1. Introduction

This paper provides an axiomatic foundation for a maxmin expected utility over a set of priors (MMEU) decision rule in an environment where the elements of choice are Savage (1954) acts. This characterization complements the original axiomatization of MMEU developed in a lottery-acts (or Anscombe and Aumann, 1963) framework by Gilboa and Schmeidler (1989). MMEU preferences are of interest primarily because they provide a natural and tractable way of modeling decision makers who display an aversion to uncertainty or ambiguity. In Casadesus-Masanell et al. (1999)CaKlOz99, we characterized an MMEU rule over Savage acts using axioms stated in terms of standard sequences, a measurement theory construction (see Krantz et al., 1971KrLuSuTv71). In contrast, the key axioms here involve statewise combinations of acts (as defined below). Statewise combinations have the advantage that they look much like the convex combinations used in Anscombe-Aumann and von Neumann-Morgenstern style theories. Thus, as in Gul (1992)Gu92, they allow one to transfer much of the intuition from these settings to a setting with Savage acts. The disadvantage of this approach is that it is less general than the one in Casadesus-Masanell et al. (1999)CaKlOz99: the theory here will imply the existence of a binary partition of the state space over which subjective expected utility holds.

The remainder of the paper presents a set of axioms and a theorem proving the equivalence between these axioms and an MMEU rule. The novel axiom is a weakening of Gul's (1992, assumption 2) act-independence condition. Results in Nakamura (1990)Na90 and Gilboa and Schmeidler (1989)GiSc89 are useful for the proofs.

2. Notation and framework

Ω is the *set of states.* A *state* in Ω is represented by ω. Σ is an algebra of subsets of Ω. *Events* are elements of Σ. $X = [m, M] \subset \mathbb{R}$, $m < M$ is the set of *prizes* or outcomes. A (Savage) *act* f is a function $f : \Omega \to X$. A *simple act* is an act with only finitely many distinct values. A simple act is Σ-measurable if $\{\omega \in \Omega \mid f(\omega) \in W\} \in \Sigma$ for all $W \subseteq X$. F is the set of all Σ-measurable acts defined as the closure in the supnorm of all Σ-measurable simple acts. A set $G \subseteq F$ is *closed* if it is closed in the supnorm. A *constant act* f is one for which $f(\omega) = x$ for all $\omega \in \Omega$, for some $x \in X$; we denote this constant act by x^* or simply x when no confusion would result. F^* is the subset of F consisting of all constant acts. For any event $B \in \Sigma$ and x, $y \in X$, $x_B y$ denotes $f \in F$ such that $f(\omega) = x$ for $\omega \in B$ and $f(\omega) = y$ for $\omega \notin B$; such acts are referred to as B-measurable. The event $\Omega - B$ is denoted B^c. For $f, g, h \in F$ and $B \in \Sigma$, if $h(\omega) \sim f(\omega)_B g(\omega)$ for all $\omega \in \Omega$ then h is a *statewise combination of f and g over the event B.* $\mathcal{P}$ is the set of all finitely additive probability measures $P : \Sigma \to [0, 1]$. Finally, $\succeq$ is a binary relation on F. Note that this environment is similar to that in Savage (1954)Sa54 with the difference that we impose more structure on the prize set (X). The important aspect of this structure is that X is connected and separable. This will allow the non-singleton set of states in our theory to be of any size, finite or infinite.

3. Axioms

Axiom 1. ***(Weak order)*** *$\succeq$ is complete and transitive.*

Definition 3.1. *An event B is ordered non-null if there exist x, y and z in X with $x \preceq y \preceq z$ such that $x_B z \nsim y_B z$. An event B is ordered non-universal if there exist x, y and z in X with $z \preceq y \preceq x$ such that $z_B x \nsim z_B y$.*

Note that since we impose restrictions on the ordering of x, y and z, our definitions of non-null and non-universal (borrowed from Nakamura, 1990Na90) are weaker than the corresponding notions in Savage (1954)Sa54. See Casadesus-Masanell et al. (1999)CaKlOz99 for an explanation of why these are the appropriate notions.

Axiom 2. ***(Structure)****(a) $x > y \Rightarrow x^* \succ y^*$. (b) There exists an event $A \in \Sigma$ such that A and A^c are ordered non-null and ordered non-universal.*

Part *(a)* is purely a simplifying assumption.

Axiom 3. ***(Continuity)*** *For all $f \in F$, the sets $M(f) = \{g \in F | \, g \succeq f\}$ and $W(f) = \{g \in F | \, f \succeq g\}$ are closed.*

Axiom 4. ***(Monotonicity)****(a) For all f, $g \in F$, if $f(\omega) \succeq g(\omega)$, for all $\omega \in \Omega$ then $f \succeq g$. (b) If $B \in \Sigma$ is ordered non-null and $z \succeq x \succ y$, then $x_B z \succ y_B z$. If $B \in \Sigma$ is ordered non-universal and $x \succ y \succeq z$, then $z_B x \succ z_B y$.*

Observe that part *(a)* is weak monotonicity, and part *(b)* is strict monotonicity on ordered non-null and non-universal events.

Axiom 5. ***(A-act-independence)*** *Let x_1, x_2, y_1, y_2, z_1 and $z_2 \in X$ and let $f = x_{1A} x_2$, $g = y_{1A} y_2$ and $h = z_{1A} z_2$. If f', $g' \in F$ are such that, for either $B = A$ or $B = A^c$, $f'(\omega) \sim h(\omega)_B \, f(\omega)$ and $g'(\omega) \sim h(\omega)_B \, g(\omega)$ for all $\omega \in \Omega$ then, $f \succeq g \Leftrightarrow f' \succeq g'$.*

This axiom imposes act-independence (as introduced in Gul, 1992Gu92) only for A-measurable acts and the events A and A^c. In words, given A-measurable acts f, g and h and given the event $B = A$ or $B = A^c$, if f' is a statewise combination of h and f over the event B and g' is a statewise combination of h and g over the event B, then preference between f and g is the same as between f' and g'. As discussed in Gul (1992)Gu92, act-independence is analogous to the independence axiom in the theory of expected utility over lotteries.

These first five axioms guarantee the existence of an expected utility representation for A-measurable acts. That is, there exists a strictly increasing and continuous function $u : X \to \mathbb{R}$ and $\rho \in (0,1)$ such that if x, y, v, $w \in X$ then

$$x_A y \succeq v_A w \Leftrightarrow \rho u(x) + (1-\rho) u(y) \geq \rho u(v) + (1-\rho) u(w).$$

Moreover, u is unique up to positive affine transformations and ρ is unique.

How do preferences extend from A-measurable acts to all acts? If act-independence is required to hold for all acts and non-null events, expected utility preferences result (see Gul, 1992Gu92 and Chew and Karni, 1994ChKa94). This act-independence is too strong for MMEU as it is incompatible, for example, with the Ellsberg Paradox (Ellsberg, 1961El61). We now develop the appropriate weakening of act-independence.

Definition 3.2. *(Ghirardato et al., 1998GhKlMa98) Two acts f and g are affinely related if there exist $\alpha \geq 0$ and $\beta \in \mathbb{R}$ such that either $u(f(\omega)) = \alpha u(g(\omega)) + \beta$ for all $\omega \in \Omega$ or $u(g(\omega)) = \alpha u(f(\omega)) + \beta$ for all $\omega \in \Omega$.*

The key is the use of statewise combinations over the event A to form what will turn out to be sets of affinely related acts. This requires the following definitions:

Definition 3.3. *A set $S \subset F$ contains all statewise combinations over the event A if f_1, $f_2 \in S$, and for all $\omega \in \Omega$, $f(\omega) \sim f_1(\omega)_A f_2(\omega)$ implies $f \in S$.*

Definition 3.4. *Fix $f \in F$. We define $S_0^f = \{f\} \cup F^*$. Let $\overline{S}^f \supseteq S_0^f$ be the smallest closed set containing all statewise combinations over the event A.*

It can be shown that given an act f, the set $\overline{S}^f$ may be constructed by the following iterative method: at each step $i = 1, 2, 3, ...$ we produce a set

$$S_i^f = \left\{ f^i \in F : f^i(\omega) \sim f_1^{i-1}(\omega)_A f_2^{i-1}(\omega), \text{ for all } \omega \in \Omega \text{ where } f_1^{i-1}, f_2^{i-1} \in S_{i-1}^f \right\}.$$

Finally, $\overline{S}^f$ is the closure of $\cup_{i=1}^{\infty} S_i^f \subseteq F$. Observe that S_1^f consists of statewise combinations over the event A of either f or a constant act with either f or a constant act. By the expected utility representation for A-measurable acts, we end up with either a constant utility act or a positive affine transformation of the state-by-state utility of f. Either way the resulting acts are affinely related to f. To form S_2^f, we take any two acts in S_1^f and combine them statewise over A. Since both of these acts are affinely related to the act f, the resulting act will also be affinely related to f. This argument plus continuity shows that $\overline{S}^f$ consists only of acts that are affinely related to f. In fact, if f is not a constant act, the set $\overline{S}^f$ contains *all* acts that are affinely related to f.

Axiom 6. ***($\overline{S}$-act-independence)*** *For any f, g, $h \in F$ such that there exist l, $k \in \{f, g\}$ for which f, $h \in \overline{S}^l$ and g, $h \in \overline{S}^k$, if f', $g' \in F$ are such that, for either $B = A$ or $B = A^c$, $f'(\omega) \sim h(\omega)_B f(\omega)$ and $g'(\omega) \sim h(\omega)_B g(\omega)$ for all $\omega \in \Omega$ then, $f \succeq g \Leftrightarrow f' \succeq g'$.*

Using the representation for A-measurable acts and the definition of $\overline{S}^f$, it can be shown that the axiom applies only if: (1) h is a constant act; or (2) h is not a constant act, but f, g and h are pairwise affinely related.

It is useful to compare $\overline{S}$-act-independence to the certainty-independence (C-independence) axiom of Gilboa and Schmeidler (1989) GiSc89:

(C-independence) *For any acts f and g, any constant act h, and any $\alpha \in (0,1)$, if f', g' are such that, $f' = \alpha f + (1-\alpha)h$ and $g' = \alpha g + (1-\alpha)h$ then, $f \succeq g \Leftrightarrow f' \succeq g'$.*

Note that the convex combination operation is defined statewise and is well-defined since acts in their setting are functions from states to probability distributions over prizes. C-independence relaxes the independence axiom of Anscombe and Aumann (1963)AnAu63 so that it is only required to hold when the third act, h, is a constant act.

