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Nonnegativity of bivariate quadratic functions on a triangle

Edmond Nadler

Department of Mathematics, Wayne State University, Detroit, MI 48202, USA

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Abstract

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A necessary and sufficient condition for the nonnegativity of a bivariate quadratic defined on a triangle is presented in terms of the Bernstein–Bézier form of the function.

Keywords. Nonnegativity, bivariate quadratic function, Bernstein–Bézier form, positivity preserving interpolation, contouring

1. Introduction

Bivariate piecewise quadratic functions defined on triangulations are often used for surface fitting and design in CAGD for a number of reasons, among them: the structure of quadratics does not allow unwanted oscillations, and the ease with which the surface can be controlled when expressed in Bernstein–Bézier form. Perhaps most important is the fact that quadratics are easier to *contour* than other functions. Contouring remains an important method for displaying the features of a surface, e.g., its smoothness, despite the ever-increasing sophistication of 3D graphics displays. Contouring quadratics over a triangle has been considered in [Marlow & Powell '76], [Farin '86], and most recently, in [Worsey & Farin '90], which presents a simple, yet robust algorithm for doing the contouring.

For contouring, and in other situations, the problem of *positivity preserving interpolation*, i.e., interpolation to nonnegative data by a nonnegative function, is often of interest. This problem could arise if one has data points on one side of a plane, and wishes to have an interpolating surface which is also on that side of the plane. For example, certain panels of an automobile body may be constrained to lie on one side of a plane to prevent them from interfering with other panels. Thus, the problem of ensuring the nonnegativity of a piecewise quadratic function defined on a triangulation is important. The purpose of this paper, then, is to develop a necessary and sufficient condition for the nonnegativity of a bivariate quadratic function on a triangle. The result was first presented in [Nadler '88].

In earlier related work, a necessary and sufficient condition for the *convexity* of bivariate polynomials in Bernstein–Bézier form was obtained in [Chang & Davis '84], with improved conditions subsequently obtained in [Chang & Feng '84] and [Wang & Liu '88]. In the latter, a sufficient condition for the related idea of positivity was also obtained.

Correspondence to: E. Nadler, Department of Mathematics, Wayne State University, Detroit, MI 48202, USA. E-mail: enadler@math.wayne.edu

Let Q be a bivariate quadratic function defined on a triangle T with vertices x_1, x_2, x_3. In the Bernstein–Bézier form, a bivariate polynomial of degree n is expressed as a linear combination of the *Bernstein basis polynomials* in the barycentric coordinates β_1, β_2, β_3 on T, as follows:

$$Q(x) = \sum_{\substack{i+j+k=n \\ i,j,k \geqslant 0}} c_{ijk} \frac{n!}{i!j!k!} \beta_1^i(x) \beta_2^j(x) \beta_3^k(x) \tag{1}$$

where the coefficients c_{ijk} are known as the *Bézier ordinates*, and are associated with the domain points $(i/n, j/n, k/n)$, expressed in barycentric coordinates. (The aforementioned paper [Chang & Davis '84] further discusses the basics of the Bernstein–Bézier form; for a more complete discussion see [Farin '86] or [Farin '90].)

For simplicity in our discussion of quadratics, let a_1, a_2, a_3 stand for the Bézier ordinates c_{200}, c_{020}, c_{002}, respectively, i.e., the ones associated with the vertices of the triangle, and let b_1, b_2, b_3 stand for c_{011}, c_{101}, c_{110}, the ones assocated with the midpoints of the edges e_i opposite x_i.

We will work mostly with the barycentric coordinates, so to be precise, let $\tilde{Q}$ denote the quadratic as a function of these barycentric coordinates $\boldsymbol{\beta} := (\beta_1, \beta_2, \beta_3)$, i.e., $\tilde{Q}(\boldsymbol{\beta}(x)) = Q(x)$. This function is a quadratic form in $\boldsymbol{\beta}$:

$$\tilde{Q}(\boldsymbol{\beta}) = \boldsymbol{\beta}^{\mathrm{T}} A \boldsymbol{\beta} \tag{2}$$

where

$$A := \begin{bmatrix} a_1 & b_3 & b_2 \\ b_3 & a_2 & b_1 \\ b_2 & b_1 & a_3 \end{bmatrix}. \tag{3}$$

Here, $\boldsymbol{\beta} \in \mathscr{S}_+ := \{(\beta_1, \beta_2, \beta_3) \in \mathscr{S}: \beta_i \geqslant 0\}$ where $\mathscr{S} := \{(\beta_1, \beta_2, \beta_3): \Sigma\beta_i = 1\}$. Our condition for the nonnegativity of Q over T will be stated in terms of the six quantities a_i and b_i, $i = 1, 2, 3$.

