Singularity of Cubic Bézier Curves

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joint work with

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Outline

Introduction

Singularity of a Parametric Curve

Bézier Curves

Singularity of Bézier Curves
Parametric Cubic Curve

\[ C(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 \]

Example ("twisted cubic"): \[ C(t) = \langle t, t^2, t^3 \rangle \]
Singularity of a Parametric Curve

*Singularity* of a curve $C(t)$: $t^*$ where $C'(t^*) = \vec{0}$

Geometrically, a *cusp*, except when also $C''(t^*) = 0$,

which for cubic can happen only when curve is a *line*

Example: $C(t) = \langle 4t^3 - 3t^2 + 1, 4t^3 - 9t^2 + 6t \rangle$, $t \in [0, 1]$

$$C'(t) = \langle 12t^2 - 6t, 12t^2 - 18t + 6 \rangle$$

$$t^* = \frac{1}{2}, \ C(t^*) = \langle \frac{3}{4}, \frac{5}{4} \rangle$$
Bézier Curves

- A representation of parametric polynomial curves

- Geometric and intuitive, facilitating creative design process

- Computationally efficient and stable

- At the core of Computer Aided Geometric Design (CAGD)
Bézier Curves of degree 1

\[ C(t) = (1 - t)P_0 + tP_1, \quad t \in [0, 1] \]
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Bézier Curves of degree 2

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Bézier Curves of degree 3

\[ P_{01} = (1 - t)P_0 + t P_1; \quad P_{12} = (1 - t)P_1 + t P_2; \quad P_{23} = (1 - t)P_2 + t P_3 \]

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\[ + t [(1 - t)[(1 - t)P_1 + t P_2] + t [(1 - t)P_2 + t P_3]] \]

\[ = (1 - t)^3t^0P_0 + 3(1 - t)^2t^1P_1 + 3(1 - t)t^2P_2 + (1 - t)^0t^3P_3 \]

\[ = \sum_{i=0}^{3} \binom{3}{i}(1 - t)^{3-i}t^i P_i \]
Bézier Curve – Definition

Degree 3:

\[ C(t) = \sum_{i=0}^{3} \binom{3}{i} (1 - t)^{3-i} t^i P_i \]

\[ = \sum_{i=0}^{3} B_i^3(t) P_i \]

where

\[ B_i^3(t) = \binom{3}{i} (1 - t)^{3-i} t^i \] is the \( i \textsuperscript{th} \) Bernstein (basis) polynomial of degree 3, and

\( P_i \) are known as (Bézier) control points.
Bézier Curve – Definition

Degree \( n \):

\[
C(t) = \sum_{i=0}^{n} \binom{n}{i} (1 - t)^{n-i} t^i P_i
\]

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Bernstein Basis Polynomials

Degree 3:

\[ \{B_i^3(t)\}_{i=0}^3 = \{(1 - t)^3, 3(1 - t)^2t, 3(1 - t)t^2, t^3\} \]
Bernstein Basis Polynomials

Degree 3:

\[ \{ B_i^3(t) \}_{i=0}^3 = \{ (1 - t)^3, \ 3(1 - t)^2t, \ 3(1 - t)t^2, \ t^3 \} \]

Partition of unity:

\[ \sum_{i=0}^{n} B_i^n(t) = \sum_{i=0}^{n} \binom{n}{i} (1 - t)^{n-i} t^i \]

\[ = (1 - t + t)^n = 1 \]
Bernstein polynomials were used by Sergei Bernstein in 1910 in his elegant proof of the Weierstrass Approximation Theorem (1885): a continuous function on a closed interval can be uniformly approximated by polynomials.

Bézier curves were first developed in the 1950s in the French automobile industry.
**Historical Notes**

- *Bernstein polynomials* were used by Sergei Bernstein in 1910 in his elegant proof of the *Weierstrass Approximation Theorem* (1885): a continuous function on a closed interval can be uniformly approximated by polynomials.

- Bézier curves were first developed in the 1950s in the French automobile industry.

Paul de Casteljau (Citroën) developed in 1959 the geometric algorithm presented – bearing his name – for evaluating points on a Bézier curve. It is the most robust and numerically stable method for evaluating polynomials, and one of the most important algorithms in CAGD.

![from de Casteljau's writings](image)

Pierre Bézier (Rénault) also worked on Bézier curves and surfaces, which are now used in most computer-aided design and computer graphics systems.
Examples of Cubic Bézier Curves

from Farin & Hansford 2000
Properties of Bézier Curves

- Endpoint interpolation: \( C(0) = P_0 \) and \( C(1) = P_n \)

- Endpoint tangency to control polygon:
  \[ C'(0) \parallel (P_1 - P_0) \] and \( C'(1) \parallel (P_n - P_{n-1}) \)

- Convex Hull Property: \( C[\{P\}] \subset \text{ConvexHull}(\{P\}) \)
  implies \( (\{P\} \text{ planar} \implies C[\{P\}] \text{ planar}) \)

- Convexity preservation for planar curves:
  \( \{P\} \text{ convex} \implies C[\{P\}] \text{ convex} \)

- Affine invariance: \( C[\Phi\{P\}] = \Phi C[\{P\}] \)

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Derivative of Bézier Curve

\[ C'(t) = \sum_{i=0}^{n} B_i^n'(t) P_i \]

\[ = \sum_{i=0}^{n} \binom{n}{i} ((1 - t)^{n-i} t^i)' P_i \]

\[ = n \sum_{i=0}^{n-1} B_i^{n-1}'(t) (P_{i+1} - P_i) \]

That is, the Bézier control points of \( C' \) are simply

\[ \{n (P_{i+1} - P_i)\}_{i=0}^{n-1} \]

Differentiate a Bézier Curve by differencing its control points!

The curve \( C' \) is known as the hodograph of the curve \( C \).
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Derivative of Bézier Curve

\[ C'(t) = \sum_{i=0}^{n} B'_i(t) P_i \]

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Differentiate a Bézier Curve by *differencing* its control points!