$\overline{S}$-act-independence and C-independence are quite similar in form, with two salient differences. First, statewise combinations over A or A^c replace convex combinations. Second, as pointed out in possibility (2) above, $\overline{S}$-act-independence applies to some h which are not constant acts. In fact, the first difference leads to the second one. In an Anscombe-Aumann framework, all probabilities in the unit interval are available. Consequently, to express the fact that preference is preserved by homogeneous transformations (of utility), one need only consider convex combinations of the act in question and a constant act. In contrast, the only probabilities that are available through statewise combinations over A are those of A and A^c. For example, if the revealed probability of A happens to be $\frac{1}{3}$ and we want to show that multiplying the utility of a pair of acts by $\frac{1}{2}$ preserves the preference ordering between them, we cannot construct the "$\frac{1}{2}$-acts" through statewise combinations over A without taking combinations of two non-constant acts. This is why we cannot restrict h to be a constant act in the $\overline{S}$-act-independence axiom.

The final axiom, act-uncertainty aversion, restricts the way that act-independence can be violated. It requires that the decision-maker weakly likes to smooth utilities across states of the world, since this leaves her less exposed to any uncertainty or ambiguity about the probability of various states. Specifically it modifies Gilboa and Schmeidler's (1989)GiSc89 uncertainty aversion axiom by replacing convex combinations with statewise combinations over A.

Axiom 7. ***(Act-uncertainty aversion)*** *For all $f, g, f' \in F$, if $f \sim g$ and $f'(\omega) \sim f(\omega)_A\, g(\omega)$ for all $\omega \in \Omega$ then, $f' \succeq f$.*

See Casadesus-Masanell et al. (1999)CaKlOz99 for a discussion relating this type of uncertainty aversion axiom to recent alternatives suggested by Epstein (1999) and Ghirardato and Marinacci (1998)GhMa98.

4. A Representation Theorem

Theorem 4.1. *Let $\succeq$ be a binary relation on F. Then $\succeq$ satisfies Axioms 1-7 if and only if there exists a continuous and strictly increasing function $u : X \to \mathbb{R}$, and a non-empty, compact and convex set $\mathcal{C}$ of finitely additive probability measures on Σ such that*

$$[f \succeq g] \Leftrightarrow \left[\min_{P \in \mathcal{C}} \int u \circ f dP \geq \min_{P \in \mathcal{C}} \int u \circ g dP\right] \text{ for all } f \text{ and } g \in F.$$

Furthermore, there exists an event $A \in \Sigma$ and a $\rho \in (0,1)$ such that $P \in \mathcal{C}$ implies $P(A) = \rho$. Moreover, u is unique up to positive affine transformations and the set $\mathcal{C}$ is unique.

Proof. We sketch sufficiency; necessity is omitted. Axioms 1-5 imply versions of the appropriate axioms of Nakamura (1990)Na90 using events A and A^c. Apply Nakamura's theorem 1 to yield an expected utility representation for A-measurable acts. The continuity axiom and structure axiom part (a) guarantee that u is continuous and strictly increasing. Let ρ be the probability of A. Let $K = u(X)$. We normalize u such that $K = [-2, 2]$.

We now construct a functional $J : F \to \mathbb{R}$ that represents preferences. For any constant act $f = x \in X$ we define $J(f) = u(x)$. For general acts $f \in F$, let $J(f) = u(c)$, where c is the certainty equivalent of f. Continuity ensures that c exists.

Let B be the space of bounded (in the supnorm), Σ-measurable, real valued functions on Ω. For $\gamma \in \mathbb{R}$, we denote by γ^* the element of B that assigns γ to every ω. Let $B(K)$ be the subset of functions in B with values in K. Observe that for $f \in F$, $u \circ f \in B(K)$, and for $d \in B(K)$ there exists $f \in F$ such that $u \circ f = d$.

Define the functional $I : B(K) \to \mathbb{R}$ by $I(u \circ f) = J(f)$. Since J represents preferences, it is clear that I does as well. It can be shown that an extension of I to B satisfies the following properties:

(i) $I(1^*) = 1$;

(ii) (I is monotonic) For all $a, b \in B$, $a \geq b$ implies $I(a) \geq I(b)$;

(iii) (I is homogeneous of degree 1) For all $b \in B$, $\alpha \geq 0$, $I(\alpha b) = \alpha I(b)$;

(iv) (I is C-independent) For all $b \in B$, $\gamma \in \mathbb{R}$, $I(b + \gamma^*) = I(b) + I(\gamma^*)$; and,

(v) (I is superadditive) For all $a, b \in B$, $I(a + b) \geq I(a) + I(b)$.

Properties (i) and (ii) on $B(K)$ are immediate. The proof of (iii) on $B(K)$ uses $\overline{S}$-act-independence heavily. Then extend I to all of B by homogeneity. This preserves homogeneity and monotonicity. Now (iv) and (v) can be shown for the extension of I by arguments similar to those in Gilboa and Schmeidler (1989). GiSc89

Properties (i) - (v) imply the existence of the minimum expectation over a set of measures representation. This is a fundamental lemma variations of which have been proved by, for example, Gilboa and Schmeidler (1989)GiSc89, Chateauneuf (1991)Ch91, and Marinacci (1997)Ma97a. ■

References

Anscombe, F.J. and R.J. Aumann, 1963, A definition of subjective probability, The Annals of Mathematical Statistics 34, 199–205.

Casadesus-Masanell, R., P. Klibanoff, and E. Ozdenoren, 1999, Maxmin expected utility over Savage acts with a set of priors, mimeo, Northwestern University.

Chateauneuf, A., 1991, On the use of capacities in modeling uncertainty aversion and risk aversion, Journal of Mathematical Economics 20, 343–369.

Chew, S.H. and E. Karni, 1994, Choquet expected utility with a finite state space: commutativity and act-independence, Journal of Economic Theory 62, 469–479.

Ellsberg, D., 1961, Risk, ambiguity, and the Savage axioms, Quarterly Journal of Economics 75, 643–669.

Epstein, L., 1999, A definition of uncertainty aversion, The Review of Economic Studies, forthcoming.

Ghirardato, P., P. Klibanoff, and M. Marinacci, 1998, Additivity with multiple priors, Journal of Mathematical Economics 30, 405–420.

Ghirardato, P. and M. Marinacci, 1998, Ambiguity made precise: a comparative foundation, Social Science Working Paper 1026, California Institute of Technology.

Gilboa, I. and D. Schmeidler, 1989, Maxmin expected utility with non-unique prior, Journal of Mathematical Economics 18, 141–153.

Gul, F., 1992, Savage's theorem with a finite number of states, Journal of Economic Theory 57, 99–110.

Krantz, D.H., R.D. Luce, P. Suppes, and A. Tversky, 1971, Foundations of measurement, volume I (Academic Press, New York).

Marinacci, M., 1997, A simple proof of a basic result for multiple priors, mimeo, U. Toronto.

Nakamura, Y., 1990, Subjective expected utility with non-additive probabilities on finite state spaces, Journal of Economic Theory 51, 346–366.

Savage, L.J., 1954, The foundations of statistics (Wiley, New York).

Extended exposition of theorem and proof for "Maxmin expected utility through statewise combinations" by Ramon Casadesus-Masanell, Peter Klibanoff and Emre Ozdenoren

5. Restatement of theorem

Theorem 5.1. *Let $\succeq$ be a binary relation on F. Then $\succeq$ satisfies Axioms 1-7 if and only if there exists a continuous and strictly increasing function $u : X \to \mathbb{R}$, and a non-empty, compact and convex set $\mathcal{C}$ of finitely additive probability measures on Σ such that*

$$[f \succeq g] \Leftrightarrow \left[\min_{P\in\mathcal{C}} \int u \circ f dP \geq \min_{P\in\mathcal{C}} \int u \circ g dP\right] \text{ for all } f \text{ and } g \in F.$$

Furthermore, there exists an event $A \in \Sigma$ and a $\rho \in (0, 1)$ such that $P \in \mathcal{C}$ implies $P(A) = \rho$. Moreover, u is unique up to positive affine transformations and the set $\mathcal{C}$ is unique.

6. Proof of theorem

In section 6.1 we prove that the axioms are sufficient and, in section 6.2, that the axioms are necessary. In section 6.3, we show that if f is not a constant act, $\overline{S}^f$ contains all acts affinely related to f, as claimed in the paper.

6.1. Sufficiency

Given any $f \in F$, denote its certainty equivalent by $m(f)$. That is, $m(f)$ is an element in F^* such that $m(f) \sim f$. For $x_1, x_2 \in X$, $B \in \Sigma$, define $m^B(x_1, x_2) \equiv m(x_{1B}x_2)$. The next lemma shows that a unique certainty equivalent exists for each act.

Lemma 6.1. *For each $f \in F$, $m(f)$ exists and is unique.*

Proof. By Axiom 4 (monotonicity), part(a), the sets $M(f)_c := \{x \in X | x \succeq f\}$ and $W(f)_c := \{x \in X | f \succeq x\}$ are non-empty. By continuity, both of these sets are closed. By weak order, $M(f)_c \cup W(f)_c = X$. Therefore, since X is connected, $M(f)_c \cap W(f)_c \neq \emptyset$ and there must be at least one $x \in X$ such that $x \sim f$. Suppose there are two such prizes, $x_1 < x_2$. Then, by the structure axiom, part (a), and weak order, $x_1 \prec x_2 \sim f$, a contradiction. ■

Definition 6.2. *We say that $\succeq$ is bounded if, for each $f \in F$, there are $x, y \in X$ such that $x \succeq f \succeq y$.*

Definition 6.3. *Given $B \in \Sigma$ which is ordered non-null and ordered non-universal and a and $b \in X$ with $a \succ b$, we define a standard sequence with respect to a and b using event B, as a sequence $\{a_1, a_2, a_3, \ldots\}$ where $a_i \in X$, for which either (i) $a \preceq a_i$ and $a_B a_i \sim b_B a_{i+1}$ for all i such that a_i is not the last element in the sequence; or (ii) $a_i \preceq b$ and $a_{iB} b \sim a_{i+1B} a$ for all i such that a_i is not the last element in the sequence.*

Consider the following axioms following Nakamura [3]:

A1. $\preceq$ on F is a bounded weak order.