Let us first consider the analogous problem in *one* variable, for two reasons. First, some of the methods carry over to the bivariate problem. And second, the actual result is used, since the bivariate quadratic restricted to a line containing each edge of the triangle is a univariate quadratic, and it is necessary that this univariate quadratic be nonnegative on each edge.

Let a, b, c denote the Bézier ordinates of a *univariate* quadratic function on a compact interval $I \subset \mathbb{R}^2$, that is, the coefficients of β_1^2, $2\beta_1\beta_2$, β_2^2, respectively, where β_1, β_2 are the univariate barycentric coordinates with respect to I. We have the following elementary result, whose proof is deferred to the next section.

Lemma 1. *Necessary and sufficient conditions for the nonnegativity of a univariate quadratic function defined on an interval I with Bézier ordinates a, b, c are*

$$a \geqslant 0, \qquad c \geqslant 0 \tag{4}$$

and

$$b \geqslant -\sqrt{ac}\,. \tag{5}$$

The general idea of the bivariate problem is then as follows. A property of the Bernstein–Bézier form is that the univariate quadratic on an edge resulting from the restriction of Q to that edge, has as its three Bézier ordinates the three Bézier ordinates of Q along that edge. That is, along the edge e_i, the Bézier ordinates of the univariate quadratic are a_j, b_i, and a_k, where here and throughout, we adopt the following

Convention. *When the indices* i, j, k *appear together in a statement, assume that* (i, j, k) *is a cyclic permutation of* $(1, 2, 3)$.

From the necessity of nonnegativity of Q on the boundary ∂T of the triangle T, and Lemma 1, then follows

Proposition 1. *The following conditions are necessary for the nonnegativity of* Q *on* T:

$$a_i \geqslant 0, \quad i = 1, 2, 3 \tag{6}$$

and

$$b_i \geqslant -\sqrt{a_j a_k}, \quad i = 1, 2, 3. \tag{7}$$

With $Q_{\min}$ the minimum value assumed by Q on T, $Q_{\min} \geqslant 0$ is, of course, necessary and sufficient for the nonnegativity of Q on T. Hence, if $Q_{\min}$ happens to be assumed on ∂T, then (6) and (7) are necessary and *sufficient* for the nonnegativity of Q on T. In case $Q_{\min}$ is *not* assumed by Q on ∂T, then (6) and (7) are only *necessary* for the nonnegativity of Q on T, and $Q_{\min} \geqslant 0$ must be explicitly required for nonnegativity of Q on T.

By examining the family of surfaces which are the graphs of bivariate quadratic functions, it was demonstrated in [Nadler '88] by geometric arguments that the value $Q_{\min}$ is *not* assumed by Q in ∂T if and only if the function is strictly convex (which means the surface is an elliptic paraboloid, concave up) with critical point x_0 in $T°$, the *interior* of T. The critical point in this case is the point where $Q_{\min}$ is assumed. Therefore, in this case, one must explicitly require for the nonnegativity of Q that $Q(x_0) \geqslant 0$.

Then explicit inequality conditions were established which are necessary and sufficient for the aforementioned properties: Q strictly convex, $x_0 \in T°$, and $Q(x_0) \geqslant 0$. These led to the desired result of necessary and sufficient conditions for the nonnegativity of Q. That result is again presented here as Theorem 1, in Section 3, however, the details of the earlier geometric arguments are not given here (although the idea of them remains behind much of the discussion). Instead, simpler arguments primarily using elementary linear algebra are given.

In the next section some further notation is introduced, and some preliminary results are presented, most notably, a condition for the existence of a critical point x_0, with a formula for it and for $Q(x_0)$. In Section 3 the necessary and sufficient condition for the nonnegativity of a bivariate quadratic function on a triangle is then proved and further discussed.

2. Preliminary results

Before proving Lemma 1, the necessary and sufficient condition for nonnegativity in the univariate case, let us first state and prove an elementary monotonicity principle of the Bernstein–Bézier form, which will be useful several times in the paper.