The curve \( C' \) is known as the *hodograph* of the curve \( C \).
Recall definition of *singularity* of a curve $C(t)$:

$t^\ast$ where $C'(t^\ast) = \vec{0}$

Apply this to Bézier Curve of degree 1:

$$C(t) = (1 - t)P_0 + t P_1$$

$$C'(t) = P_1 - P_0$$

$$= \vec{0} \quad \forall t \quad \text{iff } P_0 = P_1$$

That is, the only case of singularity of a polynomial curve of degree 1 is the trivial case when its two endpoints agree!
Singularity of Bézier Curve of degree 1

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Singularity of Bézier Curve of degree 2 – part 1

\[ C'(t^*) = \vec{0} \text{ for } n = 2: \]

\[ C(t) = B_0^2P_0 + B_1^2P_1 + B_2^2P_2 \]

\[ C'(t) = B_0^1(P_1 - P_0) + B_1^1(P_2 - P_1) \text{ by derivative formula} \]

\[ = (1 - t)(P_1 - P_0) + t(P_2 - P_1) \]

Hence, the only cases of singularity of a polynomial curve of degree 2 occur when its Bézier control points satisfy

\[ (P_1 - P_0) \parallel (P_2 - P_1) \]

i.e., they are collinear

Hence, by the Convex Hull Property, the curve actually lies on a line.
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Hence, by the \textit{Convex Hull Property}, the curve actually lies on a \textit{line}. 
Singularity of Bézier Curve of degree 2 – part 2

Equation for singularity:

\[ C'(t^*) = (1 - t^*)(P_1 - P_0) + t^*(P_2 - P_1) = \vec{0}, \quad t^* \in [0, 1] \]

For singularity, in addition to being collinear, must have \( P_0, P_1, P_2 \) “out of order”, i.e.,
\( P_0 \) between \( P_1 \) and \( P_2 \): \( t^* \in [0, \frac{1}{2}] \)

OR

\( P_2 \) between \( P_0 \) and \( P_1 \): \( t^* \in [\frac{1}{2}, 1] \)

In all cases, the singularity is at \( P_1 \); curve reverses direction there.

Special cases of coincident adjacent end control points:
- If \( P_0 = P_1 \), singularity there at \( t = 0 \)
- If \( P_1 = P_2 \), singularity there at \( t = 1 \)
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Singularity of Bézier Curve of degree 3 – part 1

Basics

\[ C'(t^*) = \vec{0} \text{ for } n = 3: \]

\[ C(t) = B_0^3P_0 + B_1^3P_1 + B_2^3P_2 + B_3^3P_3 \]

\[ \frac{1}{3}C'(t) = B_0^2(P_1 - P_0) + B_1^2(P_2 - P_1) + B_2^2(P_3 - P_2) \]

\[ = (1 - t)^2(P_1 - P_0) + 2(1 - t)t(P_2 - P_1) + t^2(P_3 - P_2) \]

Hence, the only cases of singularity of a polynomial curve of degree 3 occur when a linear combination of \((P_1 - P_0), (P_2 - P_1), (P_3 - P_2)\) equals \(\vec{0}\)

Hence, for singularity, these three vectors, and hence, \(\{P_0, P_1, P_2, P_3\}\) themselves, must be coplanar

Hence, for singularity, by the Convex Hull Property, the curve must be planar
Singularity of Bézier Curve of degree 3 – part 1

Basics

\[ C'(t^*) = \vec{0} \text{ for } n = 3: \]

\[ C(t) = B_0^3P_0 + B_1^3P_1 + B_2^3P_2 + B_3^3P_3 \]

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\[ = (1 - t)^2(P_1 - P_0) + 2(1 - t)t(P_2 - P_1) + t^2(P_3 - P_2) \]

Hence, the only cases of singularity of a polynomial curve of degree 3 occur when a linear combination of \( \{(P_1 - P_0), (P_2 - P_1), (P_3 - P_2)\} \) equals \( \vec{0} \)

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\[ C(t) = B_0^3 P_0 + B_1^3 P_1 + B_2^3 P_2 + B_3^3 P_3 \]

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Hence, for singularity, by the \textit{Convex Hull Property}, the curve must be \textit{planar}
Singularity of Bézier Curve of degree 3 – part 1

Basics

\[ C'(t^*) = \vec{0} \text{ for } n = 3: \]

\[ C(t) = B_0^3P_0 + B_1^3P_1 + B_2^3P_2 + B_3^3P_3 \]

\[ \frac{1}{3}C'(t) = B_0^2(P_1 - P_0) + B_1^2(P_2 - P_1) + B_2^2(P_3 - P_2) \]

\[ = (1 - t)^2(P_1 - P_0) + 2(1 - t)t(P_2 - P_1) + t^2(P_3 - P_2) \]

Hence, the only cases of singularity of a polynomial curve of degree 3 occur when a linear combination of \{[(P_1 - P_0), (P_2 - P_1), (P_3 - P_2)]\} equals \vec{0}

Hence, for singularity, these three vectors, and hence, \{P_0, P_1, P_2, P_3\} themselves, must be coplanar

Hence, for singularity, by the Convex Hull Property, the curve must be planar
Singularity of Bézier Curve of degree 3 – part 2

**Construct** singular curve, given some *translate* of its hodograph

\[ \Delta P_i = P_{i+1} - P_i \]

Hodograph
\[ C[\{P\}]' = C[\{3\Delta P\}] \implies \]

| singularity : \[C[\{\Delta P\}](t^*) = \vec{0}\] |

\[ C[\{\tilde{\Delta} P\}](t), \quad \tilde{\Delta} P_i = \Delta P_i + \vec{C} \]
Construct singular curve, given some *translate* of its hodograph

\[ \Delta P_i = P_{i+1} - P_i \]

**Hodograph**

\[ C[\{P\}] = C[\{3\Delta P\}] \rightarrow \]

**Singularity:** \[ C[\{\Delta P\}](t^*) = \vec{0} \]

\[ C[\{\Delta P\}](t), \Delta \tilde{P}_i = \Delta P_i + \vec{C} \]
Singularity of Bézier Curve of degree 3 – part 2

**Construct** singular curve, given some *translate* of its hodograph

\[ \Delta P_i = P_{i+1} - P_i \]

**Hodograph**

\[ C[\{P\}] = C[\{3\Delta P\}] \implies \]

**singularity**:

\[ C[\{\Delta P\}](t^*) = \vec{0} \]

\[ C[\{\widetilde{\Delta P}\}](t), \quad \widetilde{\Delta P}_i = \Delta P_i + \vec{C} \]
Singularity of Bézier Curve of degree 3 – part 2

**Construct** singular curve, given some *translate* of its hodograph

\[ \Delta P_i = P_{i+1} - P_i \]

Hodograph
\[ C[\{P\}]' = C[\{3\Delta P\}] \implies \]

singularity: \[ C[\{\Delta P\}](t^*) = \vec{0} \]

\[ C[\{\tilde{\Delta P}\}](t), \tilde{\Delta P}_i = \Delta P_i + \vec{C} \]

\( \{\tilde{\Delta P}\} \to \{\Delta P\} \ni \]
\[ C[\{\Delta P\}](t^*) = \vec{0} \]
Singularity of Bézier Curve of degree 3 – part 2

**Construct** singular curve, given some *translate* of its hodograph

\[ \Delta P_i = P_{i+1} - P_i \]

**Hodograph**

\[ C[\{P\}]' = C[\{3\Delta P\}] \implies \]

\[ \text{singularity : } C[\{\Delta P\}](t^*) = \vec{0} \]

\[ C[\{\tilde{\Delta} P\}](t), \quad \tilde{\Delta} P_i = \Delta P_i + \vec{C} \]

\[ \{\tilde{\Delta} P\} \rightarrow \{\Delta P\} \ni \]

\[ C[\{\Delta P\}](t^*) = \vec{0} \]

\[ P_0 = \vec{0} \text{ (arbitrary)} \]

\[ P_1 = P_0 + \Delta P_0 \]
Singularity of Bézier Curve of degree 3 – part 2

Construct singular curve, given some translate of its hodograph

\[ \Delta P_i = P_{i+1} - P_i \]