A2. For $f \in F$, $B \in \Sigma$ and $x, y, z \in X$, if $x_B z \preceq f \preceq y_B z$ then $f \sim a_B z$ for some $a \in X$.

A3. If $B \in \Sigma$ is ordered non-null and $x \preceq z$ and $y \preceq z$, then $x \preceq y$ if and only if $x_B z \preceq y_B z$; if $B \in \Sigma$ is ordered non-universal and $z \preceq x$ and $z \preceq y$, then $x \preceq y$ if and only if $z_B x \preceq z_B y$.

A4. If $x \preceq y$ and $B \subseteq C$ then $x_C y \preceq x_B y$.

A5. Every strictly bounded standard sequence is finite.

A6. If $x_1 \preceq x_2$ and $y_1 \preceq y_2$ with $x_i \preceq y_i$ for $i = 1, 2$, then

$$m^B (x_1, x_2)_B \, m^B (y_1, y_2) \sim m^B (x_1, y_1)_B \, m^B (x_2, y_2) .$$

The next two lemmas show that our axioms for the general case imply Nakamura's A1-A5 and A6 with $B = A$. This allows us to use several lemmas from Nakamura [3] to show that an expected utility representation for ordered A-measurable acts exists.

Lemma 6.4. *Axioms weak order, structure, continuity, monotonicity and ordered A-act-independence imply A1-A5.*

Proof.

- (Axioms⇒A1) This is implied by weak order and monotonicity.

- (Axioms⇒A2) Consider $W(f)_B = \{b \in X \mid b_B z \preceq f\}$ and $M(f)_B = \{b \in X \mid f \preceq b_B z\}$. $W(f)_B$ and $M(f)_B$ are non-empty because $x \in W(f)_B$ and $y \in M(f)_B$ by assumption. We want to show that $W(f)_B$ is closed. To do this we will show that $X - W(f)_B$ is open, where $X - W(f)_B = \{b \in X \mid b_B z \succ f\}$. Let $W(f)^c = \{g \in F \mid g \succ f\}$. Let t be in $X - W(f)_B$. (If such t does not exist then $X - W(f)_B = \emptyset$ and $W(f)_B = [m, M]$ which is closed, so we are done.) Since $t \in X - W(f)_B$, we have that $t_B z \in W(f)^c$. By continuity, we have that $W(f)^c$ is open. Hence $\exists \delta > 0$ such that $\forall g \in U_\delta(t_B z)$ we have that $g \in W(f)^c$, where $U_\delta(t_B z) = \{g \in F \mid \| g - t_B z \| < \delta\}$ ($\| \cdot \|$ denotes the supnorm). Consider now the set $V_\delta(t_B z) = \{b \in X \mid \| b_B z - t_B z \| < \delta\}$. Note that $\{g \in F \mid g = b_B z$ for some $b \in V_\delta(t_B z)\} \subset U_\delta(t_B z)$. Since $\forall g \in U_\delta(t_B z)$ we have that $g \succ f$, we must have that $\forall b \in V_\delta(t_B z)$, $b_B z \succ f$, implying that $X - W(f)_B$ is open. Hence $W(f)_B$ is closed. Similarly $M(f)_B$ is closed. By weak order, $W(f)_B \cup M(f)_B = X$. Since X is connected, $W(f)_B \cap M(f)_B \neq \emptyset$. Therefore there exists $a \in X$ such that $a_B z \sim f$.

- (Axioms⇒A3) This follows from Axiom 4 (monotonicity).

- (Axioms⇒A4) This follows from Axiom 4 (monotonicity), part (a).

- (Axioms⇒A5) Let $B \in \Sigma$ be ordered non-null and ordered non-universal and fix $a, b \in X$ such that $a \succ b$. Let $\{a_i\}$ be a strictly bounded standard sequence with respect to a and b using event B. Let's assume first that $\{a_i\}$ is such that $a \preceq a_i$ and $a_B a_i \sim b_B a_{i+1}$ for all i. Since B is ordered non-null and $b \prec a \preceq a_{i+1}$, monotonicity implies $a_B a_i \sim b_B a_{i+1} \prec a_B a_{i+1}$. Thus, $a_i \prec a_{i+1}$ for all i and $\{a_i\}$ is an increasing sequence. As $\{a_i\}$ is strictly bounded it must in particular be bounded above. Hence, if $\{a_i\}$ has infinitely many terms then it must converge. Towards a contradiction, assume $\{a_i\}$ has infinitely many terms. Let $a^* = \lim_{i\to\infty} a_i$. Consider $W(b_B a^*)$. Since for each i, $a_B a_i \sim b_B a_{i+1} \prec b_B a^*$, we have that for each i, $a_B a_i \in W(b_B a^*)$. Take limits to obtain $\lim_{i\to\infty} a_B a_i = a_B a^*$. By continuity, $a_B a^* \in W(b_B a^*)$ (i.e., $a_B a^* \preceq b_B a^*$). Now, B ordered non-null, $b \prec a$ and $\{b, a\} \preceq a^*$, together with the monotonicity axiom, implies $b_B a^* \prec a_B a^*$, which is a contradiction. So $\{a_i\}$ must have only a finite number of terms. The other case, where $\{a_i\}$ is such that $b \succeq a_i$ and $a_{iB} b \sim a_{i+1B} a$ for all i, follows from a similar argument using the fact that B is ordered non-universal and $\{a_i\}$ is decreasing and bounded below. Note that Nakamura's definition of a standard sequence also allows for decreasing $\{a_i\}$ when $a \preceq a_i$ and increasing $\{a_i\}$ when $b \succeq a_i$. It is easy to adapt the arguments just given above to show that $\{a_i\}$ must be finite in these cases as well. ■

Lemma 6.5. *Axioms weak order, structure, continuity, monotonicity and ordered A-act-independence imply that for all x_1, x_2, y_1, and $y_2 \in X$ for B is either A or A^c*

$$m^B(x_1, x_2)_B\, m^B(y_1, y_2) \sim m^B(x_1, y_1)_B\, m^B(x_2, y_2).$$

Proof. We show the result for $B = A$. The result for $B = A^c$ follows from a similar argument. The argument requires two applications of ordered A-act-independence.

Let f, g and $h \in F$ be such that:

$$\begin{aligned} f &= y_{1A} y_2 \\ g &= m^A(y_1, y_2) \\ h &= x_{1A} x_2. \end{aligned}$$

Consider f' and $g' \in F$ with, for all $\omega \in \Omega$,

$$\begin{aligned} f'(\omega) &\sim h(\omega)_A f(\omega) = \begin{cases} x_{1A} y_1 & , \omega \in A \\ x_{2A} y_2 & , \omega \in A^c \end{cases} \\ g'(\omega) &\sim h(\omega)_A g(\omega) = \begin{cases} x_{1A} m^A(y_1, y_2) & , \omega \in A \\ x_{2A} m^A(y_1, y_2) & , \omega \in A^c \end{cases}. \end{aligned}$$

Since $f \sim g$, ordered A-act-independence implies $f' \sim g'$. That is,

$$m^A(x_1, y_1)_A\, m^A(x_2, y_2) \sim m^A\left(x_1, m^A(y_1, y_2)\right)_A m^A\left(x_2, m^A(y_1, y_2)\right).$$

Now, let $\widehat{f}$, $\widehat{g}$ and $\widehat{h} \in F$ be such that:

$$\begin{aligned} \widehat{f} &= x_{1A} x_2 \\ \widehat{g} &= m^A(x_1, x_2) \\ \widehat{h} &= m^A(y_1, y_2). \end{aligned}$$

Consider $\widehat{f}'$ and $\widehat{g}' \in F$ with, for all $\omega \in \Omega$,

$$\begin{array}{rcl} \widehat{f}'(\omega) & \sim & \widehat{h}(\omega)_A \widehat{f}(\omega) = \begin{cases} m^A(y_1, y_2)_{A^c} x_1 & , \omega \in A \\ m^A(y_1, y_2)_{A^c} x_2 & , \omega \in A^c \end{cases} \\ \widehat{g}'(\omega) & \sim & \widehat{h}(\omega)_A \widehat{g}(\omega) = m^A(y_1, y_2)_{A^c} m^A(x_1, x_2). \end{array}$$

Since $\widehat{f} \sim \widehat{g}$, ordered A-act-independence implies $\widehat{f}' \sim \widehat{g}'$. Hence,

$$m^A\left(x_1, m^A(y_1, y_2)\right)_A m^A\left(x_2, m^A(y_1, y_2)\right) \sim m^A(x_1, x_2)_A m^A(y_1, y_2).$$

It follows that

$$m^A(x_1, x_2)_A m^A(y_1, y_2) \sim m^A(x_1, y_1)_A m^A(x_2, y_2).$$

■

Lemma 6.6. *There is a strictly increasing, continuous function $u : X \to \mathbb{R}$ and real numbers $\pi(A)$, $\pi(A^c) \in (0,1)$ with $\pi(A) = 1 - \pi(A^c)$ such that for all x, y, v, $w \in X$,*

$$\begin{array}{rl} & x_A y \succeq v_A w \\ \Leftrightarrow & \pi(A) u(x) + (1 - \pi(A)) u(y) \geq \pi(A) u(v) + (1 - \pi(A)) u(w). \end{array} \tag{6.1}$$

and

$$\begin{array}{rl} & x_{A^c} y \succeq v_{A^c} w \\ \Leftrightarrow & \pi(A^c) u(x) + (1 - \pi(A^c)) u(y) \geq \pi(A^c) u(v) + (1 - \pi(A^c)) u(w). \end{array} \tag{6.2}$$

Moreover, u is unique up to positive affine transformations and $\pi(A)$ is unique.

Proof. Existence of a function u and a real number $\pi(A)$ satisfying all the conditions of the lemma other than strict monotonicity and continuity, follows from Lemmas 6.4 and 6.5 and Nakamura's [3] Theorem 1.

