meromorphic** in $\Omega$ if it is holomorphic in $\Omega \setminus \{z_i\}$ and has at most poles at the $\{z_i\}$. **Remark.-** The $\{z_i\}$ must be isolated and, in particular, they form a countable collection.

- **Def.-** Suppose $f$ is holomorphic for all $|z| > R$ where $R > 0$. We say that $f$ has a **pole at infinity** if $F(z) = f(1/z)$ has a pole at $z = 0$. Similarly, $f$ has a **removable singularity** (resp. **essential singularity**) at infinity if $F(z)$ has a removable (resp. essential) singularity at $z = 0$. A meromorphic function on $C$ that is holomorphic at infinity, or has a pole at infinity, is said to be **holomorphic in the extended complex plane**.

- **Thm.-** The meromorphic functions in the extended complex plane are the rational functions. Rational functions are determined up to a constant by the location and multiplicity of the zeros and poles. **Remark.-** We really need the function to be meromorphic on $C$ to start with – consider $\exp(1/z)$.

**Argument principle and applications**

- **Thm.-** (Argument principle) Suppose $f$ is meromorphic in an open set containing a circle $C$ and its interior. If $f$ has no poles and never vanishes on $C$ then

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} \, dz = \# \{ \text{zeros of } f \text{ inside } C \} - \# \{ \text{poles of } f \text{ inside } C \}$$

where the zeros and poles are counted with multiplicity. **Proof idea.-** $f'(z)/f(z)$ has at most simple poles. Analyze the residues.

- **Thm.-** (Rouché’s theorem) Suppose $f$ and $g$ are holomorphic in an open set containing a circle $C$ and its interior. If

$$|f(z)| > |g(z)| \quad \text{for all } z \in C$$

then $f$ and $f + g$ have the same number of zeros inside of $C$.

- **Thm.-** (Open mapping theorem) Non-constant holomorphic functions are open.

- **Thm.-** (Maximum modulus principle) Non-constant holomorphic functions on a domain $\Omega$ cannot attain a maximum in $\Omega$. **Cor.-** If $f$ is holomorphic in a **bounded** domain $\Omega$ and it extends continuously onto $\partial \Omega$ then $f$ attains its maximum in $\partial \Omega$.

- **Thm.-** (Strict maximum principle) Suppose $f(z)$ is a holomorphic function on **any** domain $\Omega$ with $|f(z)| \leq M$ for all $z \in \Omega$. If $|f(z_0)| = M$ for some $z_0 \in \Omega$ then $f(z)$ is constant on $\Omega$. **Remark.-** This version does not require $f(z)$ to extend continuously onto the boundary. This is in [G].

**Homotopies and simply connected domains**

- **Thm.-** If $f$ is holomorphic in $\Omega$ and $\gamma_0 \simeq \gamma_1$ (i.e. $\gamma_0$ and $\gamma_1$ are homotopic paths in $\Omega$) then

$$\int_{\gamma_0} f(z) \, dz = \int_{\gamma_1} f(z) \, dz.$$

- **Thm.-** Any holomorphic function in a simply-connected domain has a primitive. **Cor.-** If $f$ is holomorphic in a simply-connected domain $\Omega$ then $\int_\gamma f(z) \, dz = 0$ for a closed loop $\gamma$.
The complex logarithm

- **Remark.** Let Ω be a simply connected domain with 1 ∈ Ω and 0 /∈ Ω. Then the function 1/z has a primitive \( \log_Ω(z) \) in Ω – called a branch of the logarithm – satisfying:
  (i) \( e^{\log_Ω(z)} = z \) for all \( z \in Ω \).
  (ii) \( F(r) = \log r \) whenever \( r \) is a real number near 1.
We can do just fine without 1 ∈ Ω as long as we pick our constant carefully.

- **Remark.** This allows us to define power functions \( z^α \) where \( α \in \mathbb{C} \) for simply connected domains that don’t contain 0.