Lemma 2. *A multivariate polynomial of degree n in Bernstein–Bézier form defined on a simplex in* $\mathbb{R}^k$ *is a monotone increasing function of each Bézier ordinate, i.e., increasing a Bézier ordinate increases the value of the polynomial at each point of the domain simplex. On the interior of the simplex, it is strictly monotone.*

Proof. At each point x of the simplex, with barycentric coordinates β, the value of the polynomial is a linear combination of the Bézier ordinates $c_{i_1 i_2 \cdots i_{k+1}}$, $\sum_j i_j = n$, with *nonnegative* coefficients

$$\frac{n!}{i_1! \cdots i_{k+1}!} \beta_1^{i_1} \cdots \beta_{k+1}^{i_{k+1}},$$

establishing monotonicity. On the interior of the simplex, all components of $\boldsymbol{\beta}$ are strictly positive, and thus, the above coefficients are strictly positive. □

Proceeding now with the proof of Lemma 1, we note that there are simpler proofs of this elementary lemma, but the proof is presented in the following way because some of the ideas will carry over to the bivariate case.

Proof of Lemma 1. As in the bivariate case, a univariate quadratic function P can be written $\tilde{P}(\boldsymbol{\beta}) = \boldsymbol{\beta}^{\mathrm{T}} A_2 \boldsymbol{\beta}$ where the barycentric coordinates

$$\boldsymbol{\beta} \in \mathscr{T}_+ := \{(\beta_1, \beta_2) : \beta_1, \beta_2 \geqslant 0, \beta_1 + \beta_2 = 1\}$$

and

$$A_2 := \begin{bmatrix} a & b \\ b & c \end{bmatrix}.$$

Sufficiency. $a \geqslant 0$, $c \geqslant 0$ and $|b| \leqslant \sqrt{ac}$ imply that the matrix $\boldsymbol{B}$ is postive semidefinite, so that $\boldsymbol{\beta}^{\mathrm{T}} \boldsymbol{B} \boldsymbol{\beta} \geqslant 0$ for *any* $\boldsymbol{\beta}$, in particular for $\boldsymbol{\beta} \in \mathscr{T}_+$. Then for $b > \sqrt{ac}$, the nonnegativity of P follows from the monotonicity of P in the Bézier ordinate b, by Lemma 2.

Necessity. Given $P \geqslant 0$ in I, to show $b \geqslant -\sqrt{ac}$ (5), consider the following two cases:

(i) a or $c = 0$: (5) becomes simply $b \geqslant 0$. This is necessary for $P \geqslant 0$, since if $b < 0$, one clearly has $P < 0$ on I in a neighborhood of the endpoint with the 0 Bézier ordinate.

(ii) a and $c > 0$: Use the monotonicity of P in the Bézier ordinate b to show $b < -\sqrt{ac}$ implies $P < 0$ on I. Consider the quadratic function P_0 with $b = -\sqrt{ac}$. Observe that

$$\tilde{P}_0(\hat{\boldsymbol{\beta}}_0) = 0 \quad \text{for } \hat{\boldsymbol{\beta}}_0 := \left(\frac{\sqrt{c}}{\sqrt{a} + \sqrt{c}}, \frac{\sqrt{a}}{\sqrt{a} + \sqrt{c}} \right),$$

a point in the interior of I since a and c are strictly positive. Then for P with $b < -\sqrt{ac}$, it follows from Lemma 2 that $\tilde{P}(\hat{\boldsymbol{\beta}}_0) < 0$, as desired.

Thus in both cases $P \geqslant 0$ implies $b \geqslant -\sqrt{ac}$. □

Turning now to the bivariate problem, some notation and preliminary results are established.

Under the assumption of $a_i > 0$, $i = 1, 2, 3$, and important special case, define the useful quantities

$$w_i := \frac{b_i}{\sqrt{a_j a_k}}, \quad i = 1, 2, 3 \tag{8}$$

with j and k the two elements of $\{1, 2, 3\}$ other than i, in accordance with our convention.

Motivated by the remarks near the end of Section 1, we now establish a condition for the existence of a critical point $\boldsymbol{x}_0$ of Q, and present a formula for it and for $Q(\boldsymbol{x}_0)$. Begin by decomposing $\boldsymbol{A}$:

$$\boldsymbol{A} = \boldsymbol{D}^{-1} \boldsymbol{B} \boldsymbol{D}^{-1} \tag{9}$$

where

$$\boldsymbol{B} := \begin{bmatrix} 1 & w_3 & w_2 \\ w_3 & 1 & w_1 \\ w_2 & w_1 & 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{D} := \begin{bmatrix} 1/\sqrt{a_1} & 0 & 0 \\ 0 & 1/\sqrt{a_2} & 0 \\ 0 & 0 & 1/\sqrt{a_3} \end{bmatrix}$$

so that

$$D^{-1} := \begin{bmatrix} \sqrt{a_1} & 0 & 0 \\ 0 & \sqrt{a_2} & 0 \\ 0 & 0 & \sqrt{a_3} \end{bmatrix}.$$