To see that u is strictly increasing, assume $x > v$. By part (a) of the structure axiom, we have $x \succ v$. Apply the already proven part of the lemma to x, $y = x$, v, $w = v$ to obtain $u(x) > u(v)$. Continuity of u follows from the following argument: Since u is strictly increasing, the only discontinuities can be (an at most countable number of) jumps up. Therefore, limits from above and from below exist at each point in X. Suppose there is a jump of height $\delta > 0$ at $\hat{x} \in X$. Consider the case where $u(\hat{x}) = \lim_{y \to \hat{x}^+} u(y)$. By definition of δ, $u(\hat{x}) - \delta = \lim_{y \to \hat{x}^-} u(y)$. By (6.1), if $y \prec \hat{x}$, then $y_A \hat{x} \sim x$ only if $\pi(A) u(y) + (1 - \pi(A)) u(\hat{x}) = u(x)$. Since $u(\hat{x}) - \delta = \lim_{y \to \hat{x}^-} u(y)$, for any $\varepsilon > 0$, there exists $\hat{y} \prec \hat{x}$ such that $u(\hat{y}) > u(\hat{x}) - \delta - \varepsilon$. Then, fixing $\varepsilon < \frac{1 - \pi(A)}{\pi(A)} \delta$,

$$\begin{array}{rcl} u(\hat{x}) & > & \pi(A) u(\hat{y}) + (1 - \pi(A)) u(\hat{x}) \\ & > & \pi(A)(u(\hat{x}) - \delta - \varepsilon) + (1 - \pi(A)) u(\hat{x}) \\ & = & u(\hat{x}) - \pi(A)(\delta + \varepsilon) \\ & > & u(\hat{x}) - \delta \\ & = & \lim\limits_{y \to \hat{x}^-} u(y). \end{array}$$

But this implies that $\hat{y}_A\hat{x}$ has no certainty equivalent, contradicting Lemma 6.1. The case where $u(\hat{x}) = \lim_{y\to\hat{x}^-} u(y)$ or $\lim_{y\to\hat{x}^-} u(y) < u(\hat{x}) < \lim_{y\to\hat{x}^+} u(y)$ generate a contradiction by a similar argument. This shows u can have no jumps. ■

Remark 1. *u^{-1} is continuous.*

Proof. Since u is strictly increasing and continuous, so is u^{-1}. ■

We next construct a real-valued representation of preferences over acts by fixing u and assigning each act the utility of its certainty equivalent.

Lemma 6.7. *Given a $u : X \to \mathbb{R}$ from lemma 6.6, there is a unique $J : F \to \mathbb{R}$ such that:*
(i) for all f and $g \in F$, $f \succeq g$ if and only if $J(f) \geq J(g)$;
(ii) for any constant act $f = x \in X$, $J(f) = u(x)$.

Proof. For constant acts, we uniquely define $J(\cdot)$ by (ii). For general acts $f \in F$, let $J(f) = u(m(f))$. Clearly, $J(\cdot)$ satisfies (i) and is unique. ■

Remark 2. *For any $x, y \in X$ such that $x \preceq y$,*

$$J(x_A y) = \pi(A)\, u(x) + (1 - \pi(A))\, u(y)$$

where $\pi(A)$ is given by Lemma 6.6.

Proof. Let u be the utility function used in the construction of J. By Lemma 6.6, $x_A y \sim m^A(x,y)$ implies $u\left(m^A(x,y)\right) = \pi(A)\, u(x) + (1-\pi(A))\, u(y)$. Note that J has been constructed so that $J(x_A y) = u\left(m^A(x,y)\right)$. Hence, $J(x_A y) = \pi(A)\, u(x) + (1-\pi(A))\, u(y)$. ■

Let $K = u(X)$. Since u is continuous, K is a closed interval in $\mathbb{R}$. We normalize u such that $K = [-2, 2]$. Let B be the space of bounded (in the supnorm), Σ-measurable, real valued functions on Ω. For $\gamma \in \mathbb{R}$, we denote by γ^* the element of B that assigns γ to every ω. Let $B(K)$ be the subset of functions in B with values in K. Observe that for $f \in F$, $u \circ f \in B(K)$, and for $d \in B(K)$ there exists $f \in F$ such that $u \circ f = d$. Now we use this observation to construct a functional on $B(K)$ that represents preferences.

Definition 6.8. *For $f \in F$ we define the functional $I : B(K) \to \mathbb{R}$ by $I(u \circ f) = J(f)$.*

Since J represents preferences, it is clear that I does as well. The next lemma shows that I satisfies several important properties, and that these properties may be preserved when extending I from $B(K)$ to all of B.

Lemma 6.9. *$I : B(K) \to \mathbb{R}$ may be extended to all of B in such a way that:*
(i) $I(1^*) = 1$;
(ii) (I is monotonic) For all $a, b \in B$, $a \geq b$ implies $I(a) \geq I(b)$;
(iii) (I is homogeneous of degree 1) For all $b \in B$, $\alpha \geq 0$, $I(\alpha b) = \alpha I(b)$;
(iv) (I is C-independent) For all $b \in B$, $\gamma \in \mathbb{R}$, $I(b + \gamma^*) = I(b) + I(\gamma^*)$; and,
(v) (I is superadditive) For all $a, b \in B$, $I(a + b) \geq I(a) + I(b)$.

Proof. First note that there exists $x \in X$ such that $u(x) = 1$. By construction then, $I(1^*) = J(x^*) = u(x) = 1$. Also, monotonicity of I on $B(K)$ follows directly from Axiom 4. We will now show that I is homogeneous of degree 1 on $B(K)$.

We will now show that the I functional is homogeneous of degree 1 and that $\overline{S}^f$ contains all acts that are multiples of f in terms of utility.

Lemma 6.10. *Axioms 1-6 imply I is homogeneous of degree 1 on $B(K)$, i.e. $I(\alpha b) = \alpha I(b)$ for all $b, \alpha b \in B(K)$, $\alpha \geq 0$. Moreover if some $g \in F$ is such that $u \circ g = b$ then there exists $f' \in \overline{S}^g$ such that $u \circ f' = \alpha b$.*

Proof.

It suffices to prove homogeneity for $\alpha \in [0, 1]$, as $\alpha > 1$ then follows by considering the reciprocal. First we need five lemmas:

The first lemma shows that the statewise combination over A of any two acts in the same $\overline{S}$ set is indifferent to the statewise combination over A of their certainty equivalents. Two applications of the $\overline{S}$-act-independence axiom are used to show this. This result is then used in Lemma 6.12.

Lemma 6.11. *If $h \in F$, $f_1, f_2 \in \overline{S}^h, x_1, x_2 \in X, x_1 \sim f_1, x_2 \sim f_2$ then*

$$f(\omega) \sim f_1(\omega)_A f_2(\omega), \tag{6.3}$$

and

$$x \sim x_{1A} x_2, \tag{6.4}$$

implies

$$f \sim x.$$

Proof. Let g be defined as follows:

$$g(\omega) \sim f_1(\omega)_A x_2, \tag{6.5}$$

Since $f_1, x_1, x_2 \in \overline{S}^h$, (6.4) and (6.5) imply (by $\overline{S}$-act-independence),

$$g \sim x \Leftrightarrow f_1 \sim x_1$$

To finish the proof we need to show that $f \sim g$. To see this note that $f_1, f_2, x_2 \in \overline{S}^h$. Then (6.3) and (6.5) will imply (again by $\overline{S}$-act-independence), $f_2 \sim x_2 \Leftrightarrow f \sim g$. ■

The following lemma provides a single step in the homogeneity argument. Specifically, it shows that if I satisfies homogeneity with respect to two specific coefficients, then I must also be homogeneous with respect to two specific weighted averages of these coefficients.

Lemma 6.12. *Let* $b \in B(K)$ *and* $s, t \in [0,1]$. *Let* $\underline{a} = sb$, $\overline{a} = tb$, $s' = \pi(A)s + (1-\pi(A))t$ *and* $t' = (1-\pi(A))s + \pi(A)t$. *Suppose*
(i) there exists $f_1, f_2 \in \overline{S}^g$ *for some* $g \in F$ *such that* $u \circ f_1 = \underline{a}$ *and* $u \circ f_2 = \overline{a}$,
and
(ii) $I(sb) = sI(b)$ *and* $I(tb) = tI(b)$.
Then,
(iii) there exists $h_1, h_2 \in \overline{S}^g$ *such that* $u \circ h_1 = s'b$ *and* $u \circ h_2 = t'b$,
and
(iv) $I(s'b) = s'I(b)$ *and* $I(t'b) = t'I(b)$.

Proof. Let $x_1 \sim f_1$ and $x_2 \sim f_2$ where $x_1, x_2 \in X$. Let h_1 be defined as follows: For all $\omega \in \Omega$,

$$h_1(\omega) \sim f_1(\omega)_A f_2(\omega). \tag{6.6}$$

This implies $u \circ h_1 = \pi(A)\underline{a} + (1-\pi(A))\overline{a} = s'b$.

Let $x \in X$ be defined as

$$x \sim x_{1A} x_2. \tag{6.7}$$

This implies $u(x) = \pi(A)u(x_1) + (1-\pi(A))u(x_2)$.

By Lemma 6.11, $h_1 \sim x$. So, $I(s'b) = J(h_1) = u(x) = \pi(A)u(x_1) + (1-\pi(A))u(x_2) = \pi(A)J(f_1) + (1-\pi(A))J(f_2) = \pi(A)I(\underline{a}) + (1-\pi(A))I(\overline{a}) = (\pi(A)s + (1-\pi(A))t)I(b) = s'I(b)$. Note that $h_1 \in \overline{S}^g$.

The argument for h_2, defined by, for all $\omega \in \Omega$,

$$h_2(\omega) \sim f_2(\omega)_A f_1(\omega) \tag{6.8}$$

proceeds similarly. ■

In proving homogeneity, we will need to apply Lemma 6.12 iteratively. In fact, we will want to be able to take the limit of an infinite sequence of iterations. To ensure that homogeneity is preserved in the limit, we must show that $I(\alpha b)$ is continuous in α. To this end, the next lemma shows that J is continuous.

