Short-course on symmetry and crystallography

Part 1:
Point symmetry

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Ann Arbor, June 2011
Euclidean move

**Definition 1:** An Euclidean move $\mathcal{T} = \{A, b\}$ is a linear transformation that leaves space invariant:

$$x \mapsto \mathcal{T}(x) = Ax + b$$

Here $x$ is a vector, $A$ an $3 \times 3$ orthogonal Matrix and $b$ a $3$-vector.

**Question:** Euclidean moves form a ?-dimensional space.
**Product of Euclidean moves**

**Definition 2:** The product of two transformations $T_1 = \{A_1, b_1\}$ and $T_2 = \{A_2, b_2\}$ is: $T_2 \circ T_1 = \{A_2A_1, A_2b_1 + b_2\}$
(Note: $T_1$ is applied first!)

**Definition 3:** The order of a transformation $T$ is the smallest integer $n$ such that $T^n(x) = T \circ T \circ T \circ \cdots \circ T(x) = x$
One can also say this transformation is $n$-fold.

**Observations:**

1. The inverse is: $T^{-1} = \{A^{-1}, -A^{-1}b\}$
   (Check: $T \circ T^{-1} = T^{-1} \circ T = 1$)

2. Every transformation of finite order $n$ (i.e. $T^n = 1$) leaves at least one point invariant.
Group

Definition

A group is a set, $G$, together with an operation $\cdot$ (called the group law of $G$) that combines any two elements $a$ and $b$ to form another element, denoted $a \cdot b$ or $ab$. To qualify as a group, the set and operation, $(G, \cdot)$, must satisfy four requirements known as the group axioms:^[4]

Closure

For all $a, b$ in $G$, the result of the operation, $a \cdot b$, is also in $G$.^[b][1]

Associativity

For all $a, b$ and $c$ in $G$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

Identity element

There exists an element $e$ in $G$, such that for every element $a$ in $G$, the equation $e \cdot a = a \cdot e = a$ holds. The identity element of a group $G$ is often written as 1 or $1_G$,^[5] a notation inherited from the multiplicative identity.

Inverse element

For each $a$ in $G$, there exists an element $b$ in $G$ such that $a \cdot b = b \cdot a = 1_G$.

Formal definition of symmetry group

**Definition 4:**
- A symmetry of an object in space (cluster, tiling, lattice, ...) is an Euclidean move that leaves the object indistinguishable.
- A symmetry group is a group of symmetries.

**Definition 5:** The order of a group is equal to the number of elements:

\[ |G| = |\{ g \in G \}| \]
Types of symmetries

\[ x \mapsto \mathcal{T}(x) = Ax + b \]

**Normal form:**

\[
A = \begin{pmatrix}
\pm 1 & 0 & 0 \\
0 & \cos(\alpha) & -\sin(\alpha) \\
0 & \sin(\alpha) & \cos(\alpha)
\end{pmatrix}
\]

**Classification:**

1) \( b = 0 \ or \ b \neq 0? \)
2) Angle \( \alpha \).
3) Eigenvalues of \( A \).

**Basic types:**

Identity = 1,
(i) Reflection, (ii) Rotation, (iii) Translation

**Composite types:**

(iv) Glide reflection, (v) Rotoreflection (Inversion),
(iv) Helical symmetry
Reflexion/mirror symmetry ($S^2 = 1$)

Ambigramm (segerman.org)

Kyoto, June 2008
(n-fold) Rotational symmetry ($S^n = 1$)

Ambigramm (segerman.org), $n = 2$

Flag, $n = 3$

Mandala, $n = 6$
Translational symmetry ($n > 1: S^n \neq 1$)

SEM image of the wing of a Papilio butterfly

Giant's causeway, Northern Ireland
Composite Symmetries

Rotation + Reflection = Rotoreflection (Inversion)

Translation + Reflection = Glide reflection

Translation + Rotation = Helical symmetry
Here:

• The group $G$ is a set of Euclidean moves.
• The set $X$ is the three-dimensional space.
• An Euclidean move acts on 3D space as an affine transformation.
Orbits and stabilizers

Consider a group $G$ acting on a set $X$. The orbit of a point $x$ in $X$ is the set of elements of $X$ to which $x$ can be moved by the elements of $G$. The orbit of $x$ is denoted by $Gx$:

$$Gx = \{g \cdot x \mid g \in G\}.$$ 

The defining properties of a group guarantee that the set of orbits of (points $x$ in) $X$ under the action of $G$ form a partition of $X$. The associated equivalence relation is defined by saying $x \sim y$ if and only if there exists a $g$ in $G$ with $g \cdot x = y$. The orbits are then the equivalence classes under this relation; two elements $x$ and $y$ are equivalent if and only if their orbits are the same, i.e. $Gx = Gy$.