Then define

$$\phi := \det B = \frac{\det A}{a_1 a_2 a_3} = 1 + 2w_1w_2w_3 - w_1^2 - w_2^2 - w_3^2 \tag{10}$$

and

$$C := \operatorname{adj} B = \begin{bmatrix} 1 - w_1^2 & w_1w_2 - w_3 & w_3w_1 - w_2 \\ w_1w_2 - w_3 & 1 - w_2^2 & w_2w_3 - w_1 \\ w_3w_1 - w_2 & w_2w_3 - w_1 & 1 - w_3^2 \end{bmatrix}. \tag{11}$$

Note that

$$ADCD = \phi I, \tag{12}$$

which follows from (9) and the fact that $BC = \phi I$. Also, let $u := (1, 1, 1)$.

Lemma 3. *If $u^{\mathrm{T}}DCDu \neq 0$, then there exists a critical point x_0 of Q with barycentric coordinates*

$$\beta_0 = \frac{DCDu}{u^{\mathrm{T}}DCDu}, \tag{13}$$

and

$$\tilde{Q}(\beta_0) \equiv Q(x_0) = \frac{\phi}{u^{\mathrm{T}}DCDu}. \tag{14}$$

Remark. It is easy to show that $u^{\mathrm{T}}DCDu = 0$ if and only if the determinant of the Hessian of Q equals 0, which means that the surface is a parabolic cylinder (trough-shaped) rather than the usual elliptic paraboloid (bowl-shaped) or hyperbolic paraboloid (saddle-shaped).

Proof. First, note that under the hypothesis of the lemma, β_0 as given in (13) is *defined*. Hence to show the existence of a critical point with coordinates given by (13), it suffices to show that the expression in (13) satisfies an equation which characterizes a critical point. One way of characterizing the critical point x_0 of a quadratic function is that the function is symmetric about it:

$$Q(x) = Q(2x_0 - x) \quad \forall x \in \mathbb{R}^2.$$

This can be written in terms of the barycentric coordinates, after some simplification, as

$$\beta^{\mathrm{T}}A\beta_0 \stackrel{?}{=} \beta_0^{\mathrm{T}}A\beta_0 \quad \forall \beta \in \mathcal{S}. \tag{15}$$

To show that β_0 as given in (13) does indeed satisfy (15), observe that (with β_0 as in (13)),

$$A\beta_0 = \frac{\phi u}{u^{\mathrm{T}}DCDu},$$

which follows immediately from (12). Substituting this in (15), one then has

$$\frac{\beta^{\mathrm{T}}u}{u^{\mathrm{T}}DCDu} \stackrel{?}{=} \frac{\beta_0^{\mathrm{T}}u}{u^{\mathrm{T}}DCDu} \quad \forall \beta \in \mathcal{S},$$

which obviously is true. Thus a critical point of Q does exist, with barycentric coordinates given by (13).

Then with $\boldsymbol{\beta}_0$ as given in (13), again using (12), we calculate $\tilde{Q}(\boldsymbol{\beta}_0)$ to be as in (14), establishing the second part of the lemma. □

Finally, we make the following observation, whose proof is trivial.

Lemma 4. *$w_i \geqslant -1$, $i = 1, 2, 3$, and $\psi \leqslant 0$ imply $w_i \leqslant 1$, $i = 1, 2, 3$.*

Having introduced some notation and proven our lemmas, we are now ready to state and prove the main results of the paper.

3. The necessary and sufficient conditions for nonnegativity

Theorem 1 [Nadler '88]. *Necessary and sufficient conditions for the nonnegativity of a bivariate quadratic Q on a triangle T, with Bézier ordinates a_i and b_i, $i = 1, 2, 3$, as described above, are*

$$a_i \geqslant 0, \quad i = 1, 2, 3, \tag{6}$$

$$b_i \geqslant -\sqrt{a_j a_k}, \quad i = 1, 2, 3, \tag{7}$$

and

$$\det A \geqslant 0 \quad or \quad \sqrt{a_1 a_2 a_3} + \sum_{i=1}^{3} b_i \sqrt{a_i} \geqslant 0. \tag{16a,b}$$

The necessity of (6) and (7), which ensure that $Q \geqslant 0$ on ∂T, has been established in Proposition 1. As remarked in Section 1, the additional condition (16) was first established by geometric methods in [Nadler '88] to ensure that the minimum value of Q on T is nonnegative in case this minimum value is not assumed in ∂T.