Lemma 6.13. $J : F \to \mathbb{R}$ *is continuous.*

Proof. Let A be an arbitrary open set in $[u(m), u(M)] \subset \mathbb{R}$. Since u continuous and strictly increasing, J is onto $[u(m), u(M)]$ by construction. Let O_i, $i = 1, 2, ...$ be a collection of disjoint open intervals such that $\cup_{i\in\Lambda} O_i = A$, where $\Lambda \subset \mathbb{Z}_{++}$. Let $O_i = (\underline{o}_i, \overline{o}_i)$. Fix $g_i \in J^{-1}(\underline{o}_i)$ and $g_i' \in J^{-1}(\overline{o}_i)$. Then, $J^{-1}((\underline{o}_i, \overline{o}_i)) = \{f \in F | f \succ g_i\} \cap \{f \in F | g_i' \succ f\}$. By the continuity axiom each of the sets in this intersection is open, therefore their intersection is also open. Note now that $J^{-1}(A) = \cup_{i\in\Lambda} J^{-1}((\underline{o}_i, \overline{o}_i))$, and thus it is an open set. Hence, we have that $J^{-1}(\cdot)$ maps open sets to open sets, therefore $J(\cdot)$ is continuous. ■

Lemma 6.14. $I(\alpha b)$ *is continuous in* α *for* $b \in B(K)$, $\alpha \in (0,1]$.

Proof. Let $g \in F$ be such that $u \circ g = b$. Let $\alpha,\ \alpha' \in (0,1]$. Define $f',\ f \in F$ such that, for all $\omega \in \Omega$,

$$\begin{aligned} &u\left(f'\left(\omega\right)\right) = \alpha' u\left(g\left(\omega\right)\right) \text{ and,} \\ &u\left(f\left(\omega\right)\right) = \alpha u\left(g\left(\omega\right)\right). \end{aligned}$$

Thus,

$$\begin{aligned} &f'\left(\omega\right) = u^{-1}\left(\alpha' u\left(g\left(\omega\right)\right)\right) \text{ and,} \\ &f\left(\omega\right) = u^{-1}\left(\alpha u\left(g\left(\omega\right)\right)\right). \end{aligned}$$

By Remark 1, $u^{-1}\left(\cdot\right)$ is continuous. Therefore, for each $\varepsilon > 0$, there exists a $\delta > 0$ such that if $\left|\alpha' u\left(g\left(\omega\right)\right) - \alpha u\left(g\left(\omega\right)\right)\right| < \delta$ then $\left|u^{-1}\left(\alpha' u\left(g\left(\omega\right)\right)\right) - u^{-1}\left(\alpha u\left(g\left(\omega\right)\right)\right)\right| = \left|f'\left(\omega\right) - f\left(\omega\right)\right| < \varepsilon$. That is, if $\left|\alpha' - \alpha\right| < \frac{\delta}{\left|u(g(\omega))\right|}$ then $\left|f'\left(\omega\right) - f\left(\omega\right)\right| < \varepsilon$. Note that $\left|u\left(g\left(\omega\right)\right)\right| \leq 2$ by our normalization of u. Therefore, if $\left|\alpha' - \alpha\right| < \frac{\delta}{2}$ then $\left|f'\left(\omega\right) - f\left(\omega\right)\right| < \varepsilon$. Since this is true for any ω, if $\left|\alpha' - \alpha\right| < \frac{\delta}{2}$ then $\left\|f' - f\right\| < \varepsilon$.

Fix $\gamma > 0$. By continuity of J, there exists $\psi > 0$ such that if $\left\|f' - f\right\| < \psi$ then $\left|J\left(f'\right) - J\left(f\right)\right| = \left|I\left(\alpha' b\right) - I\left(\alpha b\right)\right| < \gamma$. By continuity of u^{-1} and the above argument, there exists $\lambda > 0$ such that if $\left|\alpha' - \alpha\right| < \lambda$ then $\left\|f' - f\right\| < \psi$. Therefore, $I\left(\alpha b\right)$ is continuous in α for $b \in B\left(K\right)$, $\alpha \in (0,1]$. ■

Now we show that if I satisfies homogeneity with respect to two specific coefficients, then I must also be homogeneous with respect to the average of these coefficients. To do this we take limits of the convex combinations used in Lemma 6.12. Notice that this limiting argument is needed because a "half-half" combination may only be able to be reached in the limit if the weight on A or A^c is not a power of $\frac{1}{2}$.

Lemma 6.15. *Let $b \in B(K)$ and $s_0, t_0 \in [0,1]$. Let $\underline{a} = s_0 b$ and $\overline{a} = t_0 b$. Suppose*
(i) there exists $f_1^0, f_2^0 \in \overline{S}^g$ for some $g \in F$ such that $u \circ f_1^0 = \underline{a}$ and $u \circ f_2^0 = \overline{a}$
and,
(ii) $I(s_0 b) = s_0 I(b)$ and $I(t_0 b) = t_0 I(b)$.
Then
(iii) $I(\frac{s_0+t_0}{2} b) = \frac{s_0+t_0}{2} I(b)$ and there exists $f \in \overline{S}^g$ such that $u \circ f = \frac{s_0+t_0}{2} b$.

Proof. Suppose w.l.o.g. that $s_0 < t_0$. Define $\{s_i\}$ and $\{t_i\}$ by $s_i = \pi(A)s_{i-1} + (1-\pi(A))t_{i-1}$ and $t_i = (1-\pi(A))s_{i-1} + \pi(A)t_{i-1}$ for $i = 1, 2, 3...$

We will show that $\lim_{i\to\infty} s_i = \lim_{i\to\infty} t_i = \frac{s_0+t_0}{2}$. The proof will proceed in three cases.

Case 1: $\pi(A) = \frac{1}{2}$.

If $\pi(A) = \frac{1}{2}$, then $s_i = t_i = \frac{s_0+t_0}{2}$ for all i. So $\lim_{i\to\infty} s_i = \lim_{i\to\infty} t_i = \frac{s_0+t_0}{2}$.

Case 2: $1 > \pi(A) > \frac{1}{2}$.

We will use an induction argument to prove this case. Observe that $s_1 < \frac{s_0+t_0}{2} < t_1$ and $s_1 + t_1 = s_0 + t_0$. Suppose for all $k \leq n$, $s_k < \frac{s_0+t_0}{2} < t_k$ and $s_k + t_k = s_0 + t_0$. We now show $s_{n+1} < \frac{s_0+t_0}{2} < t_{n+1}$ and $s_{n+1} + t_{n+1} = s_0 + t_0$. Since $s_n + t_n = s_0 + t_0$, s_n and t_n are equidistant from $\frac{s_0+t_0}{2}$. Thus $s_{n+1} = \pi(A)s_n + (1-\pi(A))t_n < \frac{s_0+t_0}{2} < (1-\pi(A))s_n + \pi(A)t_n = t_{n+1}$. Also, $s_{n+1} + t_{n+1} = (\pi(A) + 1 - \pi(A))s_n + (\pi(A) + 1 - \pi(A))t_n = s_n + t_n = s_0 + t_0$. So for any $j = 0, 1, 2, ...$, $s_j < \frac{s_0+t_0}{2} < t_j$ and $s_j + t_j = s_0 + t_0$. Observe that $s_j > s_{j-1}$ and $t_j < t_{j-1}$ for all $j = 1, 2, 3, ...$ Since $\{s_i\}$ and $\{t_i\}$ are monotone bounded sequences, $\lim_{i\to\infty} s_i$ and $\lim_{i\to\infty} t_i$

exist. Furthermore $\lim_{i\to\infty} s_i \leq \frac{s_0+t_0}{2} \leq \lim_{i\to\infty} t_i$. Let $\overline{s} = \lim_{i\to\infty} s_i$ and $\overline{t} = \lim_{i\to\infty} t_i$. Suppose $\overline{s} < \overline{t}$. Fix any $\varepsilon > 0$. There exists $N(\varepsilon)$ such that for all $n \geq N(\varepsilon)$, $\overline{s} - s_n < \varepsilon$ and $t_n - \overline{t} < \varepsilon$. Consider $s_{n+1} = \pi(A)s_n + (1-\pi(A))t_n > \pi(A)(\overline{s}-\varepsilon) + (1-\pi(A))\overline{t} = \pi(A)\overline{s} + (1-\pi(A))\overline{t} - \varepsilon\pi(A)$. But for ε small enough $s_{n+1} > \overline{s}$, a contradiction. Therefore $\overline{s} = \overline{t} = \frac{s_0+t_0}{2}$.

Case 3: $\frac{1}{2} > \pi(A) > 0$.

Observe that $t_1 < \frac{s_0+t_0}{2} < s_1$, $s_1 + t_1 = s_0 + t_0$, $s_2 < \frac{s_0+t_0}{2} < t_2$ and $s_2 + t_2 = s_0 + t_0$. Using arguments similar to those in case 2 one can show:

$$\begin{aligned} &s_j < \tfrac{s_0+t_0}{2} < t_j, \quad s_j + t_j = s_0 + t_0, \quad j \text{ even},\\ &s_j > \tfrac{s_0+t_0}{2} > t_j, \quad s_j + t_j = s_0 + t_0, \quad j \text{ odd}. \end{aligned}$$

Then by the argument in case 2 applied to even and odd subsequences, $\lim_{j\to\infty} s_j = \lim_{j\to\infty} t_j = \frac{s_0+t_0}{2}$ for j even and, $\lim_{j\to\infty} s_j = \lim_{j\to\infty} t_j = \frac{s_0+t_0}{2}$ for j odd. Thus $\lim_{j\to\infty} s_j = \lim_{j\to\infty} t_j = \frac{s_0+t_0}{2}$.