For every $x$ in $X$, we define the stabilizer subgroup of $x$ (also called the isotropy group or little group) as the set of all elements in $G$ that fix $x$:

$$G_x = \{g \in G \mid g \cdot x = x\}.$$ 

- The orbit consists of all points that are equivalent under symmetry.
- The stabilizer consists of all symmetries that leave a point invariant.
**Point symmetries**

**Definition 6:**
A point symmetry is a symmetry which leaves a point $x_0$ invariant: $\mathcal{T}(x_0) = x_0$

**Observations:**
- Translations cannot be point symmetries.
- Symmetries with finite order are point symmetries.
- Symmetries with infinite order cannot be point symmetries.
  (Note: Some sources consider spherical and cylindrical symmetry point symmetries.)
Point group

**Definition 7:** A point group is a group of point symmetries, which leave a common point $x_0$ invariant.

**Observation:**

1. A point group is a finite subgroup of $O(3)$, the space of three dimensional orthogonal matrices.  
   Note: $O(3) = \{ A \in \mathbb{R}^{3\times3} : A^T A = 1 \}$  
   $SO(3) = \{ A \in \mathbb{R}^{3\times3} : A^T A = 1, \det(A) = 1 \}$

2. If two symmetries have no common invariant point, then they generate a group of infinite order. (Exercise)

**Classification strategy:** Determine finite subgroups of $SO(3)$. Then extend them into $O(3)$. 
Comparing groups

**Definition 8:** Two subgroups $H_1$ and $H_2$ of a group $G$ are *conjugated*, if there exists a $g \in G$, such that:

$$H_2 = g^{-1} H_1 g$$

(Exercise: Show that conjugated subgroups are isomorphic.)

**Example:** $G = \text{O}(3)$. Two point groups are conjugated, if there is a change of basis that maps them into each other.
Classification of 2D point groups (up to conjugacy)

Normal form of an orthogonal Matrix in O(2):

\[ A = \pm \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix} \]

Cyclic groups: \( C_1, C_2, C_3, \ldots \) where \( C_n \) consists of all rotations about a fixed point by multiples of \( 360/n \).

Dihedral groups: \( D_1, D_2, D_3, D_4, \ldots \) where \( D_n \) (of order \( 2n \)) consists of the rotations in \( C_n \) together with reflections in \( n \) axes that pass through the fixed point.
Proper point groups in 3D (subgroups of SO(3))

- Cyclic groups: \( C_n \) with order \( n \)
- Dihedral groups: \( D_n \) with order \( 2n \)
- Tetrahedral group \( T \) with order 12. Octahedral group \( O \) with order 24. Icosahedral group \( I \) with order 60.

<table>
<thead>
<tr>
<th>Polyhedron</th>
<th>Vertices</th>
<th>Edges</th>
<th>Faces</th>
<th>Schl&quot;afli symbol</th>
<th>Vertex configuration</th>
<th>Historically corresponding element</th>
</tr>
</thead>
<tbody>
<tr>
<td>tetrahedron</td>
<td>4</td>
<td>6</td>
<td>4</td>
<td>{3, 3}</td>
<td>3.3.3</td>
<td>Fire</td>
</tr>
<tr>
<td>cube</td>
<td>8</td>
<td>12</td>
<td>6</td>
<td>{4, 3}</td>
<td>4.4.4</td>
<td>Earth</td>
</tr>
<tr>
<td>octahedron</td>
<td>6</td>
<td>12</td>
<td>8</td>
<td>{3, 4}</td>
<td>3.3.3.3</td>
<td>Air</td>
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<tr>
<td>dodecahedron</td>
<td>20</td>
<td>30</td>
<td>12</td>
<td>{5, 3}</td>
<td>5.5.5</td>
<td>Ether, Universe</td>
</tr>
<tr>
<td>icosahedron</td>
<td>12</td>
<td>30</td>
<td>20</td>
<td>{3, 5}</td>
<td>3.3.3.3.3</td>
<td>Water</td>
</tr>
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</table>
### Role of dimension