- **Thm.** Let \( f(z) \) be a nowhere vanishing function holomorphic in a simply connected domain Ω. Then there exists a function \( g(z) \) on Ω such that
  \[ f(z) = e^{g(z)}. \]

  **Proof idea.** Take \( g(z) = \int_γ \frac{f'(z)}{f(z)}dz + c_0 \).

Hurwitz’s Theorem [G]

- **Def.** A sequence \( \{f_k(z)\} \) of holomorphic functions on a domain Ω is said to converge normally to \( f(z) \) if \( \{f_k(z)\} \) converges uniformly on each closed disk contained in Ω – or, equivalently, on every compact set contained in Ω. Also equivalently, if around every point in Ω there is a neighbourhood on which the convergence is uniform.

- **Thm.** (Hurwitz) Suppose \( \{f_k(z)\} \) is a sequence of analytic functions that converges normally to \( f(z) \) on a domain Ω, and that \( f(z) \) has a zero of order \( n \) at \( z_0 \in Ω \). Then there exists a \( ρ > 0 \) such that for \( k \gg 0 \) the function \( f_k(z) \) has exactly \( n \) zeros inside the disk \( \{|z − z_0| < ρ\} \), counting multiplicities.

- **Def.** A holomorphic function on Ω is univalent if it is one-to-one – i.e. if it is conformal onto some other domain.

- **Cor.** If a sequence \( \{f_k(z)\} \) of univalent functions converges normally to \( f(z) \) then \( f(z) \) is either univalent or constant.

§4: The Schwarz Lemma [G]

The Schwarz Lemma

- **Thm.** (Schwarz Lemma) Let \( f(z) \) be analytic for \(|z| < 1\) and suppose that \(|f(z)| ≤ 1\) for all \(|z| < 1\) and \( f(0) = 0 \). Then \(|f(z)| ≤ |z|\) for \(|z| < 1\). Furthermore, if equality holds at some point \( z_0 \neq 0 \) then \( f(z) = λz \) for some \(|λ| = 1\). **Proof idea.** Write \( f(z) = zg(z) \) and apply maximum principle to \( g(z) \) for \(|z| < r\) where \( 0 < r < 1 \).

  - **Cor.** If \( f(z) \) is analytic for \(|z − z_0| < r\) and \(|f(z)| ≤ M\) for \(|z − z_0| < r\) then \(|f(z)| ≤ M/r|z − z_0|\), where equality holds if and only if \( f(z) \) is a multiple of \( z − z_0 \).

  - **Cor.** Let \( f(z) \) be analytic for \(|z| < 1\). If \(|f(z)| ≤ 1\) for \(|z| < 1\) and \( f(0) = 0 \) then \(|f'(0)| ≤ 1\) with equality if and only if \( f(z) = λz \) for some \(|λ| = 1\).
Conformal Self-Maps of the Unit Disk

- **Lemma.** If \( g(z) \) is a conformal self-map of the (open) unit disk \( \mathbb{D} \) with \( g(0) = 0 \) then \( g(z) = e^{i\phi}z \).

- **Thm.** The conformal self-maps of the unit disk \( \mathbb{D} \) are of the form

\[
f(z) = e^{i\phi} \frac{z - a}{1 - \bar{a}z}
\]

where \( 0 \leq \phi < 2\pi \) and \( a \in \mathbb{D} \). Moreover, \( \phi \) and \( a \) give a one-to-one correspondence between conformal self-maps of \( \mathbb{D} \) and \( \mathbb{D} \times \partial \mathbb{D} \) – where \( a = f^{-1}(0) \), \( \phi = \arg f'(0) \).

- **Thm.** (Pick’s lemma) If \( f(z) \) is analytic and \( |f(z)| < 1 \) for \( |z| < 1 \) then

\[
|f'(z)| \leq \frac{1 - |f(z)|^2}{1 - |z|^2} \quad |z| < 1.
\]

**Proof idea.** Use a conformal self-map so that, after composition, we map 0 to 0 and then use Schwarz’s Lemma.

- **Def.** A finite Blaschke product is a rational function of the form

\[
B(z) = e^{i\phi} \left( \frac{z - a_1}{1 - \bar{a}_1z} \right) \cdots \left( \frac{z - a_n}{1 - \bar{a}_n z} \right)
\]

where the \( a_i \in \mathbb{D} \) and \( 0 \leq \phi \leq 2\pi \).