By lemma 6.12 we know that if $s_{i-1}, t_{i-1} \in [0,1]$, if there exists $f_1^{i-1}, f_2^{i-1} \in \overline{S}^g$such that $u \circ f_1^{i-1} = s_{i-1}b$ and $u \circ f_2^{i-1} = t_{i-1}b$ and if $I(s_{i-1}b) = s_{i-1}I(b)$ and $I(t_{i-1}b) = t_{i-1}I(b)$ then there exists $f_1^i, f_2^i \in \overline{S}^g$such that $u \circ f_1^i = s_i b$ and $u \circ f_2^i = t_i b$ and $I(s_i b) = s_i I(b)$ and $I(t_i b) = t_i I(b)$. But then by induction for any $i \in \{0,1,2,...\}$ we have $I(s_i b) = s_i I(b)$ and $I(t_i b) = t_i I(b)$. By lemma 6.14 we have homogeneity for the limit so $I(\frac{s_0+t_0}{2}b) = \frac{s_0+t_0}{2}I(b)$. Now let $f = \lim_{i\to\infty} f_1^i$. Since $f_1^i \in \overline{S}^g$ for all i and $\overline{S}^g$ is closed, then $f \in \overline{S}^g$. Furthermore u continuous implies $u \circ f = \lim_{i\to\infty} u \circ f_1^i = \lim_{i\to\infty} s_i b = \frac{s_0+t_0}{2}b$. ■

Now we complete the proof of Lemma 6.10. Let $b \in B(K)$. Let $g \in F$ be such that $u \circ g = b$ and $z \in X$ be such that $u(z) = 0$. We want to show that for any $\alpha \in [0,1]$, if $a = \alpha b$ then $I(a) = \alpha I(b)$. First, we will show this for all $\alpha = \frac{t}{2^k}$ where $k = 0,1,2,...$ and $t = 0,1,2,3,...,2^k - 1$. We will also show that for any such α there exists $f \in \overline{S}^g$ such that $u \circ f = \alpha b$. The proof will be by induction. First we will show that the statement is true for $k = 0$. This is true since $I(0^*) = J(u^{-1}(0)^*) = u(u^{-1}(0)) = 0$. Assume that the statement is true for some $k - 1 \in \{1,2,3,..\}$, in other words for all $t \in \{0,1,2,...,2^{k-1}-1\}$, assume that $I(\frac{t}{2^{k-1}}b) = \frac{t}{2^{k-1}}I(b)$ and there exists $f_t \in \overline{S}^g$ such that $u \circ f_t = \frac{t}{2^{k-1}}b$. Now we want to show that the statement is true for k, or for all $t \in \{0,1,2,...,2^k-1\}$, we want to show that $I(\frac{t}{2^k}b) = \frac{t}{2^k}I(b)$ and there exists $f_t \in \overline{S}^g$ such that $u \circ f_t = \frac{t}{2^k}b$. But by the induction hypothesis we need to only show this for $t \in \{1,3,5,...,2^k-1\}$ since for t even $\frac{t}{2^k} = \frac{t/2}{2^{k-1}}$. Since $\frac{t}{2^k}$, t odd, is exactly halfway between $\frac{t-1}{2^k}$ and $\frac{t+1}{2^k}$ lemma 6.15 shows that homogeneity for the even t implies homogeneity for the odd t. Furthermore the same lemma also implies that for all t there exists $f_t \in \overline{S}^g$ such that $u \circ f_t = \frac{t}{2^k}b$.

Now that we have homogeneity for all $\alpha \in \{\frac{t}{2^k} : k = 0,1,2,...$ and $t = \{0,1,2,...,2^k-1\}$, we want to extend this to all $\alpha \in [0,1]$. Since for any α we can find $\alpha_j \in \{\frac{t}{2^k} : k = 0,1,2,...$ and $t = 0,1,2,3,...,2^k-1\}$ such that $\lim_{j\to\infty}\alpha_j = \alpha$, homogeneity follows for the limit by lemma 6.14. Furthermore, for each α_j we know that there exists $f_j \in \overline{S}^g$ such that $u \circ f_j = \alpha_j b$. Since $\overline{S}^g$ is closed $f' = \lim_{j\to\infty} f_j \in \overline{S}^g$ and $u \circ f' = \alpha b$ since u is continuous.

This shows that I is homogeneous of degree 1 on $B(K)$. Next, we extend $I(\cdot)$ to all of B by homogeneity. Such an extension preserves homogeneity and monotonicity. ■

To show C-independence of I we need the next lemma.

Lemma 6.16. *Let* $f, g \in F$, $h \in F^*$ *and* $\alpha \in (0,1)$. *If*
(i) $f' \in F$ *is such that, for all* $\omega \in \Omega$,

$$u(f'(\omega)) = \alpha u(h) + (1-\alpha)u(f(\omega))$$

and
(ii) $g' \in F$ *is such that, for all* $\omega \in \Omega$,

$$u(g'(\omega)) = \alpha u(h) + (1-\alpha)u(g(\omega))$$

then

$$f \succeq g \Leftrightarrow f' \succeq g'.$$

Proof. Suppose $f, g \in F$, $h \in F^*$ and $\alpha \in (0,1)$. Assume that
(i) $f' \in F$ is such that for all $\omega \in \Omega$

$$u(f'(\omega)) = \alpha u(h) + (1-\alpha)u(f(\omega))$$

and
(ii) $g' \in F$ is such that for all $\omega \in \Omega$

$$u(g'(\omega)) = \alpha u(h) + (1-\alpha)u(g(\omega)).$$

There exists $\varepsilon > 0$ such that $\varepsilon \max\{\frac{1-\alpha}{1-\pi(A)}, \frac{\alpha}{\pi(A)}\} < 1$. By Lemma 6.10 there exists $l \in \overline{S}^f$ and $k \in \overline{S}^g$ such that for all $\omega \in \Omega$, $u(l(\omega)) = \varepsilon\frac{1-\alpha}{1-\pi(A)}u(f(\omega))$ and $u(k(\omega)) = \varepsilon\frac{1-\alpha}{1-\pi(A)}u(g(\omega))$. Let $t \in F^*$ be such that $u(t) = \varepsilon\frac{\alpha}{\pi(A)}u(h)$.

Now since $l, t \in \overline{S}^f$ and $k, t \in \overline{S}^g$, we can apply $\overline{S}$-act-independence. In particular let $f'', g'' \in F$ be such that $f''(\omega) \sim t_A l(\omega)$ and $g''(\omega) \sim t_A k(\omega)$ for all $\omega \in \Omega$. Note that by our representation for A-measurable acts $u(f''(\omega)) = \pi(A)u(t) + (1-\pi(A))u(l(\omega)) = \varepsilon u(f'(\omega))$ for all $\omega \in \Omega$. Similarly $u(g''(\omega)) = \pi(A)u(t) + (1-\pi(A))u(k(\omega)) = \varepsilon u(g'(\omega))$ for all $\omega \in \Omega$.

Now observe that

$$\begin{aligned} f' \succeq g' &\Leftrightarrow J(f') \geq J(g') \\ &\Leftrightarrow I(u \circ f') \geq I(u \circ g') \\ &\Leftrightarrow \varepsilon I(u \circ f') \geq \varepsilon I(u \circ g') \\ &\Leftrightarrow I(\varepsilon u \circ f') \geq I(\varepsilon u \circ g') \\ &\Leftrightarrow I(u \circ f'') \geq I(u \circ g'') \\ &\Leftrightarrow J(f'') \geq J(g'') \\ &\Leftrightarrow f'' \succeq g'' \end{aligned}$$

where the middle equivalence follows from homogeneity of I on $B(K)$. By $\overline{S}$-act-independence, $f'' \succeq g'' \Leftrightarrow l \succeq k$. Finally, using homogeneity again, $l \succeq k \Leftrightarrow I(\varepsilon\frac{1-\alpha}{1-\pi(A)}u \circ f) \geq I(\varepsilon\frac{1-\alpha}{1-\pi(A)}u \circ g) \Leftrightarrow \varepsilon\frac{1-\alpha}{1-\pi(A)}I(u \circ f) \geq \varepsilon\frac{1-\alpha}{1-\pi(A)}I(u \circ g) \Leftrightarrow f \succeq g$. Therefore $f' \succeq g' \Leftrightarrow f \succeq g$. ■

We now demonstrate C-independence of I. Consider $a \in B$ and $\gamma \in \mathbb{R}$. By homogeneity, we may assume without loss of generality that $\max\left(\frac{1}{1-\pi(A)}, \frac{1}{\pi(A)}\right) a \in B(K)$ and $\max\left(\frac{1}{1-\pi(A)}, \frac{1}{\pi(A)}\right) \gamma^* \in B(K)$. Note that by the structure of $B(K)$ (in particular the fact that K is an interval around 0), it follows that $\frac{1}{1-\pi(A)} a \in B(K)$ and $\frac{1}{\pi(A)} \gamma^* \in B(K)$. Define $\beta = I\left(\frac{1}{1-\pi(A)} a\right)$. By homogeneity, $\beta = \frac{1}{1-\pi(A)} I(a)$. Let $f \in F$ be such that $u \circ f = \frac{1}{1-\pi(A)} a$. Let y, $z \in X$ satisfy $u(y) = \beta$ and $u(z) = \frac{1}{\pi(A)} \gamma$. By construction of I, $J(f) = \beta$ and $J(y) = u(y) = \beta$, implying $f \sim y$. Now, let $g' \in F^*$ be the constant act such that, for all $\omega \in \Omega$,

$$u(g'(\omega)) = \pi(A) u(z) + (1 - \pi(A)) u(y).$$

Thus, $u(g'(\omega)) = \gamma + (1 - \pi(A)) \beta = I(\gamma^*) + I(a)$.

Now, let $f' \in F$ be an act such that, for all $\omega \in \Omega$,

$$u(f'(\omega)) = \pi(A) u(z) + (1 - \pi(A)) u(f(\omega)).$$

By Lemma 6.16 and the previously noted fact that $f \sim y$, we have $f' \sim g'$. Therefore, $I(a + \gamma^*) = J(f') = J(g') = I(a) + I(\gamma^*)$ and I is C-Independent.

To show superadditivity of I, we need the following lemma.

Lemma 6.17. *Let $f, g \in F$ and $\alpha \in (0,1)$. Suppose $f \sim g$. If $h \in F$ is such that, for all $\omega \in \Omega$,*

$$u(h(\omega)) = \alpha u(g(\omega)) + (1-\alpha) u(f(\omega))$$

then

$$h \succeq f.$$

Proof. Suppose $f, g \in F$ and $\alpha \in (0,1)$, $f \sim g$. Assume that $h \in F$ is such that for all $\omega \in \Omega$

$$u(h(\omega)) = \alpha u(g(\omega)) + (1-\alpha) u(f(\omega)).$$

There exists $\varepsilon > 0$ such that $\varepsilon \max\{\frac{1-\alpha}{1-\pi(A)}, \frac{\alpha}{\pi(A)}\} < 1$. The proof will proceed in three cases.