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\[ \{\tilde{\Delta}P\} \rightarrow \{\Delta P\} \ni \]
\[ C[\{\Delta P\}](t^*) = \vec{0} \]

\[ P_0 = \vec{0} \text{ (arbitrary)} \]
\[ P_1 = P_0 + \Delta P_0 \]
\[ P_2 = P_1 + \Delta P_1 \]
Singularity of Bézier Curve of degree 3 – part 2

**Construct** singular curve, given some *translate* of its hodograph

\[ \Delta P_i = P_{i+1} - P_i \]

Hodograph

\[ C[\{P\}]' = C[\{3\Delta P\}] \implies \]

\[ \text{singularity: } C[\{\Delta P\]}(t^*) = \vec{0} \]

\[ C[\{\Delta P\}](t), \quad \Delta P_i = \Delta P_i + \vec{C} \]

\[ \{\Delta P\} \rightarrow \{\Delta P\} \ni \]

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\[ P_0 = \vec{0} \text{ (arbitrary)} \]

\[ P_1 = P_0 + \Delta P_0 \]

\[ P_2 = P_1 + \Delta P_1 \]

\[ P_3 = P_2 + \Delta P_2 \]
Singularity of Bézier Curve of degree 3 – part 2

Construct singular curve, given some translate of its hodograph

\[ \Delta P_i = P_{i+1} - P_i \]

Hodograph

\[ C[\{P\}]' = C[\{3\Delta P\}] \implies \text{singularity: } C[\{\Delta P\]}(t^*) = \vec{0} \]

\[ C[\{\tilde{\Delta} P\}](t), \tilde{\Delta} P_i = \Delta P_i + \vec{C} \]

\[ \{\tilde{\Delta} P\} \rightarrow \{\Delta P\} \ni C[\{\Delta P\}](t^*) = \vec{0} \]

\[ P_0 = \vec{0} \text{ (arbitrary)} \]
\[ P_1 = P_0 + \Delta P_0 \]
\[ P_2 = P_1 + \Delta P_1 \]
\[ P_3 = P_2 + \Delta P_2 \]

\[ C[\{P\}] \text{ singular, with} \]
\[ C[\{P\}'](t^*) = \vec{0} \]
Examples of singular cubics with various values of $t^*$, using the construction:

$$
\begin{align*}
P_2 &= P_3 P_1 / \Delta P_0 P_0 / \Delta P_1 P_1 / \Delta P_2 P_2
\end{align*}
$$

$t_{\text{sing}} = 0.5$
Examples of singular cubics with various values of $t^*$, using the construction:

$$t_{\text{sing}} = 0.6$$
Examples of singular cubics with various values of \( t^* \), using the construction:

\[
\begin{align*}
P_2 &= P_3 \\
P_1 &= P_0 / \Delta P_0 \\
P_0 &= P_1 / \Delta P_1 \\
P_0 &= P_2 / \Delta P_2
\end{align*}
\]

\( t_{\text{sing}} = 0.7 \)
Examples of singular cubics with various values of $t^*$, using the construction:

$t_{\text{sing}} = 0.8$
Examples of singular cubics with various values of $t^*$, using the construction:

$t_{\text{sing}}=0.9$
Examples of singular cubics with various values of $t^*$, using the construction:

$$P_2 = P_3$$

tangent@sing. $\parallel (P_3 - P_1)$

$t_{\text{sing}}=1$. 
Examples of singular cubics with various values of $t^*$, using the construction:

\[ P_1 = P_0 \]

\[ \text{tangent@sing. } \parallel (P_2 - P_0) \]

\[ t_{\text{sing}} = 0 \]
Examples of singular cubics with various values of $t^*$, using the construction:
Examples of singular cubics with various values of $t^*$, using the construction:

$t_{\text{sing}} = 0.2$
Examples of singular cubics with various values of $t^*$, using the construction:

$$t_{\text{sing}} = 0.3$$
Examples of singular cubics with various values of $t^*$, using the construction:
Examples of singular cubics with various values of $t^*$, using the construction:

$t_{\text{sing}} = 0.5$
Singularity of Bézier Curve of degree 3 – part 4a

Solution for singularity using Bézier singularity condition

Define points

\[ O = \ell(P_0, P_3 - P_2) \cap \ell(P_3, P_1 - P_0) \]
\[ R = \ell(P_1, P_3 - P_2) \cap \ell(P_2, P_1 - P_0) \]

where \( \ell(P, V) \) is the line defined by point \( P \) and vector \( V \).

From the geometry above, \( R - O = \Delta P_0 - \Delta P_2 \),

\[ \Delta P_0 \parallel (P_3 - O) \text{ and } \Delta P_2 \parallel (P_0 - O) \]

\[ \Delta P_0 = x(P_3 - O), \quad x = \frac{\det(\Delta P_2, \Delta P_0)}{\det(\Delta P_2, P_3 - P_0)} \quad (1) \]
\[ \Delta P_2 = -y(P_0 - O), \quad y = \frac{\det(\Delta P_0, \Delta P_2)}{\det(\Delta P_0, P_3 - P_0)} \quad (2) \]

with \((x, y)\) capturing the essential shape of the control polygon.

Under the Bézier singularity condition \( C[\{\Delta P\}](t^*) = 0 \), \((1), (2) \rightarrow (x, y) = \left( \frac{2t^*}{3t^* - 1}, \frac{2(1-t^*)}{2-3t^*} \right) \), which satisfies

\[ (x - \frac{4}{3})(y - \frac{4}{3}) = \frac{4}{9} \quad (*) \]
Define points
\[ O = \ell(P_0, P_3 - P_2) \cap \ell(P_3, P_1 - P_0) \]
\[ R = \ell(P_1, P_3 - P_2) \cap \ell(P_2, P_1 - P_0) \]
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\[ \Delta P_2 = -y(P_0 - O), \quad y = \frac{\det(\Delta P_0, \Delta P_2)}{\det(\Delta P_0, P_3 - P_0)} \quad (2) \]

with \((x, y)\) capturing the essential shape of the control polygon.

Under the Bézier singularity condition \( C[\{\Delta P\}] (t^*) = \vec{0} \), \((1), (2) \rightarrow (x, y) = \left( \frac{2t^*}{3t^* - 1}, \frac{2(1-t^*)}{2-3t^*} \right) \), which satisfies

\[ (x - \frac{4}{3})(y - \frac{4}{3}) = \frac{4}{9} \quad (*) \]
Singularity of Bézier Curve of degree 3 – part 4a

**Solution** for singularity using Bézier singularity condition

Define points

\[ O = \ell(P_0, P_3 - P_2) \cap \ell(P_3, P_1 - P_0) \]
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with \((x, y)\) capturing the essential shape of the control polygon.