As in the univariate case, this theorem naturally breaks into two cases: (i) one or more of the $a_i = 0$, and (ii) all the $a_i > 0$. We deal quickly with the first case in Proposition 2, and the more important second case is designated as Theorem 1′.

Proposition 2. *Given $a_i = 0$ for $i = 1, 2$, or 3, the following are necessary and sufficient for the nonnegativity of a bivariate quadratic function Q on T:*

$$a_j,\ a_k \geqslant 0 \tag{$\tilde{6}$}$$

and

$$b_i \geqslant -\sqrt{a_j a_k}, \qquad b_j,\ b_k \geqslant 0. \tag{$\tilde{7}$}$$

Notice that (16b) becomes $b_j\sqrt{a_j} + b_k\sqrt{a_k} \geqslant 0$ is implied by ($\tilde{6}$) and ($\tilde{7}$). Hence (16) is redundant and not needed here.

Proof. In Proposition 1, (6) and (7) were seen to be *necessary* for the nonnegativity of Q on T (since they characterize nonnegativity of Q on ∂T), and therefore so are their special cases (i.e., in the case of $a_i = 0$), ($\tilde{6}$) and ($\tilde{7}$).

For *sufficiency* of ($\tilde{6}$) and ($\tilde{7}$) for nonnegativity, assume that b_j, $b_k = 0$ and $b_i = -\sqrt{a_j a_k}$, i.e., the lower bounds of the b_i in ($\tilde{7}$), and prove that Q is nonnegative under these assumptions. It will then follow from the monotonicity of Q in the b_i (Lemma 2), that Q is nonnegative in general.

Under the above assumptions, with $i = 1$, say, for definiteness,

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & a_2 & -\sqrt{a_2 a_3} \\ 0 & -\sqrt{a_2 a_3} & a_3 \end{bmatrix},$$

which is positive semidefinite since its principal submatrices

$$A, \quad \begin{bmatrix} a_2 & -\sqrt{a_2 a_3} \\ -\sqrt{a_2 a_3} & a_3 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & a_3 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & a_2 \end{bmatrix}, \quad [0], \quad [a_2], \quad \text{and} \quad [a_3] \tag{17}$$

all have nonnegative determinant. Since then $\boldsymbol{\beta}^{\mathrm{T}} A \boldsymbol{\beta} \geqslant 0$ for *all* $\boldsymbol{\beta} \in \mathbb{R}^3$, in particular one has that $\boldsymbol{\beta}^{\mathrm{T}} A \boldsymbol{\beta} \geqslant 0$ for $\boldsymbol{\beta} \in \mathscr{S}_+$, i.e., that Q is nonnegative on T. □

Now, with w_i and ϕ introduced in equations (8) and (10) of Section 2, and with

$$\psi := \frac{\sqrt{a_1 a_2 a_3} + \sum_{i=1}^{3} b_i \sqrt{a_i}}{\sqrt{a_1 a_2 a_3}} = 1 + w_1 + w_2 + w_3 \tag{18}$$

(noting that the numerators in the definitions of ϕ and ψ equal the left-hand sides of the inequalities (16), respectively), (7) and (16) are simply expressed as (7′) and (16′) in the following theorem dealing with the important case of the a_i positive.

Theorem 1′. *Assuming* $a_i > 0$, $i = 1, 2, 3$, *the following are necessary and sufficient for the nonnegativity of* Q *on* T:

$$w_i \geqslant -1, \quad i = 1, 2, 3 \tag{7′}$$

and

$$\phi \geqslant 0 \quad or \quad \psi \geqslant 0. \tag{16′a,b}$$

Proof. (i) *Necessity.* $Q \geqslant 0$ on T implies (7′), by Proposition 1. To prove (16′), assume $\psi < 0$ and show $\phi \geq 0$. This is done by showing that there exists a critical point $\boldsymbol{x}_0$, that this point $\boldsymbol{x}_0 \in T$, and that $\phi = kQ(\boldsymbol{x}_0)$ for some positive constant k.

First, establish that $\boldsymbol{x}_0$ exists and is in T (in fact, in T°), by showing that all three of its barycentric coordinates are positive. By Lemma 3, the barycentric coordinates of $\boldsymbol{x}_0$ are given by $\boldsymbol{\beta}_0 = \boldsymbol{DCDu}/\boldsymbol{u}^{\mathrm{T}}\boldsymbol{DCDu}$ (13), *provided the denominator is non-zero.* The required positive constant k mentioned above will turn out to be this non-zero denominator.