Case 1: $\alpha = \pi(A)$. This implies that $I\left(\varepsilon \frac{\alpha}{\pi(A)} u \circ g\right) = I\left(\varepsilon \frac{1-\alpha}{1-\pi(A)} u \circ f\right)$. Let $l, k \in F$ be such that for all $\omega \in \Omega$, $u(l(\omega)) = \varepsilon \frac{1-\alpha}{1-\pi(A)} u(f(\omega))$ and $u(k(\omega)) = \varepsilon \frac{\alpha}{\pi(A)} u(g(\omega))$. By assumption $l \sim k$. Let $h' \in F$ be such that $h'(\omega) \sim k(\omega)_A l(\omega)$ for all ω. By act-uncertainty aversion, $h' \succeq k$. Observe that $\varepsilon I(u \circ h) = I(u \circ h') \geq I(u \circ k) = \varepsilon I(u \circ f)$. So $h \succeq f$.

Case 2: $\alpha > \pi(A)$. This implies that $I\left(\varepsilon \frac{\alpha}{\pi(A)} u \circ g\right) > I\left(\varepsilon \frac{1-\alpha}{1-\pi(A)} u \circ f\right)$. Let $\gamma = I\left(\varepsilon \frac{\alpha}{\pi(A)} u \circ g\right) - I\left(\varepsilon \frac{1-\alpha}{1-\pi(A)} u \circ f\right) > 0$. There exists $0 < \delta < 1$ such that $I(\delta \varepsilon \frac{1-\alpha}{1-\pi(A)} u \circ f + (\delta\gamma)^*) \in B(K)$. Let $k \in F$ be such that for all $\omega \in \Omega$, $u(k(\omega)) = \delta \varepsilon \frac{1-\alpha}{1-\pi(A)} u(f(\omega)) + \delta\gamma$. By the C-independence of I, $I(u \circ k) = I(\delta \varepsilon \frac{1-\alpha}{1-\pi(A)} u \circ f + (\delta\gamma)^*) = I(\delta \varepsilon \frac{1-\alpha}{1-\pi(A)} u \circ f) + \delta\gamma =$

$I\left(\delta\varepsilon\frac{\alpha}{\pi(A)}u\circ g\right)$. Let $n \in F$ be such that for all $\omega \in \Omega$, $u(n(\omega)) = \delta\varepsilon\frac{\alpha}{\pi(A)}u(g(\omega))$. Let $h'' \in F$ be such that $h''(\omega) \sim n(\omega)_A k(\omega)$ for all ω. By act-uncertainty aversion, $h'' \succeq k$. Observe that

$$\begin{aligned} I(u\circ h') &= I(\pi(A)u\circ n + (1-\pi(A))u\circ k) \\ &= I(\delta\varepsilon\alpha u\circ g + \delta\varepsilon(1-\alpha)u\circ f + ((1-\pi(A))\delta\gamma)^*) \\ &= I(\delta\varepsilon u\circ h) + (1-\pi(A))\delta\gamma \\ &= \delta\varepsilon I(u\circ h) + (1-\pi(A))\delta\gamma \end{aligned}$$

and,

$$I(u\circ k) = I(\delta\varepsilon\frac{1-\alpha}{1-\pi(A)}u\circ f) + \delta\gamma.$$

These imply that $\delta\varepsilon I(u\circ h) + (1-\pi(A))\delta\gamma \geq I(\delta\varepsilon\frac{1-\alpha}{1-\pi(A)}u\circ f) + \delta\gamma$. Substitute $\gamma = I\left(\varepsilon\frac{\alpha}{\pi(A)}u\circ g\right) - I\left(\varepsilon\frac{1-\alpha}{1-\pi(A)}u\circ f\right)$ in the previous expression and use homogeneity to obtain $I(u\circ h) \geq I(u\circ f)$. So $h \succeq f$.

Case 3: $\alpha < \pi(A)$. This case can be proved using an argument that is similar to the one used in proving Case 2. ■

Finally, we show that I is superadditive. Consider a, $b \in B$. As above, by homogeneity we may assume without loss of generality that $\max\left(\frac{1}{1-\pi(A)}, \frac{1}{\pi(A)}\right) a \in B(K)$ and $\max\left(\frac{1}{1-\pi(A)}, \frac{1}{\pi(A)}\right) b \in B(K)$. Specifically, this implies $\frac{1}{1-\pi(A)}a \in B(K)$ and $\frac{1}{\pi(A)}b \in B(K)$. Let acts f, $g \in F$ be such that $u\circ f = \frac{1}{\pi(A)}b$ and $u\circ g = \frac{1}{1-\pi(A)}a$. The argument proceeds by considering the possible orderings of $I\left(\frac{1}{1-\pi(A)}a\right)$ and $I\left(\frac{1}{\pi(A)}b\right)$.

Case 1: $I\left(\frac{1}{1-\pi(A)}a\right) = I\left(\frac{1}{\pi(A)}b\right)$. Then $f \sim g$. Define the act f' by, for all $\omega \in \Omega$,

$$u(f'(\omega)) = (1-\pi(A))u(g(\omega)) + \pi(A)u(f(\omega)).$$

Thus, $u\circ f' = \pi(A)(u\circ f) + (1-\pi(A))(u\circ g) = b + a$ and $J(f') = I(u\circ f') = I(a+b)$. By Lemma 6.17, we have that $f' \succeq f$. Therefore, $I(a+b) = J(f') \geq J(f) = \frac{1}{\pi(A)}I(b) = \left(\frac{1-\pi(A)+\pi(A)}{\pi(A)}\right)I(b) = \left(\frac{1-\pi(A)}{\pi(A)}\right)I(b) + I(b) = I(a) + I(b)$, since $I(a) = (1-\pi(A))I\left(\frac{1}{\pi(A)}b\right)$.

Case 2: $I\left(\frac{1}{\pi(A)}b\right) > I\left(\frac{1}{1-\pi(A)}a\right)$. Let $\gamma = I\left(\frac{1}{\pi(A)}b\right) - I\left(\frac{1}{1-\pi(A)}a\right) > 0$. Let $\frac{1}{1-\pi(A)}c = \frac{1}{1-\pi(A)}a + \gamma^*$. By C-independence of I, $I\left(\frac{1}{1-\pi(A)}c\right) = I\left(\frac{1}{1-\pi(A)}a\right) + \gamma = I\left(\frac{1}{\pi(A)}b\right)$. By case 1, $I(c+b) \geq I(c) + I(b)$. But $I(c+b) = I(a + (1-\pi(A))\gamma^* + b) = I(a+b) + (1-\pi(A))\gamma$ by C-independence. Similarly, $I(c) = I(a + (1-\pi(A))\gamma^*) = I(a) + (1-\pi(A))\gamma$. Thus, $I(a+b) + (1-\pi(A))\gamma = I(c+b) \geq I(c) + I(b) = I(a) + I(b) + (1-\pi(A))\gamma$. Thus, $I(a+b) \geq I(a) + I(b)$.

The third and final case, where $I\left(\frac{1}{\pi(A)}b\right) < I\left(\frac{1}{1-\pi(A)}a\right)$, is proved similarly. This shows that I is superadditive and completes the proof of the lemma. ■

The importance of Lemma 6.9 is made clear by the next result which states that such an I may be written as the minimum expectation over a compact and convex set of finitely additive probability measures.

Lemma 6.18. *Let $I : B \to \mathbb{R}$ be a functional satisfying:*

(i) $I(1^*) = 1$;

(ii) $I(a) \geq I(b)$ if $a \geq b$ for all $a, b \in B$;

(iii) $I(a+b) \geq I(a) + I(b)$ for all $a, b \in B$;

(iv) $I(\alpha a + \beta 1^*) = \alpha I(a) + \beta$ for all $a \in B$, $\alpha \geq 0$ and $\beta \in \mathbb{R}$.

Then there exists a unique convex and w-compact set $\mathcal{C} \subseteq \mathcal{P}$ such that*

$$I(a) = \min_{P \in \mathcal{C}} \int a dP \quad \textit{for all } a \in B.$$

Proof. See Gilboa and Schmeidler [2], lemma 3.5 and the argument for uniqueness in the proof of their theorem 1. ■

Observe that *(i)* - *(v)* in lemma 6.9 imply that I satisfies *(i)* - *(iv)* of lemma 6.18. Therefore, we may represent $\succeq$ on F by $J(f) = I(u \circ f) = \min_{P \in \mathcal{C}} \int u \circ f dP$ with $\mathcal{C}$ unique, convex and w*-compact and u strictly increasing, continuous and unique up to positive affine transformations. This representation together with the representation (6.1) in lemma 6.6 imply that $\max_{P \in \mathcal{C}} P(A) = \pi(A)$ and $0 < \max_{P \in \mathcal{C}} P(A) < 1$, $\max_{P \in \mathcal{C}} P(A^c) = \pi(A^c)$ and $0 < \max_{P \in \mathcal{C}} P(A^c) < 1$, and $\pi(A) + \pi(A^c) = 1$. But this implies $P(A) = \pi(A)$ for all $P \in \mathcal{C}$. This proves sufficiency of the axioms in theorem 5.1.

6.2. Necessity

Lemmas 6.19-6.27 together prove necessity in Theorem 5.1. Note that any real-valued representation implies Axiom 1 (weak order).

Lemma 6.19. *The representation in Theorem 5.1 $\Rightarrow$ Axiom 2 (structure), part (a).*

Proof. Suppose $x > y$. Since u is strictly increasing, $u(x) > u(y)$. Therefore, $\min_{P \in \mathcal{C}} \int u(x) dP = u(x) > u(y) = \min_{P \in \mathcal{C}} \int u(y) dP$, which implies $x^* \succ y^*$. ■

Lemma 6.20. *The representation in Theorem 5.1 $\Rightarrow$Axiom 2 (structure), part (b).*

Proof. Consider the event A referred to in the representation theorem. We will show that such event is ordered non-null and ordered non-universal.

Recall that an event B is ordered non-null if there exist x, y and $z \in X$ with $x \preceq y \preceq z$ such that $x_B z \prec y_B z$.