Under the Bézier singularity condition \( C[\{\Delta P\}](t^*) = \vec{0} \), (1), (2) \( \rightarrow \)

\[(x, y) = \left( \frac{2t^*}{3t^*-1}, \frac{2(1-t^*)}{2-3t^*} \right), \quad \text{which satisfies} \quad (x - \frac{4}{3})(y - \frac{4}{3}) = \frac{4}{9} \quad (*)\]
Singularity of Bézier Curve of degree 3 – part 4a

**Solution** for singularity using Bézier singularity condition

Define points

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\[ \Delta P_2 = -y(P_0 - O), \quad y = \frac{\text{det}(\Delta P_0, \Delta P_2)}{\text{det}(\Delta P_0, P_3 - P_0)} \quad (2) \]

with \((x, y)\) capturing the essential shape of the control polygon.

Under the Bézier singularity condition \( C[\{\Delta P\}] (t^*) = \vec{0} \), \((1), (2) \rightarrow \)

\[ (x, y) = \left( \frac{2t^*}{3t^* - 1}, \frac{2(1-t^*)}{2-3t^*} \right), \quad \text{which satisfies } \left( x - \frac{4}{3} \right) \left( y - \frac{4}{3} \right) = \frac{4}{9} \quad (*) \]
Singularity of Bézier Curve of degree 3 – part 4a

**Solution** for singularity using Bézier singularity condition

Define points

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\[ \Delta P_2 = -y(P_0 - O), \quad y = \frac{\text{det}(\Delta P_0, \Delta P_2)}{\text{det}(\Delta P_0, P_3 - P_0)} \]  \hfill (2)

with \( (x, y) \) capturing the essential shape of the control polygon.

Under the Bézier singularity condition \( C[\{\Delta P\}] (t^*) = \vec{0} \), (1), (2) \( \rightarrow \)

\[ (x, y) = \left( \frac{2t^*}{3t^* - 1}, \frac{2(1-t^*)}{2-3t^*} \right), \quad \text{which satisfies} \quad \left( x - \frac{4}{3} \right) \left( y - \frac{4}{3} \right) = \frac{4}{9} \quad (\ast) \]

Condition (\( \ast \)) for singularity was found by [Su & Liu 1990] using other methods that did not make essential use of the Bézier form.
Singularity of Bézier Curve of degree 3 – part 4a

Solution for singularity using Bézier singularity condition

Define points

\[ O = \ell(P_0, P_3 - P_2) \cap \ell(P_3, P_1 - P_0) \]
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\[ \Delta P_2 = -y(P_0 - O), \quad y = \frac{\text{det}(\Delta P_0, \Delta P_2)}{\text{det}(\Delta P_0, P_3 - P_0)} \quad (2) \]

with \( (x, y) \) capturing the essential shape of the control polygon.

Under the Bézier singularity condition \( C[\{\Delta P\}](t^*) = \vec{0} \), \((1), (2) \longrightarrow \)

\[ (x, y) = \left( \frac{2t^*}{3t^* - 1}, \frac{2(1-t^*)}{2-3t^*} \right), \quad \text{which satisfies} \]

\[ (x - \frac{4}{3})(y - \frac{4}{3}) = \frac{4}{9} \quad (*) \]

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Singularity of Bézier Curve of degree 3 – part 4a

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\[ \Delta P_2 = -y(P_0 - O), \quad y = \frac{\det(\Delta P_0, \Delta P_2)}{\det(\Delta P_0, P_3 - P_0)} \] \quad (2)

with \((x, y)\) capturing the essential shape of the control polygon.

Under the Bézier singularity condition \( C[\{\Delta P\}](t^*) = \vec{0} \), (1), (2) \( \rightarrow \)

\[(x, y) = \left( \frac{2t^*}{3t^* - 1}, \frac{2(1-t^*)}{2-3t^*} \right), \quad \text{which satisfies} \quad (x - \frac{4}{3})(y - \frac{4}{3}) = \frac{4}{9} \quad (\ast)\]

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Singularity of Bézier Curve of degree 3 – part 4a

Solution for singularity using Bézier singularity condition

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with \((x, y)\) capturing the essential shape of the control polygon.

Under the Bézier singularity condition \( C[\{\Delta P\}](t^*) = 0 \), (1), (2) \( \longrightarrow \)

\[ (x, y) = \left( \frac{2t^*}{3t^*-1}, \frac{2(1-t^*)}{2-3t^*} \right), \quad \text{which satisfies} \]
\[ (x - \frac{4}{3}) (y - \frac{4}{3}) = \frac{4}{9} \quad (\ast) \]

Condition \((\ast)\) for singularity was found by [Su & Liu 1990] using other methods that did not make essential use of the Bézier form.
Singularity of Bézier Curve of degree 3 – part 4a

**Solution** for singularity using Bézier singularity condition

Define points

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with \((x, y)\) capturing the essential shape of the control polygon.

Under the Bézier singularity condition \( C[\{\Delta P\}](t^*) = \vec{0} \), (1), (2) \( \rightarrow \)

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Singularity of Bézier Curve of degree 3 – part 4a

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\[ \Delta P_0 = x(P_3 - O), \ x = \frac{\text{det}(\Delta P_2, \Delta P_0)}{\text{det}(\Delta P_2, P_3 - P_0)} \]  
\[ (1) \]
\[ \Delta P_2 = -y(P_0 - O), \ y = \frac{\text{det}(\Delta P_0, \Delta P_2)}{\text{det}(\Delta P_0, P_3 - P_0)} \]
\[ (2) \]

with \((x, y)\) capturing the essential shape of the control polygon.

Under the Bézier singularity condition \( C[\{\Delta P\}](t^*) = \vec{0} \), \( (1), (2) \rightarrow \)

\[ (x, y) = \left( \frac{2t^*}{3t^*-1}, \frac{2(1-t^*)}{2-3t^*} \right), \text{ which satisfies } \]
\[ (x - \frac{4}{3})(y - \frac{4}{3}) = \frac{4}{9} \]  
\[ (*) \]

Condition \((*)\) for singularity was found by [Su & Liu 1990] using other methods that did not make essential use of the Bézier form.
Singularity of Bézier Curve of degree 3 – part 4a

**Solution** for singularity using Bézier singularity condition

Define points

\[ O = \ell(P_0, P_3 - P_2) \cap \ell(P_3, P_1 - P_0) \]
\[ R = \ell(P_1, P_3 - P_2) \cap \ell(P_2, P_1 - P_0) \]

where \( \ell(P, V) \) is the line defined by point \( P \) and vector \( V \).