The diagonal entries $1 - w_i^2$ of C, as given in (11), are *nonnegative* since $w_i \geqslant -1$ implies by Lemma 4 that $w_i \leqslant 1$. The positivity of the off-diagonal entries $w_i w_j - w_k$ of C follows easily from $w_i \geqslant -1$ and the assumed negativity of ψ. It then follows that the diagonal and off-diagonal entries of $\boldsymbol{DCD}$ are also nonnegative and positive, respectively. Hence all three components of the vector $\boldsymbol{DCDu}$ are positive, and hence $\boldsymbol{u}^{\mathrm{T}}\boldsymbol{DCDu} > 0$ and thus, the hypothesis of Lemma 3 is satisfied. Hence $\boldsymbol{x}_0$ indeed exists and has barycentric coordinates given by (13), all three of whose components are positive. Thus $\boldsymbol{x}_0 \in T^{\circ}$.

Now, by Lemma 3, $Q(\boldsymbol{x}_0) = \phi/\boldsymbol{u}^{\mathrm{T}}\boldsymbol{DCDu}$ (14), and the denominator has just been seen to be positive. Thus, $\boldsymbol{u}^{\mathrm{T}}\boldsymbol{DCDu}$ is the positive constant k sought above; and so we see that for Q to be nonnegative on T, it must be the case that $\phi \geqslant 0$, completing the proof of *necessity* of (16′).

Remark. Under the assumption $\psi < 0$, it is not difficult to see by a further calculation that $\boldsymbol{x}_0$ is, in fact, in a certain triangle contained in T, with one vertex on each edge of T, given in barycentric coordinates as follows:

$$\left\{\left(0, \frac{\sqrt{a_3}}{\sqrt{a_2}+\sqrt{a_3}}, \frac{\sqrt{a_2}}{\sqrt{a_2}+\sqrt{a_3}}\right), \left(\frac{\sqrt{a_3}}{\sqrt{a_3}+\sqrt{a_1}}, 0, \frac{\sqrt{a_1}}{\sqrt{a_3}+\sqrt{a_1}}\right), \left(\frac{\sqrt{a_2}}{\sqrt{a_1}+\sqrt{a_2}}, \frac{\sqrt{a_1}}{\sqrt{a_1}+\sqrt{a_2}}, 0\right)\right\}.$$

These are precisely the points of their respective edges e_i where $Q = 0$ in case $w_i = -1$, as can be seen for example, in the proof of Lemma 1 (*Necessity*) for the univariate case, where they are analogous to $\hat{\beta}_0$.

(ii) *Sufficiency*. First show Q is nonnegative in case $\psi \leqslant 0$, and then proceed to show it for $\psi > 0$ with a monotonicity argument.

To prove $Q \geqslant 0$ on T for $\psi \leqslant 0$, it suffices to prove that A is positive semidefinite, for which it suffices to show the principal submatrices (see (17)) have nonnegative determinant, as in the proof of Proposition 2. The three 1×1 principal submatrices have nonnegative (in fact, positive) determinant. The determinants of the 2×2 submatrices are

$$a_j a_k - b_i^2 = a_j a_k (1 - w_i^2), \quad i = 1, 2, 3,$$

and these are nonnegative since $w_i \geqslant -1$ and $\psi \leqslant 0$ imply by Lemma 4 that $w_i \leqslant 1$. And to establish that $\det A \geqslant 0$, recall from (10) that $\det A = a_1 a_2 a_3 \phi$, so it suffices to show that $\phi \geqslant 0$. This is easily seen in both the cases $\psi < 0$ and $\psi = 0$: in the first case by the assumption of (16′), and in the second, from (7′) and the identity

$$\phi \equiv 2(w_1 + 1)(w_2 + 1)(w_3 + 1) - \psi^2, \tag{19}$$

which is easily seen by a direct calculation. Thus the determinants of all principal submatrices are nonnegative, and thus, A is positive semidefinite, as desired.