- **Thm.** If \( f(z) \) is continuous for \( |z| \leq 1 \) and analytic for \( |z| < 1 \) and \( |f(z)| = 1 \) for \( |z| = 1 \) then \( f(z) \) is a finite Blaschke product. **Proof idea.** Consider \( B(z) \), the finite Blaschke product that has the same zeros – with same multiplicities – as \( f(z) \). Then \( B(z)/f(z) \) and \( f(z)/B(z) \) extend holomorphically \( \mathbb{D} \to \mathbb{D} \) with modulus 1 on the boundary.

§5: Conformal Mappings [G]

- **Remark.** The Möbius transformation \( z \mapsto (z - i)/(z + i) \) maps the upper half-plane \( \mathbb{H} \) conformally onto the unit disk \( \mathbb{D} \).

- A sector can be mapped onto \( \mathbb{H} \) by the help of a power function, and from there to the unit disk if necessary.

- A strip can be rotated to be a horizontal strip. Then \( e^z \) maps horizontal strips to sectors.

- A lunar domain is a domain whose boundary consists of two circles (or line) segments. If \( z_0, z_1 \) are the points of intersection, map \( z_0 \) to 0 and \( z_1 \to \infty \) using a Möbius transformation. We then get a sector.
§6: Compact families of meromorphic functions [G]

Arzelà-Ascoli Theorem

• Def.- Let $E \subseteq \mathbb{C}$ be a subset and $\mathcal{F}$ be a family of functions on $E$. We say $\mathcal{F}$ is equicontinuous at $z_0 \in E$ if for all $\epsilon > 0$ there exists some $\delta > 0$ such that whenever $|z - z_0| < \delta$ then for all $f \in \mathcal{F}$ $|f(z) - f(z_0)| < \epsilon$. We say $\mathcal{F}$ is uniformly bounded on $E$ if there is some $M > 0$ such that $|f(z)| \leq M$ for all $z \in E$ and $f \in \mathcal{F}$.

• Thm.- (Arzelà-Ascoli - $\mathbb{C}$-version) Let $\Omega \subseteq \mathbb{C}$ be a domain and $\mathcal{F}$ be a family of continuous functions on $\Omega$ that is uniformly bounded on compacts. Then the following are equivalent:
  (i) $\mathcal{F}$ is equicontinuous on $\Omega$.
  (ii) $\mathcal{F}$ is normally sequentially compact, i.e. every sequence in $\mathcal{F}$ has a subsequence that converges normally.

• Thm.- (Arzelà-Ascoli - $\hat{\mathbb{C}}$-version) Let $\Omega \subseteq \mathbb{C}$ be a domain and $\mathcal{F}$ be a family of continuous functions from $D$ to $\hat{\mathbb{C}}$. Then the following are equivalent:
  (i) $\mathcal{F}$ is equicontinuous on $\Omega$.
  (ii) $\mathcal{F}$ is normally sequentially compact.

• Remark.- In the last theorem we use the spherical metric on $\hat{\mathbb{C}}$. Observe that no boundedness assumptions are needed on the $\hat{\mathbb{C}}$-version – I suspect because there is a general version where we only need the target to be compact.

Compactness of families of functions

• Lemma.- If $\mathcal{F}$ is a family of analytic functions on a domain $\Omega$ such that $\mathcal{F}'$, the family of derivatives of functions in $\mathcal{F}$, is uniformly bounded then $\mathcal{F}$ is equicontinuous at every point in $\Omega$.

• Thm.- (Montel – weak version) Suppose $\mathcal{F}$ is a family of analytic functions on a domain $\Omega$ that is uniformly bounded on compacts. Then every sequence in $\mathcal{F}$ has a normally convergent subsequence. Proof idea.- Using Cauchy estimates we show $\mathcal{F}'$ is uniformly bounded on compacts, thus $\mathcal{F}$ is equicontinuous. Then use Arzelà-Ascoli to obtain a subsequence that converges – but this sequence may depend on the compact, so we need to use a diagonalization argument.