From the geometry above, \( R - O = \Delta P_0 - \Delta P_2 \),

\[ \Delta P_0 \parallel (P_3 - O) \text{ and } \Delta P_2 \parallel (P_0 - O) \]
\[ \Delta P_0 = x(P_3 - O), \quad x = \frac{\det(\Delta P_2, \Delta P_0)}{\det(\Delta P_2, P_3 - P_0)} \quad (1) \]
\[ \Delta P_2 = -y(P_0 - O), \quad y = \frac{\det(\Delta P_0, \Delta P_2)}{\det(\Delta P_0, P_3 - P_0)} \quad (2) \]

with \((x, y)\) capturing the essential shape of the control polygon.

Under the Bézier singularity condition \( C[\{\Delta P\}](t^*) = 0 \), (1), (2) \( \rightarrow \)

\[ (x, y) = \left( \frac{2t^*}{3t^*-1}, \frac{2(1-t^*)}{2-3t^*} \right), \quad \text{which satisfies} \]
\[ (x - \frac{4}{3}) (y - \frac{4}{3}) = \frac{4}{9} \quad (*) \]

**Additional case**: special doubly degenerate case of \((x, y) = (0, 0) \Rightarrow \)

\( P_1 = P_0 \& P_2 = P_3 \Rightarrow \text{singular at } t = 0 \& t = 1 \)
Singularity of Bézier Curve of degree 3 – part 4a

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\[ O = \ell(P_0, P_3 - P_2) \cap \ell(P_3, P_1 - P_0) \]
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\[ \Delta P_0 \parallel (P_3 - O) \text{ and } \Delta P_2 \parallel (P_0 - O) \]
\[ \Delta P_0 = x(P_3 - O), \quad x = \frac{\det(\Delta P_2, \Delta P_0)}{\det(\Delta P_2, P_3 - P_0)} \] (1)
\[ \Delta P_2 = -y(P_0 - O), \quad y = \frac{\det(\Delta P_0, \Delta P_2)}{\det(\Delta P_0, P_3 - P_0)} \] (2)

with \((x, y)\) capturing the essential shape of the control polygon.

Under the Bézier singularity condition \( C[\{\Delta P\}](t^*) = \vec{0} \), (1), (2) \( \rightarrow \)

\[ (x, y) = \left( \frac{2t^*}{3t^* - 1}, \frac{2(1-t^*)}{2-3t^*} \right), \quad \text{which satisfies } \]
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Additional case: special doubly degenerate case of \((x, y) = (0, 0) \quad \Rightarrow \)

\[ P_1 = P_0 \text{ & } P_2 = P_3 \Rightarrow \text{singular at } t = 0 \text{ & } t = 1 \]

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Bézier Curve Singularity
Singularity of Bézier Curve of degree 3 – part 4b

**Summary** of main result

Define affine coordinates \((x, y)\) of the control polygon of a cubic Bézier curve by \(R - O = (P_3 - O)x + (P_0 - O)y\) ; see graph at bottom left.

The curve has a singularity at \(t = t^*\) iff

\[
(x, y) = \left(\frac{2t^*}{3t^*-1}, \frac{2(1-t^*)}{2-3t^*}\right),
\]

which satisfies

\[
(x - \frac{4}{3}) (y - \frac{4}{3}) = \frac{4}{9}
\]

or two singularities at \(t^* \in \{0, 1\}\), for the case \((x, y) = (0, 0)\).
Singularity of Bézier Curve of degree 3 – part 4b

Summary of main result

Define affine coordinates \((x, y)\) of the control polygon of a cubic Bézier curve by \(R - O = (P_3 - O)x + (P_0 - O)y\); see graph at bottom left.

The curve has a singularity at \(t = t^*\) iff

\[
(x, y) = \left(\frac{2t^*}{3t^*-1}, \frac{2(1-t^*)}{2-3t^*}\right),
\]

which satisfies

\[
(x - \frac{4}{3})(y - \frac{4}{3}) = \frac{4}{9}
\]

or two singularities at \(t^* \in \{0, 1\}\), for the case \((x, y) = (0, 0)\).
Singularity of Bézier Curve of degree 3 – part 4b

**Summary** of main result

Define affine coordinates \((x, y)\) of the control polygon of a cubic Bézier curve by \(R - O = (P_3 - O)x + (P_0 - O)y\); see graph at bottom left.

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\[
(x, y) = \left(\frac{2t^*}{3t^*-1}, \frac{2(1-t^*)}{2-3t^*}\right), \text{ which satisfies } \left(x - \frac{4}{3}\right) \left(y - \frac{4}{3}\right) = \frac{4}{9}
\]

or two singularities at \(t^* \in \{0, 1\}\), for the case \((x, y) = (0, 0)\).

The \(x\)-\(y\) hyperbola, with some values of \(t^*\):
Examples of singular solution

Singular cubic curves with various values of $t^* \in [0, 1]$, with $P_0, P_3$, and the directions of $P_1 - P_0$ and $P_3 - P_2$ fixed.

$t_{\text{sing}} = 0.5$
Examples of singular solution

Singular cubic curves with various values of $t^* \in [0, 1]$, with $P_0, P_3$, and the directions of $P_1 - P_0$ and $P_3 - P_2$ fixed.

$t_{\text{sing}} = 0.55$
Examples of singular solution

Singular cubic curves with various values of $t^* \in [0, 1]$, with $P_0, P_3$, and the directions of $P_1 - P_0$ and $P_3 - P_2$ fixed.
Examples of singular solution

Singular cubic curves with various values of \( t^* \in [0, 1] \), with \( P_0, P_3 \), and the directions of \( P_1 - P_0 \) and \( P_3 - P_2 \) fixed.

\[ t_{\text{sing}} = 0.65 \]
Examples of singular solution

Singular cubic curves with various values of $t^* \in [0, 1]$, with $P_0, P_3$, and the directions of $P_1 - P_0$ and $P_3 - P_2$ fixed.

$t_{\text{sing}} = 0.7$
**Examples** of singular solution

Singular cubic curves with various values of $t^* \in [0, 1]$, with $P_0, P_3$, and the directions of $P_1 - P_0$ and $P_3 - P_2$ fixed.

$t_{\text{sing}} = 0.75$
**Examples** of singular solution

Singular cubic curves with various values of $t^* \in [0, 1]$, with $P_0, P_3$, and the *directions* of $P_1 - P_0$ and $P_3 - P_2$ fixed.

$$t_{\text{sing}} = 0.8$$
**Examples of singular solution**

Singular cubic curves with various values of $t^* \in [0, 1]$, with $P_0, P_3$, and the *directions* of $P_1 - P_0$ and $P_3 - P_2$ fixed.

$$t_{\text{sing}} = 0.85$$
Examples of singular solution

Singular cubic curves with various values of $t^* \in [0, 1]$, with $P_0, P_3$, and the directions of $P_1 - P_0$ and $P_3 - P_2$ fixed.