Now, to extend the result to the case of $\psi > 0$, we shall show that for any quadratic function Q with $\psi > 0$ (i.e., $w_1 + w_2 + w_3 > -1$) and satisfying (7′) (i.e., $w_1, w_2, w_3 \geqslant -1$), there exists a quadratic function $Q_0 \leqslant Q$ on T with $\psi = 0$ and again satisfying (7′). Since we have just seen $Q_0 \geqslant 0$ on T, Q will thus also be nonnegative. Q_0 will be chosen to have the same values of a_1, a_2, a_3 as Q, so everything can be expressed in terms of the quantities w_i. With $\boldsymbol{w} := (w_1, w_2, w_3)$ associated with Q, and $\boldsymbol{w}_0$ with Q_0, one has by Lemma 2 that for $Q_0 \leqslant Q$ on T, it suffices for $\boldsymbol{w}_0 \leqslant \boldsymbol{w}$ componentwise. Thus for any $\boldsymbol{w}$ with $(1, 1, 1) \cdot \boldsymbol{w} > -1$ and components all $\geqslant -1$, it suffices to show that *there exists* $\boldsymbol{w}_0$ *with* $(1, 1, 1) \cdot \boldsymbol{w}_0 = -1$ *and, again, components all* $\geqslant -1$, *with* $\boldsymbol{w}_0 \leqslant \boldsymbol{w}$ *componentwise*. In other words, given $\boldsymbol{w}$ with $w_1 + w_2 + w_3 > -1$ and $w_1, w_2, w_3 \geqslant -1$, show that the sets

$$\begin{aligned} \mathscr{P} &:= \{(x_1, x_2, x_3) : x_1 + x_2 + x_3 = -1\}, \\ \mathscr{Q} &:= \{(x_1, x_2, x_3) : x_i \geqslant -1, i = 1, 2, 3\}, \\ \mathscr{R} &:= \{(x_1, x_2, x_3) : x_i \leqslant w_i, i = 1, 2, 3\} \end{aligned} \tag{20}$$

have nonempty intersection. It is convenient to work in the plane $\mathscr{P}$, showing that $\mathscr{Q} \cap \mathscr{P}$ and $\mathscr{R} \cap \mathscr{P}$ are nondisjoint. These two sets are triangles in $\mathscr{P}$, with vertices

$$(1, -1, -1), \quad (-1, 1, -1), \quad (-1, -1, 1) \tag{21}$$

and

$$(-1 - w_2 - w_3, w_2, w_3), \quad (w_1, -1 - w_1 - w_3, w_3), \quad (w_1, w_2, -1 - w_1 - w_2), \tag{22}$$

respectively. They are indeed nondisjoint, as the point

$$\tilde{w} := \left(\frac{w_1 - w_2 - w_3 - 1}{w_1 + w_2 + w_3 + 3}, \frac{-w_1 + w_2 - w_3 - 1}{w_1 + w_2 + w_3 + 3}, \frac{-w_1 - w_2 + w_3 - 1}{w_1 + w_2 + w_3 + 3} \right) \tag{23}$$

belongs to both sets, since its components $\tilde{w}_i \in [-1, w_i]$ for $i = 1, 2, 3$, as is easily verified by a straightforward calculation.

This point $\tilde{w}$ is the intersection of any pair of lines determined by a pair of corresponding vertices, as ordered in (21) and (22), of the two triangles $\mathscr{C} \cap \mathscr{P}$ and $\mathscr{R} \cap \mathscr{P}$, respectively. This pair of triangles happens to be *perspective from the point* $\tilde{w}$, meaning that all three such lines pass through the common point $\tilde{w}$.

Thus $\tilde{w}$ can serve as the required point w_0 (and turns out to be the *only* possible w_0 in certain cases), and thus, for any w with $\psi > 0$ and satisfying (7′), we have shown that there exists a w_0 with $\psi = 0$ and satisfying (7′), such that $w_0 \leqslant w$ componentwise, as desired, establishing nonnegativity of Q on T in this final case. □

Remark. After this paper was complete, the following two relevant papers have come to the author's attention. [Hadeler '83] using algebraic methods completely different than those used here, establishes necessary and sufficient conditions for the *copositivity of a* 3×3 *matrix*, which amount to those of Theorem 1 for the nonnegativity of bivariate quadratic polynomials on a triangle. The equivalence of nonnegativity of multivariate quadratics on a simplex and copositivity of matrices is pointed out in [Micchelli & Pinkus '89], which, it should be noted, acknowledges the earlier presentation of the result of Theorem 1 in [Nadler '88].