Platonic solids in 4D:

<table>
<thead>
<tr>
<th>Names</th>
<th>Family</th>
<th>Schläfli symbol</th>
<th>Vertices</th>
<th>Edges</th>
<th>Faces</th>
<th>Cells</th>
<th>Vertex figures</th>
<th>Dual polytope</th>
<th>Symmetry group</th>
</tr>
</thead>
<tbody>
<tr>
<td>5-cell pentatope</td>
<td>simplex (n-simplex)</td>
<td>{3,3,3}</td>
<td>5</td>
<td>10</td>
<td>10 triangles</td>
<td>5 tetrahedra</td>
<td>tetrahedra</td>
<td>(self-dual)</td>
<td>A₄</td>
</tr>
<tr>
<td>hyperpyramid hypertetrahedron</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8-cell Tesseract</td>
<td>hypercube (n-cube)</td>
<td>{4,3,3}</td>
<td>16</td>
<td>32</td>
<td>24 squares</td>
<td>8 cubes</td>
<td>tetrahedra</td>
<td>16-cell</td>
<td>B₄</td>
</tr>
<tr>
<td>hypercube 4-cube</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>16-cell orthoplex</td>
<td>cross-polytope (n-orthoplex)</td>
<td>{3,3,4}</td>
<td>8</td>
<td>24</td>
<td>32 triangles</td>
<td>16 tetrahedra</td>
<td>octahedra</td>
<td>tesseract</td>
<td>B₄</td>
</tr>
<tr>
<td>hyperoctahedron 4-orthoplex</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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</tr>
<tr>
<td>24-cell octoplex</td>
<td>{3,4,3}</td>
<td>24</td>
<td>96</td>
<td>96</td>
<td>24 triangles</td>
<td>24 octahedra</td>
<td>cubes</td>
<td>(self-dual)</td>
<td>F₄</td>
</tr>
<tr>
<td>polyoctahedron</td>
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<td></td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>120-cell dodecaplex</td>
<td>{5,3,3}</td>
<td>600</td>
<td>1200</td>
<td>720</td>
<td>720 pentagons</td>
<td>120 dodecahedra</td>
<td>tetrahedra</td>
<td>600-cell</td>
<td>H₄</td>
</tr>
<tr>
<td>hyperdodecahedron polydodecahedron</td>
<td></td>
<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>600-cell tetroplex</td>
<td>{3,3,5}</td>
<td>120</td>
<td>720</td>
<td>1200</td>
<td>1200 triangles</td>
<td>600 tetrahedra</td>
<td>icosahedra</td>
<td>120-cell</td>
<td>H₄</td>
</tr>
<tr>
<td>hypericosahedron polytetrahedron</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Higher dimensions: Only simplex, hypercube, cross-polytope.
<table>
<thead>
<tr>
<th>Description of operation</th>
<th>Schoenflies symbol</th>
<th>Hermann–Mauguin symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>Counterclockwise rotation of $360^\circ/n$ about axis</td>
<td>$C_1, C_2, C_3, C_4$</td>
<td>$1, 2, 3, 4$</td>
</tr>
<tr>
<td>Reflection through a plane</td>
<td>$\sigma$</td>
<td>$m$</td>
</tr>
<tr>
<td>Rotation of $360^\circ$ about any axis.</td>
<td>$E = C_1$</td>
<td>$1$</td>
</tr>
<tr>
<td>All objects and geometric figures possess this element</td>
<td></td>
<td></td>
</tr>
<tr>
<td>All points inverted through a center of symmetry</td>
<td>$i$</td>
<td>$I$</td>
</tr>
<tr>
<td>Rotation of $360^\circ/n$ followed by reflection in a plane perpendicular to the axis</td>
<td>$S_1, S_2, S_3, S_4$</td>
<td></td>
</tr>
<tr>
<td>Rotation of $360^\circ/n$ followed by inversion through a point on the axis</td>
<td></td>
<td>$1, 2, 3, 4$</td>
</tr>
</tbody>
</table>
Classification of 3D point groups – Part I