$t_{\text{sing}} = 0.9$
Examples of singular solution

Singular cubic curves with various values of \( t^* \in [0, 1] \), with \( P_0, P_3 \), and the directions of \( P_1 - P_0 \) and \( P_3 - P_2 \) fixed.

\[ t_{\text{sing}} = 0.95 \]
Examples of singular solution

Singular cubic curves with various values of $t^* \in [0, 1]$, with $P_0, P_3$, and the directions of $P_1 - P_0$ and $P_3 - P_2$ fixed.
Examples of singular solution

Singular cubic curves with various values of $t^* \in [0, 1]$, with $P_0, P_3$, and the directions of $P_1 - P_0$ and $P_3 - P_2$ fixed.

$t_{\text{sing}} = 0$
Examples of singular solution

Singular cubic curves with various values of $t^* \in [0, 1]$, with $P_0, P_3$, and the directions of $P_1 - P_0$ and $P_3 - P_2$ fixed.

$t_{\text{sing}} = 0.05$
Examples of singular solution

Singular cubic curves with various values of $t^* \in [0, 1]$, with $P_0, P_3$, and the directions of $P_1 - P_0$ and $P_3 - P_2$ fixed.

t_{\text{sing}}=0.1
Examples of singular solution

Singular cubic curves with various values of \( t^* \in [0, 1] \), with \( P_0, P_3 \), and the directions of \( P_1 - P_0 \) and \( P_3 - P_2 \) fixed.

\[ t_{\text{sing}} = 0.15 \]
Examples of singular solution

Singular cubic curves with various values of $t^* \in [0, 1]$, with $P_0, P_3$, and the directions of $P_1 - P_0$ and $P_3 - P_2$ fixed.

$$t_{\text{sing}} = 0.2$$
Examples of singular solution

Singular cubic curves with various values of $t^* \in [0, 1]$, with $P_0, P_3$, and the directions of $P_1 - P_0$ and $P_3 - P_2$ fixed.
Examples of singular solution

Singular cubic curves with various values of $t^* \in [0, 1]$, with $P_0, P_3$, and the directions of $P_1 - P_0$ and $P_3 - P_2$ fixed.

![Diagram of Bézier curve with points $P_0$, $P_1$, $P_2$, $P_3$, and $O$.]
Examples of singular solution

Singular cubic curves with various values of $t^* \in [0, 1]$, with $P_0, P_3$, and the directions of $P_1 - P_0$ and $P_3 - P_2$ fixed.

$t_{\text{sing}} = 0.35$
Examples of singular solution

Singular cubic curves with various values of $t^* \in [0, 1]$, with $P_0, P_3$, and the directions of $P_1 - P_0$ and $P_3 - P_2$ fixed.

$t_{\text{sing}} = 0.4$
Examples of singular solution

Singular cubic curves with various values of $t^* \in [0, 1]$, with $P_0, P_3$, and the directions of $P_1 - P_0$ and $P_3 - P_2$ fixed.

$$t_{\text{sing}} = 0.45$$
Examples of singular solution

Singular cubic curves with various values of \( t^* \in [0, 1] \), with \( P_0, P_3 \), and the directions of \( P_1 - P_0 \) and \( P_3 - P_2 \) fixed.

\[ t_{\text{sing}} = 0.5 \]
Examples of cubic curves in $x$-$y$ space

A "tour" of cubic curves in other regions of $x$-$y$ space, again, with $P_0, P_3$, and the directions of $P_1 - P_0$ and $P_3 - P_2$ fixed.

$t_{\text{sing}} = 0.5$
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Examples of cubic curves in \(x\)-\(y\) space

A "tour" of cubic curves in other regions of \(x\)-\(y\) space, again, with \(P_0, P_3\), and the directions of \(P_1 - P_0\) and \(P_3 - P_2\) fixed.
Examples of cubic curves in $x$-$y$ space

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A "tour" of cubic curves in other regions of $x$-$y$ space, again, with $P_0, P_3$, and the directions of $P_1 - P_0$ and $P_3 - P_2$ fixed.

$t_{\text{sing}} = 0.5$

A complete description cubic curve shapes in $x$-$y$ space is given in [Su & Liu 1990].
Seen $t^* = 0 \implies P_0 = P_1$; $t^* = 1 \implies P_2 = P_3$.
For $t^* \in (0, 1)$, can also regard singularity as coincident end-control points (from e.g., Farin & Hansford 2000).

\[ C = C^- \cup C^+, \text{ with domains } [0, \hat{t}] \text{ & } [\hat{t}, 1], \]
control points \( \{P_i^-\} \text{ & } \{P_i^+\}, i = 0, 1, 2, 3, \) respectively,
with $P_3^- = P_0^+ = C(\hat{t})$

If $\hat{t} = t^*$, then, also, $P_2^- = P_3^-$ & $P_1^+ = P_0^+ \implies P_2^- = P_1^+$

de Casteljau algorithm, revisited:
**Subdivision:**
\[
\{P_i^-\} = \{P_0, P_{01}, P_{012}, P_{0123}\} \\
\{P_i^+\} = \{P_{0123}, P_{123}, P_{23}, P_3\} \\
P_{0123} = C(t^*)
\]

\[
P_2^- = P_1^+ \implies P_{012} = P_{123} : \\
[ (1 - t^*)P_{01} + t^* P_{12} ] = [ (1 - t^*)P_{12} + t^* P_{23} ]
\]
\[
\ldots \quad C[\{P\}]'(t^*) = 0
\]
Singularity of Bézier Curve of degree 3 – part 6a

Interval interior like endpoints – coincident end-control points

Seen $t^* = 0 \implies P_0 = P_1$; $t^* = 1 \implies P_2 = P_3$.

For $t^* \in (0, 1)$, can also regard singularity as coincident end-control points (from e.g., Farin & Hansford 2000).

$C = C^- \cup C^+$, with domains $[0, \hat{t}]$ & $[\hat{t}, 1]$, control points $\{P^-_i\}$ & $\{P^+_i\}, i = 0, 1, 2, 3$, respectively, with $P^-_3 = P^+_0 = C(\hat{t})$

If $\hat{t} = t^*$, then, also, $P^-_2 = P^-_3 \& P^+_1 = P^+_0 \implies P^-_2 = P^+_1$

de Casteljau algorithm, revisited:

Subdivision:

$\{P^-_i\} = \{P_0, P_{01}, P_{012}, P_{0123}\}$
$\{P^+_i\} = \{P_{0123}, P_{123}, P_{23}, P_3\}$
$P_{0123} = C(t^*)$
Singularity of Bézier Curve of degree 3 – part 6a

Interval **interior like endpoints** – coincident end-control points

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$C = C^- \cup C^+$, with domains $[0, \hat{t}]$ & $[\hat{t}, 1]$, control points $\{P^-_i\}$ & $\{P^+_i\}, i = 0, 1, 2, 3$, respectively, with $P^-_3 = P^+_0 = C(\hat{t})$

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de Casteljau algorithm, revisited:

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$\{P^-_i\} = \{P_0, P_{01}, P_{012}, P_{0123}\}$

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$P_{0123} = C(t^*)$
Singularity of Bézier Curve of degree 3 – part 6a

Interval **interior like endpoints** – coincident end-control points

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For $t^* \in (0, 1)$, can also regard singularity as coincident end-control points (from e.g., Farin & Hansford 2000).