With ϕ or ψ required by (16′) to be nonnegative for the nonnegativity of Q, it is of some interest to examine the actual permissible values of ϕ and ψ, which reflect the relationship (19) between ϕ and ψ, as well as the other constraint $w_i \geqslant -1$, $i = 1, 2, 3$ (7′). These permissible values are given as follows:

$$\tfrac{1}{27}(\psi - 4)^2(2\psi + 1) \geqslant \phi \geqslant \begin{cases} 0 & \text{for } \psi \in [-\frac{1}{2}, 0), \\ -\psi^2 & \text{for } \psi \in [0, \infty) \end{cases} \tag{24}$$

as can easily be seen by fixing ψ and making use of (19) and (7′). The resulting region, for ψ fixed, in w_1–w_2–w_3 space, is a triangle; for example, for $\psi = 0$ it is the triangle $\mathscr{C} \cap \mathscr{P}$ described in the proof of Theorem 1′ (*Sufficiency*) (see (20), (21)). The graph of the whole region described by (24) is depicted in Fig. 1.

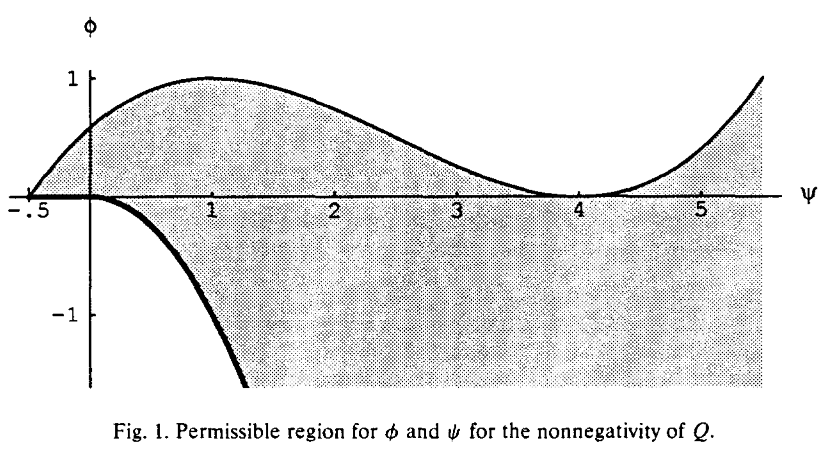

Fig. 1. Permissible region for ϕ and ψ for the nonnegativity of Q.

The condition (24) alone is necessary, but not sufficient, for nonnegativity. Evidently, condition (16′) is satisfied if (24) is, but (24) is not sufficient for the nonnegativity of Q because (7′) may be violated: exactly two of the w_i can be less than -1. If one explicitly adjoins condition (7′) to (24), the resulting set of conditions is necessary and sufficient. Thus, a precise set of conditions, but one more difficult to check than those of Theorem 1′, is given in the following

Corollary 1. *Assuming* $a_i > 0$, $i = 1, 2, 3$, *(7′) and (24) are necessary and sufficient for the nonnegativity of* Q *on* T.

In its original formulation in Section 1, the problem was naturally partitioned into the two cases of the minimum value $Q_{\min}$ of Q on T being assumed or not assumed on the boundary ∂T. We conclude by providing a connection of the above results to this original formulation.

Recall that in the proof of Theorem 1′ (*Necessity*) it had been remarked that for $\psi < 0$, the unique minimum of Q in T is in the interior of T, so that in this case, $Q_{\min}$ is *not* assumed on ∂T. For $\psi \geqslant 0$ one cannot in general say without further information whether or not $Q_{\min}$ is assumed on ∂T. But, one *can* do so in the important 'borderline' case of $Q_{\min} = 0$, i.e., for functions Q that have 0 as their minimum value on T. In this case it is not difficult to show that for $\psi \geqslant 0$, $Q_{\min}$ (i.e., 0) *is* assumed on ∂T. Moreover, these two cases (i.e., $\phi < 0$ and $\phi \geqslant 0$) of this borderline case are described in the ψ–ϕ plane by the 'lower boundary' curve of the 'nonnegativity region' depicted as the **bold** curve in Fig. 1 (and described by equality in the second inequality of (24)), as follows.

For Q *such that* $a_1, a_2, a_3 > 0$, $Q_{\min} = 0$, *and:*

$$\text{(i)}\ \forall x \in \partial T,\ Q(x) > Q_{\min},\quad (\psi, \phi) \in \{(x, 0) : x \in [-\tfrac{1}{2}, 0)\},$$

$$\text{(ii)}\ \exists x_0 \in \partial T \ni Q(x_0) = Q_{\min},\quad (\psi, \phi) \in \{(x, -x^2) : x \in [0, \infty)\}.$$

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