The 7 infinite series are:

- $C_n$ (nn) of order $n$ - **n-fold rotational symmetry** (abstract group $\mathbb{Z}_n$); for $n = 1$: no symmetry (trivial group)
- $C_{nh}$ ($n^*$) of order $2n$ (for odd $n$ abstract group $\mathbb{Z}_{2n} = \mathbb{Z}_n \times \mathbb{Z}_2$, for even $n$ abstract group $\mathbb{Z}_n \times \mathbb{Z}_2$)
- $C_{nv}$ ("nn") of order $2n$ - **pyramidal symmetry** (abstract group $\text{Dih}_n$); in biology $C_{2v}$ is called **biradial symmetry**.
- $D_n$ (22n) of order 2n - **dihedral symmetry** (abstract group $\text{Dih}_n$)
- $S_{2n}$ (nx) of order 2n (not to be confused with **symmetric groups**, for which the same notation is used; abstract group $\mathbb{Z}_{2n}$)
- $D_{nh}$ ("22n") of order 4n - **prismatic symmetry** (for odd $n$ abstract group $\text{Dih}_{2n} = \text{Dih}_n \times \mathbb{Z}_2$; for even $n$ abstract group $\text{Dih}_n \times \mathbb{Z}_2$)
- $D_{nd}$ (or $D_{nv}$) (2*nt) - **antiprismatic symmetry** of order 4n (abstract group $\text{Dih}_{2n}$)

**Involutional** symmetry (abstract group $\mathbb{Z}_2$):

- $C_i$ - **inversion symmetry**
- $C_2$ - 2-fold rotational symmetry
- $C_s$ - reflection symmetry, also called **bilateral symmetry**.

http://en.wikipedia.org/wiki/Point_groups_in_three_dimensions
Exercise 1

FIG. 2-15  A molecule with symmetry.

FIG. 2-16  CH₃CCl₃ molecule. (a) symmetry (continued on p. 32).
TABLE 2-2 GROUP $C_{2v}$ MULTIPLICATION TABLE

<table>
<thead>
<tr>
<th></th>
<th>$E$</th>
<th>$C_2$</th>
<th>$\sigma_v$</th>
<th>$\sigma_v'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E$</td>
<td>$E$</td>
<td>$C_2$</td>
<td>$\sigma_v$</td>
<td>$\sigma_v'$</td>
</tr>
<tr>
<td>$C_2$</td>
<td>$C_2$</td>
<td>$E$</td>
<td>$\sigma_v'$</td>
<td>$\sigma_v$</td>
</tr>
<tr>
<td>$\sigma_v$</td>
<td>$\sigma_v$</td>
<td>$\sigma_v$</td>
<td>$E$</td>
<td>$C_2$</td>
</tr>
<tr>
<td>$\sigma_v'$</td>
<td>$\sigma_v'$</td>
<td>$\sigma_v$</td>
<td>$C_2$</td>
<td>$E$</td>
</tr>
</tbody>
</table>