$C = C^- \cup C^+$, with domains $[0, \hat{t}] \& [\hat{t}, 1]$, control points $\{P_i^-\}$ & $\{P_i^+\}$, $i = 0, 1, 2, 3$, respectively, with $P_3^- = P_0^+ = C(\hat{t})$

If $\hat{t} = t^*$, then, also, $P_2^- = P_3^- \& P_1^+ = P_0^+ \implies P_2^- = P_1^+$

de Casteljau algorithm, revisited:

**Subdivision:**

$\{P_i^-\} = \{P_0, P_{01}, P_{012}, P_{0123}\}$

$\{P_i^+\} = \{P_{0123}, P_{123}, P_{23}, P_3\}$

$P_{0123} = C(t^*)$

$P_2^- = P_1^+ \implies P_{012} = P_{123}$:

$[(1 - t^*)P_{01} + t^*P_{12}] = [(1 - t^*)P_{12} + t^*P_{23}]$

$\ldots$ $C[\{P\}](t^*) = \vec{0}$
Singularity of Bézier Curve of degree 3 – part 6a

Interval *interior like endpoints* – coincident end-control points

Seen $t^* = 0 \implies P_0 = P_1$; $t^* = 1 \implies P_2 = P_3$.

For $t^* \in (0, 1)$, can also regard singularity as coincident end-control points (from e.g., Farin & Hansford 2000).

$C = C^- \cup C^+$, with domains $[0, \hat{t}]$ & $[\hat{t}, 1]$, control points $\{P_i^-\}$ & $\{P_i^+\}$, $i = 0, 1, 2, 3$, respectively, with $P_3^- = P_0^+ = C(\hat{t})$

If $\hat{t} = t^*$, then, also, $P_2^- = P_3^-$ & $P_1^+ = P_0^+ \implies P_2^- = P_1^+$

de Casteljau algorithm, revisited:

**Subdivision:**

$\{P_i^-\} = \{P_0, P_{01}, P_{012}, P_{0123}\}$

$\{P_i^+\} = \{P_{0123}, P_{123}, P_{23}, P_3\}$

$P_{0123} = C(t^*)$

$P_2^- = P_1^+ \implies P_{012} = P_{123}$:

$[(1 - t^*)P_{01} + t^* P_{12}] = [(1 - t^*)P_{12} + t^* P_{23}]$

$\ldots \quad C[\{P\}]'(t^*) = \vec{0}$
Singularity of Bézier Curve of degree 3 – part 6a

Interval **interior like endpoints** – coincident end-control points

Seen $t^* = 0 \implies P_0 = P_1$; $t^* = 1 \implies P_2 = P_3$.

For $t^* \in (0, 1)$, can also regard singularity as coincident end-control points (from e.g., Farin & Hansford 2000).

$C = C^- \cup C^+$, with domains $[0, \hat{t}]$ & $[\hat{t}, 1]$,

control points $\{P_i^\pm\}, i = 0, 1, 2, 3$, respectively, with $P_3^- = P_0^+ = C(\hat{t})$

If $\hat{t} = t^*$, then, also, $P_2^- = P_3^-$ & $P_1^+ = P_0^+ \implies P_2^- = P_1^+$

de Casteljau algorithm, revisited:

**Subdivision:**

$\{P_i^-\} = \{P_0, P_{01}, P_{012}, P_{0123}\}$

$\{P_i^+\} = \{P_{0123}, P_{123}, P_{23}, P_3\}$

$P_{0123} = C(t^*)$

$P_2^- = P_1^+ \implies P_{012} = P_{123}$:

$[(1 - t^*)P_{01} + t^* P_{12}] = [(1 - t^*)P_{12} + t^* P_{23}]$

$\ldots \quad C[\{P\}]'(t^*) = \vec{0}$
Singularity of Bézier Curve of degree 3 – part 6a

Interval interior like endpoints – coincident end-control points

Seen \( t^* = 0 \implies P_0 = P_1 \); \( t^* = 1 \implies P_2 = P_3 \).

For \( t^* \in (0, 1) \), can also regard singularity as coincident end-control points (from e.g., Farin & Hansford 2000).

\[
C = C^- \cup C^+, \text{ with domains } [0, \hat{t}] \text{ & } [\hat{t}, 1],
\]

control points \( \{P_i^-\} \) & \( \{P_i^+\} \), \( i = 0, 1, 2, 3 \), respectively,
with \( P_3^- = P_0^+ = C(\hat{t}) \)

If \( \hat{t} = t^* \), then, also, \( P_2^- = P_3^- \) & \( P_1^+ = P_0^+ \implies P_2^- = P_1^+ \)

de Casteljau algorithm, revisited:

**Subdivision:**

\[
\{P_i^-\} = \{P_0, P_{01}, P_{012}, P_{0123}\}
\]

\[
\{P_i^+\} = \{P_{0123}, P_{123}, P_{23}, P_3\}
\]

\( P_{0123} = C(t^*) \)

\[
P_2^- = P_1^+ \implies P_{012} = P_{123}:
\[
[(1 - t^*)P_{01} + t^* P_{12}] = [(1 - t^*)P_{12} + t^* P_{23}]
\]

\[
\ldots \quad C[\{P\}]'(t^*) = 0
\]
**Example:** Interval interior like endpoints

Convergence of de Casteljau points $P_{012}$ and $P_{123}$ as $t \to t^*$:

$t_{\text{sing}} = 0.55$
Singularity of Bézier Curve of degree 3 – part 6b

**Example:** Interval interior like endpoints

Convergence of de Casteljau points $P_{012}$ and $P_{123}$ as $t \to t^*$:

$t=0.35$
Example: Interval interior like endpoints

Convergence of de Casteljau points $P_{012}$ and $P_{123}$ as $t \to t^*$:

$$t = 0.45$$
Example: Interval interior like endpoints

Convergence of de Casteljau points $P_{012}$ and $P_{123}$ as $t \to t^*$:

$t=0.5$
Example: Interval interior like endpoints

Convergence of de Casteljau points $P_{012}$ and $P_{123}$ as $t \to t^*$:

$t = 0.525$

The diagram shows the points $P_0$, $P_1$, $P_2$, and $P_3$ forming a Bézier curve, with intermediate points $P_{01}$, $P_{012}$, $P_{0123}$, $P_{12}$, and $P_{123}$ as $t$ approaches $t^*$. The convergence is illustrated as these points approach a specific point $C(0.525)$ on the curve.
Example: Interval interior like endpoints

Convergence of de Casteljau points $P_{012}$ and $P_{123}$ as $t \to t^*$:

$t=0.55$
Singularity of Bézier Curve of degree 3 – part 6c

Interval endpoints like interior – cusp

Seen how singularity in the interior of the parameter interval \([0, 1]\) is like one on the ends: characterized by coincident end control points (of the two control polygons of de Casteljau subdivision).