• Sample application: Fix a domain $\Omega$ and a point $z_0 \in \Omega$. We consider the family $\mathcal{F}$ of analytic functions $f$ on $\Omega$ with $|f(z)| \leq 1$ for all $z \in \Omega$. Then the supremum sup$\{|f'(z_0)| : f \in \mathcal{F}\}$ is attained. (c.f. Ahlfors function).

Marty’s Theorem

• We extend the notion of normal convergence to meromorphic functions by using the spherical metric on $\hat{\mathbb{C}}$.

• Thm.- If a sequence $\{f_n(z)\}$ of meromorphic functions converges normally to $f(z)$ on a domain $\Omega$ then $f(z)$ is either meromorphic or $f(z) \equiv \infty$. If the initial $\{f_n(z)\}$ were analytic then either $f(z)$ is analytic or $f(z) \equiv \infty$.

• Def.- A family $\mathcal{F}$ of meromorphic functions on $\Omega$ is said to be a normal family if every sequence in $\mathcal{F}$ has a subsequence that converges normally in $\Omega$.

• Def.- Given a meromorphic function $f$, regarded as a map $\Omega \to \hat{\mathbb{C}}$, its spherical derivative at the point $z$ is
  $$f^\#(z) := \frac{2|f'(z)|}{1 + |f(z)|^2}.$$
• Lemma.- If \( f_k \to f \) normally on \( \Omega \) then \( f_k^{\#} \to f^{\#} \) normally on \( \Omega \).

• Thm.- (Marty) A family \( \mathcal{F} \) of meromorphic functions on \( \Omega \) is normal if and only if the family of spherical derivatives is bounded uniformly on compacts.

**Strong Montel and Picard**

• Thm.- (Zalcman’s Lemma) Suppose \( \mathcal{F} \) is a family of meromorphic functions on a domain \( \Omega \) that is not normal. Then there exist points \( z_n \in \Omega \) with \( z_n \to z \in \Omega \), \( \rho_n > 0 \) with \( \rho_n \to 0 \) and functions \( f_n \in \mathcal{F} \) such that \( g_n(\zeta) := f_n(z_n + \rho_n \zeta) \) converges normally to a meromorphic function \( g(\zeta) \) on \( \mathbb{C} \) with \( g^{\#}(0) = 1 \) and \( g^{\#}(\zeta) \leq 1 \) for \( \zeta \in \mathbb{C} \).

• Def.- A family \( \mathcal{F} \) of meromorphic functions on \( \Omega \) omits a value \( w_0 \in \hat{\mathbb{C}} \) if \( w_0 \not\in f(\Omega) \) for all \( f \in \mathcal{F} \).

• Thm.- (Montel – strong) A family \( \mathcal{F} \) of meromorphic functions on a domain \( \Omega \) that omits three values of \( \hat{\mathbb{C}} \) is normal.

• Def.- Suppose \( f \) is meromorphic on a punctured neighbourhood of \( z_0 \). A value \( w_0 \in \hat{\mathbb{C}} \) is an omitted value at \( z_0 \) if there exists some \( \delta > 0 \) such that \( f(z) \neq w_0 \) for all \( 0 < |z - z_0| < \delta \). Thus \( w_0 \) is not an omitted value if there is a sequence \( z_n \to z_0 \) with \( f(z_n) = w_0 \).

• Thm.- (Picard’s big theorem) Suppose \( f(z) \) is meromorphic on a punctured neighborhood of \( z_0 \). If \( f(z) \) omits three values at \( z_0 \) then \( f(z) \) extends to be meromorphic at \( z_0 \) – i.e. \( z_0 \) is a pole or removable.

• Thm.- (Picard’s little theorem) A nonconstant entire function assumes every value in the complex plane with at most one exception.

**References**


[G]: T. Gamelin, Complex Analysis.

[F]: G.B. Folland, Real Analysis.