TABLE 2-3 GROUP $C_{3v}$ MULTIPLICATION TABLE

<table>
<thead>
<tr>
<th></th>
<th>$E$</th>
<th>$C_3$</th>
<th>$C_3^2$</th>
<th>$\sigma_v$</th>
<th>$\sigma_v'$</th>
<th>$\sigma''$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E$</td>
<td>$E$</td>
<td>$C_3$</td>
<td>$C_3^2$</td>
<td>$\sigma_v$</td>
<td>$\sigma_v'$</td>
<td>$\sigma''$</td>
</tr>
<tr>
<td>$C_3$</td>
<td>$C_3$</td>
<td>$C_3^2$</td>
<td>$E$</td>
<td>$\sigma_v'$</td>
<td>$\sigma_v$</td>
<td>$\sigma_v$</td>
</tr>
<tr>
<td>$C_3^2$</td>
<td>$C_3^2$</td>
<td>$E$</td>
<td>$C_3$</td>
<td>$\sigma_v$</td>
<td>$\sigma_v$</td>
<td>$\sigma_v'$</td>
</tr>
<tr>
<td>$\sigma_v$</td>
<td>$\sigma_v$</td>
<td>$\sigma_v$</td>
<td>$\sigma''$</td>
<td>$E$</td>
<td>$C_3^2$</td>
<td>$C_3$</td>
</tr>
<tr>
<td>$\sigma_v'$</td>
<td>$\sigma_v'$</td>
<td>$\sigma_v'$</td>
<td>$\sigma_v'$</td>
<td>$C_3$</td>
<td>$E$</td>
<td>$C_3^2$</td>
</tr>
<tr>
<td>$\sigma''$</td>
<td>$\sigma''$</td>
<td>$\sigma''$</td>
<td>$\sigma''$</td>
<td>$C_3^2$</td>
<td>$C_3$</td>
<td>$E$</td>
</tr>
</tbody>
</table>

FIG. 2-18 $CH_2Cl_2$, $C_{2v}$ symmetry.

FIG. 2-16 $CH_3CCl_3$ molecule. (c) $C_{3v}$ symmetry.
FIG. 2-17  \( \text{CHCl}_2\text{CHCl}_2 \), \( C_{2h} \) symmetry.

### TABLE 2-4 GROUP \( C_{2h} \) MULTIPLICATION TABLE

<table>
<thead>
<tr>
<th></th>
<th>( \text{E} )</th>
<th>( C_2 )</th>
<th>( \sigma_h )</th>
<th>( i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{E} )</td>
<td>( \text{E} )</td>
<td>( C_2 )</td>
<td>( \sigma_h )</td>
<td>( i )</td>
</tr>
<tr>
<td>( C_2 )</td>
<td>( C_2 )</td>
<td>( \text{E} )</td>
<td>( i )</td>
<td>( \sigma_h )</td>
</tr>
<tr>
<td>( \sigma_h )</td>
<td>( \sigma_h )</td>
<td>( i )</td>
<td>( E )</td>
<td>( C_2 )</td>
</tr>
<tr>
<td>( i )</td>
<td>( i )</td>
<td>( \sigma_h )</td>
<td>( C_2 )</td>
<td>( \text{E} )</td>
</tr>
</tbody>
</table>
Exercise 2

FIG. 2-19  Linear molecules, group $C_{\infty v}$. (Compare Fig. 2-23.)  FIG. 2-23  Linear molecules, group $D_{\infty h}$. (Compare Fig. 2-19.)
Exercise 3

FIG. 2-21  *Three point groups for \( \text{C}_2\text{H}_6 \).* (a) (continued on p. 38).
Exercise 4

FIG. 2-25 \( N_4S_4 \), a molecule with \[ \text{symmetry}. \]

Point symmetry?
The 7 remaining point groups:

- **T (332)** of order 12 - **chiral tetrahedral symmetry**. Rotation group for a regular tetrahedron.
- **T_d (*332)** of order 24 – **full tetrahedral symmetry**. Full symmetry group of a regular tetrahedron.
- **T_h (3*2)** of order 24 – **pyritohedral symmetry**. Symmetry of a volleyball.
- **O (432)** of order 24 – **chiral octahedral symmetry**. Rotation group for a regular octahedron/cube.
- **O_h (*432)** of order 48 - **full octahedral symmetry**. Full symmetry group of a regular octahedron/cube.
- **I (532)** of order 60 – **chiral icosahedral symmetry**. Rotation group for a regular dodecahedron/icosahedron.
- **I_h (*532)** of order 120 - **full icosahedral symmetry**. Full symmetry group of a regular dodecahedron/icosahedron.
FIG. 2-27 \( \text{CH}_4 \), symmetry \( T_d \).

FIG. 2-26 A regular tetrahedron inscribed in a cube.

FIG. 2-28 A regular octahedron inscribed in a cube. The \( C_3 \) and \( C_4 \) rotation axes are shown. Not shown are six \( C_2 \) axes parallel to the face diagonals of the cube.