Also, singularity at ends is like one in the interior: exhibits a cusp.

\[ \ldots \text{if parameter interval is extended beyond} \, [0, 1]: \]
Singularity of Bézier Curve of degree 3 – part 6c

Interval **endpoints like interior** – cusp

Seen how singularity in the **interior** of the parameter interval $[0, 1]$ is like one on the **ends**: characterized by coincident end control points (of the two control polygons of de Casteljau subdivision).

Also, singularity at **ends** is like one in the **interior**: exhibits a **cusp**

... if parameter interval is extended beyond $[0, 1]$:
Seen how singularity in the *interior* of the parameter interval $[0, 1]$ is like one on the *ends*: characterized by coincident end control points (of the two control polygons of de Casteljau subdivision).

Also, singularity at *ends* is like one in the *interior*: exhibits a *cusp*.

... if parameter interval is extended *beyond* $[0, 1]$:
Interval **endpoints like interior** – cusp

Seen how singularity in the *interior* of the parameter interval $[0, 1]$ is like one on the *ends*: characterized by coincident end control points (of the two control polygons of de Casteljau subdivision)

Also, singularity at *ends* is like one in the *interior*: exhibits a *cusp* . . . if parameter interval is extended *beyond* $[0, 1]$: 

![Diagram showing singularity at parameter interval beyond [0, 1]](image-url)
Seen how singularity in the *interior* of the parameter interval \([0, 1]\) is like one on the *ends*: characterized by coincident end control points (of the two control polygons of de Casteljau subdivision)

Also, singularity at *ends* is like one in the *interior*: exhibits a *cusp*

\[ t_{\text{sing}} = 0.05 \]
Singularity of Bézier Curve of degree 3 – part 6c

Interval endpoints like interior – cusp

Seen how singularity in the interior of the parameter interval $[0, 1]$ is like one on the ends: characterized by coincident end control points (of the two control polygons of de Casteljau subdivision)

Also, singularity at ends is like one in the interior: exhibits a cusp

... if parameter interval is extended beyond $[0, 1]$:
Singularity of Bézier Curve of degree 3 – part 6c

Interval *endpoints like interior* – cusp

Seen how singularity in the *interior* of the parameter interval [0, 1] is like one on the *ends*: characterized by coincident end control points (of the two control polygons of de Casteljau subdivision)

Also, singularity at *ends* is like one in the *interior*: exhibits a *cusp* . . . if parameter interval is extended *beyond* [0, 1]:

\[ t_{\text{sing}}=0.2 \]
Interval *endpoints like interior* – *cusp*

Seen how singularity in the *interior* of the parameter interval $[0, 1]$ is like one on the *ends*: characterized by coincident end control points (of the two control polygons of de Casteljau subdivision).

Also, singularity at *ends* is like one in the *interior*: exhibits a *cusp*.

... if parameter interval is extended *beyond* $[0, 1]$:

$$t_{\text{sing}} = 0.3$$
Interval endpoints like interior – cusp

Seen how singularity in the interior of the parameter interval $[0, 1]$ is like one on the ends: characterized by coincident end control points (of the two control polygons of de Casteljau subdivision)

Also, singularity at ends is like one in the interior: exhibits a cusp

... if parameter interval is extended beyond $[0, 1]$:

\[
t_{\text{sing}} = 0.4
\]

![Diagram of Bézier curve singularity](image-url)
Singularity of Bézier Curve of degree 3 – part 6c

Interval **endpoints like interior** – cusp

Seen how singularity in the *interior* of the parameter interval $[0, 1]$ is like one on the *ends*: characterized by coincident end control points (of the two control polygons of de Casteljau subdivision).

Also, singularity at *ends* is like one in the *interior*: exhibits a cusp.

... if parameter interval is extended *beyond* $[0, 1]$: 

\[ t_{\text{sing}} = 0.5 \]
Singularity of Bézier Curve of degree 3 – part 6c

Interval endpoints like interior – cusp

Seen how singularity in the interior of the parameter interval [0, 1] is like one on the ends: characterized by coincident end control points (of the two control polygons of de Casteljau subdivision)

Also, singularity at ends is like one in the interior: exhibits a cusp

... if parameter interval is extended beyond [0, 1]:

\[ t_{\text{sing}}=0.6 \]
Interval **endpoints like interior** – cusp

Seen how singularity in the **interior** of the parameter interval $[0, 1]$ is like one on the **ends**: characterized by coincident end control points (of the two control polygons of de Casteljau subdivision)

Also, singularity at **ends** is like one in the **interior**: exhibits a **cusp**

\[ t_{\text{sing}} = 0.7 \]

\[ P_0 \]
\[ P_1 \]
\[ P_2 \]
\[ P_3 \]
Singularity of Bézier Curve of degree 3 – part 6c

Interval **endpoints like interior** – **cusp**

Seen how singularity in the *interior* of the parameter interval \([0, 1]\) is like one on the *ends*: characterized by coincident end control points (of the two control polygons of de Casteljau subdivision)

Also, singularity at *ends* is like one in the *interior*: exhibits a **cusp**

... if parameter interval is extended *beyond* \([0, 1]\):

\[
t_{\text{sing}} = 0.8
\]
Singularity of Bézier Curve of degree 3 – part 6c

Interval **endpoints like interior** – cusp

Seen how singularity in the *interior* of the parameter interval $[0, 1]$ is like one on the *ends*: characterized by coincident end control points (of the two control polygons of de Casteljau subdivision)

Also, singularity at *ends* is like one in the *interior*: exhibits a *cusp*... if parameter interval is extended *beyond* $[0, 1]$:  

$t_{\text{sing}} = 0.9$
Singularity of Bézier Curve of degree 3 – part 6c

Interval endpoints like interior – cusp

Seen how singularity in the interior of the parameter interval \([0, 1]\) is like one on the ends: characterized by coincident end control points (of the two control polygons of de Casteljau subdivision)

Also, singularity at ends is like one in the interior: exhibits a cusp

\[
\text{if parameter interval is extended beyond } [0, 1]:
\]

\[
t_{\text{sing}}=0.95
\]
Seen how singularity in the *interior* of the parameter interval 
\([0, 1]\) is like one on the *ends*: characterized by coincident end
control points (of the two control polygons of de Casteljau subdivision).

Also, singularity at *ends* is like one in the *interior*: exhibits a *cusp*

\[ \text{if parameter interval is extended beyond} \ [0, 1]: \]

\[ t_{\text{sing}}=1. \]
Summary

- Parametric polynomial curves of degree 3 are useful.
- Need to understand their singularities
- Bézier form is the best way to represent a parametric polynomial curve.
- Use Bézier form to describe singularities of parametric polynomial curves of degrees 1, 2, 3.

Current and future related work
- Curvature of curves
- Singularity of surfaces
- $G^1$ surface fitting in the presence of T-junction
For Further Reading I

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