FIG 2-29 \( \text{SF}_6 \), symmetry \( O_h \).
## Archimedean solids – Part 1

<table>
<thead>
<tr>
<th>Name (Vertex configuration)</th>
<th>Transparent</th>
<th>Solid</th>
<th>Net</th>
<th>Faces</th>
<th>Edges</th>
<th>Vertices</th>
<th>Symmetry group</th>
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<tbody>
<tr>
<td>truncated tetrahedron (3.6.6)</td>
<td><img src="image1" alt="Animation" /></td>
<td><img src="image2" alt="Image" /></td>
<td><img src="image3" alt="Image" /></td>
<td>8</td>
<td>4 triangles 4 hexagons</td>
<td>18</td>
<td>12</td>
</tr>
<tr>
<td>cuboctahedron (3.4.3.4)</td>
<td><img src="image4" alt="Animation" /></td>
<td><img src="image5" alt="Image" /></td>
<td><img src="image6" alt="Image" /></td>
<td>14</td>
<td>8 triangles 6 squares</td>
<td>24</td>
<td>12</td>
</tr>
<tr>
<td>truncated cube or truncated hexahedron (3.8.8)</td>
<td><img src="image7" alt="Animation" /></td>
<td><img src="image8" alt="Image" /></td>
<td><img src="image9" alt="Image" /></td>
<td>14</td>
<td>8 triangles 6 octagons</td>
<td>36</td>
<td>24</td>
</tr>
<tr>
<td>truncated octahedron (4.6.6)</td>
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<td><img src="image11" alt="Image" /></td>
<td><img src="image12" alt="Image" /></td>
<td>14</td>
<td>6 squares 8 hexagons</td>
<td>36</td>
<td>24</td>
</tr>
<tr>
<td>rhombicuboctahedron or small rhombicuboctahedron (3.4.4.4)</td>
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<td><img src="image14" alt="Image" /></td>
<td><img src="image15" alt="Image" /></td>
<td>26</td>
<td>8 triangles 18 squares</td>
<td>48</td>
<td>24</td>
</tr>
<tr>
<td>truncated cuboctahedron or great rhombicuboctahedron (4.6.8)</td>
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<td><img src="image17" alt="Image" /></td>
<td><img src="image18" alt="Image" /></td>
<td>26</td>
<td>12 squares 8 hexagons 6 octagons</td>
<td>72</td>
<td>48</td>
</tr>
<tr>
<td>Name (Vertex configuration)</td>
<td>Transparent</td>
<td>Solid</td>
<td>Net</td>
<td>Faces</td>
<td>Edges</td>
<td>Vertices</td>
<td>Symmetry group</td>
</tr>
<tr>
<td>-----------------------------</td>
<td>-------------</td>
<td>-------</td>
<td>-----</td>
<td>-------</td>
<td>-------</td>
<td>----------</td>
<td>----------------</td>
</tr>
<tr>
<td>snub cube or snub hexahedron or snub cuboctahedron (2 chiral forms) (3.3.3.3.4)</td>
<td><img src="image1.png" alt="Image" /></td>
<td><img src="image2.png" alt="Image" /></td>
<td><img src="image3.png" alt="Image" /></td>
<td>38</td>
<td>32 triangles 6 squares</td>
<td>60</td>
<td>24</td>
</tr>
<tr>
<td>icosidodecahedron (3.5.3.5)</td>
<td><img src="image4.png" alt="Image" /></td>
<td><img src="image5.png" alt="Image" /></td>
<td><img src="image6.png" alt="Image" /></td>
<td>32</td>
<td>20 triangles 12 pentagons</td>
<td>60</td>
<td>30</td>
</tr>
<tr>
<td>truncated dodecahedron (3.10.10)</td>
<td><img src="image7.png" alt="Image" /></td>
<td><img src="image8.png" alt="Image" /></td>
<td><img src="image9.png" alt="Image" /></td>
<td>32</td>
<td>20 triangles 12 decagons</td>
<td>90</td>
<td>60</td>
</tr>
<tr>
<td>truncated icosahedron (5.6.6)</td>
<td><img src="image10.png" alt="Image" /></td>
<td><img src="image11.png" alt="Image" /></td>
<td><img src="image12.png" alt="Image" /></td>
<td>32</td>
<td>12 pentagons 20 hexagons</td>
<td>90</td>
<td>60</td>
</tr>
<tr>
<td>rhombicosidodecahedron or small rhombicosidodecahedron (3.4.5.4)</td>
<td><img src="image13.png" alt="Image" /></td>
<td><img src="image14.png" alt="Image" /></td>
<td><img src="image15.png" alt="Image" /></td>
<td>62</td>
<td>20 triangles 30 squares 12 pentagons</td>
<td>120</td>
<td>60</td>
</tr>
<tr>
<td>truncated icosidodecahedron or great rhombicosidodecahedron (4.6.10)</td>
<td><img src="image16.png" alt="Image" /></td>
<td><img src="image17.png" alt="Image" /></td>
<td><img src="image18.png" alt="Image" /></td>
<td>62</td>
<td>30 squares 20 hexagons 12 decagons</td>
<td>180</td>
<td>120</td>
</tr>
<tr>
<td>snub dodecahedron or snub icosidodecahedron (2 chiral forms) (3.3.3.3.5)</td>
<td><img src="image19.png" alt="Image" /></td>
<td><img src="image20.png" alt="Image" /></td>
<td><img src="image21.png" alt="Image" /></td>
<td>92</td>
<td>80 triangles 12 pentagons</td>
<td>150</td>
<td>60</td>
</tr>
</tbody>
</table>
Determination of the point group of an object in space

1. Object linear: $C_{\infty v}$ or $D_{\infty h}$.
2. High symmetry, non-axial: $T$, $T_h$, $T_d$, $O$, $O_h$, $I$, $I_h$.
3. No rotation axis: $C_1$, $C_i$, $C_s$.
4. Determine the symmetry element with highest order and use the following table:

<table>
<thead>
<tr>
<th>Group Order</th>
<th>n</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$C_n$</td>
<td>$C_1$</td>
<td>$C_2$</td>
<td>$C_3$</td>
<td>$C_4$</td>
<td>$C_6$</td>
</tr>
<tr>
<td>$2n$</td>
<td>$C_{nv}$</td>
<td>$C_{1v}=C_{1h}$</td>
<td>$C_{2v}$</td>
<td>$C_{3v}$</td>
<td>$C_{4v}$</td>
<td>$C_{6v}$</td>
</tr>
<tr>
<td>$2n$</td>
<td>$C_{nh}$</td>
<td>$C_{1h}$</td>
<td>$C_{2h}$</td>
<td>$C_{3h}$</td>
<td>$C_{4h}$</td>
<td>$C_{6h}$</td>
</tr>
<tr>
<td>$2n$</td>
<td>$D_n$</td>
<td>$D_{1}=C_2$</td>
<td>$D_2$</td>
<td>$D_3$</td>
<td>$D_4$</td>
<td>$D_6$</td>
</tr>
<tr>
<td>$4n$</td>
<td>$D_{nh}$</td>
<td>$D_{1h}=C_{2v}$</td>
<td>$D_{2h}$</td>
<td>$D_{3h}$</td>
<td>$D_{4h}$</td>
<td>$D_{6h}$</td>
</tr>
<tr>
<td>$4n$</td>
<td>$D_{nd}$</td>
<td>$D_{1d}=C_{2h}$</td>
<td>$D_{2d}$</td>
<td>$D_{3d}$</td>
<td>$D_{4d}$</td>
<td>$D_{6d}$</td>
</tr>
<tr>
<td>$n$</td>
<td>$S_n$</td>
<td>$S_{1}=C_{1h}$</td>
<td>$S_2$</td>
<td>$S_{3}=C_{3h}$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- **n**: orthogonal rotations
- **2n**: horizontal mirror
- **4n**: horizontal mirror, vertical mirror
- **6**: vertical mirror
Example: Carolyn’s packings of small spheres on a big sphere

1. Six trimers of spheres arrange on the vertices of an octahedron into two different orientations.
2. What are the point groups? Ignore the numerical inaccuracy (fluctuations in the orientation).
Exam questions, part 1

• What is a symmetry?
• How does a symmetry act on Euclidean space?
• What types of symmetries are there?
• What is a point symmetry and a point symmetry group?
• What does it mean to classify point groups?
• What points groups are there in 2D and 3D?
• How can you identify the point group of an object?