Pick $x, y, z \in X$ such that $x \prec y \prec z$, so that $u(x) < u(y) < u(z)$. Consider the following ordered binary acts: $f = x_A z$ and $g = y_A z$. Now,

$$\min_{P \in \mathcal{C}} \int u \circ f dP = \max_{P \in \mathcal{C}} P(A) u(x) + \left(1 - \max_{P \in \mathcal{C}} P(A)\right) u(z)$$

and

$$\min_{P \in \mathcal{C}} \int u \circ g dP = \max_{P \in \mathcal{C}} P(A) u(y) + \left(1 - \max_{P \in \mathcal{C}} P(A)\right) u(z)$$

Since $0 < \max_{P\in\mathcal{C}} P(A)$, we have $\min_{P\in\mathcal{C}} \int u\circ f dP < \min_{P\in\mathcal{C}} \int u\circ g dP$. But this implies that $x_A z \prec y_A z$. Hence A is ordered non-null.

Recall that an event B is ordered non-universal if there exist x, y and $z \in X$ with $z \preceq y \preceq x$ such that $z_B y \prec z_B x$.

Pick x, y, $z \in X$ such that $z \prec y \prec x$, so that $u(z) < u(y) < u(x)$. Consider the following ordered binary acts: $f = z_A y$ and $g = z_A x$. Now,

$$\min_{P\in\mathcal{C}} \int u\circ f dP = \max_{P\in\mathcal{C}} P(A)\, u(z) + \left(1 - \max_{P\in\mathcal{C}} P(A)\right) u(y)$$

and

$$\min_{P\in\mathcal{C}} \int u\circ g dP = \max_{P\in\mathcal{C}} P(A)\, u(z) + \left(1 - \max_{P\in\mathcal{C}} P(A)\right) u(x)$$

Since $\max_{P\in\mathcal{C}} P(A) < 1$, we have $\min_{P\in\mathcal{C}} \int u\circ f dP < \min_{P\in\mathcal{C}} \int u\circ g dP$. But this implies that $z_A y \prec z_A x$. Hence A is ordered non-universal. Noting that $0 < P(A) < 1$ implies the same argument holds for A^c. ■

Lemma 6.21. *The representation in Theorem 5.1 ⇒Axiom 3 (continuity).*

Proof. Let $f \in F$. We want to show that the sets $M(f) = \{g \in F | \, g \succeq f\}$ and $W(f) = \{g \in F | \, f \succeq g\}$ are closed (in the supnorm). Let $\{g_n\}_{n=1}^{\infty} \in M(f)$ such that $g_n \to g$ (in the supnorm). Want to show that $g \in M(f)$. Note that $\int u\circ g_n dP \geq \min_{P\in\mathcal{C}} \int u\circ f dP$ for all $P \in \mathcal{C}$ and all n.

Since $g_n \to g$, and since $u(\cdot)$ is (uniformly) continuous, $u\circ g_n \to u\circ g$ in the supnorm. Now, since $|u(g_n(\omega))| \leq |u(M)|$ for all $\omega \in \Omega$, the dominated convergence theorem ([1], pp.124-125) implies $\lim_{n\to\infty} \int u\circ g_n dP = \int u\circ g dP$ for all $P \in \mathcal{C}$. Hence, $\min_{P\in\mathcal{C}} \int u\circ g dP \geq \min_{P\in\mathcal{C}} \int u\circ f dP$. A similar argument applies to $W(f)$. ■

Lemma 6.22. *The representation in Theorem 5.1 ⇒Axiom 4 (monotonicity), part (a).*

Proof. Suppose for some f, $g \in F$, $f(\omega) \succeq g(\omega)$, for all $\omega \in \Omega$. Then, $u(f(\omega)) \geq u(g(\omega))$, for all $\omega \in \Omega$. Thus, $\min_{P\in\mathcal{C}} \int u\circ g dP \leq \int u\circ g d\overline{P} \leq \int u\circ f d\overline{P}$, for all $\overline{P} \in \mathcal{C}$. Hence, $\min_{P\in\mathcal{C}} \int u\circ g dP \leq \min_{P\in\mathcal{C}} \int u\circ f dP$. ■

Lemma 6.23. *The representation in Theorem 5.1 ⇒Axiom 4 (monotonicity), part (b).*

Proof. First of all, note that if $B \in \Sigma$ is ordered non-null, then there is at least one probability measure $\widehat{P} \in \mathcal{C}$ with $\widehat{P}(B) > 0$. To see this, suppose $B \in \Sigma$ is ordered non-null but that for all $P \in \mathcal{C}$, $P(B) = 0$. Then, given any x, y, $z \in X$ such that $x \preceq y \preceq z$ we have $\min_{P\in\mathcal{C}} \int u\circ f dP = \min_{P\in\mathcal{C}} \int u\circ g dP$. This implies that the acts $f = x_A z$ and $g = y_A z$ are indifferent. But this contradicts B being ordered non-null. Let $\alpha = \max_{P\in\mathcal{C}} P(B) > 0$. Let x, y, $z \in X$ be such that $z \succeq x$ and $z \succeq y$. Now,

$$\begin{aligned}
&x \succ y \\
&\Leftrightarrow u(x) > u(y) \\
&\Leftrightarrow \alpha u(x) + (1-\alpha) u(z) > \alpha u(y) + (1-\alpha) u(z) \\
&\Leftrightarrow x_B z \succ y_B z.
\end{aligned}$$

The case in which $B \in \Sigma$ is ordered non-universal is proved by a similar argument noting that B ordered non-universal implies $\max_{P \in \mathcal{C}} P(B) < 1$. ■

Lemma 6.24. *The representation in Theorem 5.1 ⇒Axiom 5 (A-act-independence).*

Proof. Let x_1, x_2, y_1, y_2, z_1 and $z_2 \in X$ and let $f = x_{1A}x_2$, $g = y_{1A}y_2$ and $h = z_{1A}z_2$. Suppose f', $g' \in F$ are such that,

$$f'(\omega) \sim h(\omega)_A f(\omega) \text{ for all } \omega \in \Omega$$

and

$$g'(\omega) \sim h(\omega)_A g(\omega) \text{ for all } \omega \in \Omega.$$

By the representation $P(A) = \rho \in (0,1)$ for all $P \in \mathcal{C}$. Then,

$$\begin{aligned}
& f' \succeq g' \\
& \Leftrightarrow \rho u(m^A(z_1, x_1)) + (1-\rho)u(m^A(z_2, x_2)) \geq \rho u(m^A(z_1, y_1)) + (1-\rho)u(m^A(z_2, y_2)) \\
& \Leftrightarrow \rho^2 u(z_1) + \rho(1-\rho)u(x_1) + \rho(1-\rho)u(z_2) + (1-\rho)^2 u(x_2) \geq \\
& \quad \rho^2 u(z_1) + \rho(1-\rho)u(y_1) + \rho(1-\rho)u(z_2) + (1-\rho)^2 u(y_2) \\
& \Leftrightarrow \rho u(x_1) + (1-\rho)u(x_2) \geq \rho u(y_1) + (1-\rho)u(y_2) \\
& \Leftrightarrow f \succeq g.
\end{aligned}$$

Since $P(A^c) = 1 - \rho \in (0,1)$ for all $P \in \mathcal{C}$, the above argument may be repeated replacing A with A^c. This proves A-act-independence. ■

Lemma 6.25. *The representation in Theorem 5.1 ⇒Axiom 6 ($\overline{S}$-act-independence).*

Proof. We will first prove the following lemma which states that the representation evaluates all acts in $\overline{S}^l$ using the same probability measure.

Lemma 6.26. *Let $f, l \in F$ where $f \in \overline{S}^l$, if $P^l \in \arg\min_{P \in \mathcal{C}} \int u \circ l dP$, then $P^l \in \arg\min_{P \in \mathcal{C}} \int u \circ f dP$.*

Proof. Given S^l, let S^l_n be the set of acts obtained after n iterations in the statewise combination process. That is,

$$\begin{aligned}
& S^l_1 = \left\{ f^1 \in F : f^1(\omega) \sim f^0_1(\omega)_A f^0_2(\omega), \text{ for all } \omega \in \Omega \text{ where } f^0_1, f^0_2 \in S^l_0 \right\} \\
& S^l_2 = \left\{ f^2 \in F : f^2(\omega) \sim f^1_1(\omega)_A f^1_2(\omega), \text{ for all } \omega \in \Omega \text{ where } f^1_1, f^1_2 \in S^l_1 \right\} \\
& \vdots \\
& S^l_n = \left\{ f^n \in F : f^n(\omega) \sim f^{n-1}_1(\omega)_A f^{n-1}_2(\omega), \text{ for all } \omega \in \Omega \text{ where } f^{n-1}_1, f^{n-1}_2 \in S^l_{n-1} \right\}
\end{aligned}$$

First we will show that the representation evaluates any act $f \in \cup_{n=1}^{\infty} S^l_n$ using measure P^l. This is shown by induction.

Suppose $f \in S^l_1$. Then we can find acts f^0_1, $f^0_2 \in S^l_0$ such that for each $\omega \in \Omega$, $f(\omega) \sim f^0_1(\omega)_A f^0_2(\omega)$. By the representation, $u(f(\omega)) = \rho u(f^0_1(\omega)) + (1-\rho) u(f^0_2(\omega))$. Hence, for all $P \in \mathcal{C}$

$$\int u \circ f dP = \rho \int u \circ f^0_1 dP + (1-\rho) \int u \circ f^0_2 dP.$$

Note that the right hand side of this expression is minimized by choosing $P = P^l$. Thus, $P^l \in \arg\min_{P\in\mathcal{C}} \int u \circ f dP$ for all $f \in S^l_1$.

Fix $k \geq 1$. Assume $f \in S^l_n$, for some $n \leq k$, implies $P^l \in \arg\min_{P\in\mathcal{C}} \int u \circ f dP$. Consider an act $f \in S^l_{k+1}$. Then there exist acts f^k_1, $f^k_2 \in S^l_k$ such that for all $\omega \in \Omega$, $f(\omega) \sim f^k_1(\omega)_A f^k_2(\omega)$. By the representation, $u(f(\omega)) = \rho u\left(f^k_1(\omega)\right) + (1-\rho) u\left(f^k_2(\omega)\right)$. Hence, for all $P \in \mathcal{C}$

$$\int u \circ f dP = \rho \int u \circ f^k_1 dP + (1-\rho) \int u \circ f^k_2 dP.$$

By the induction hypothesis, P^l minimizes the right hand side. Therefore, $P^l \in \arg\min_{P\in\mathcal{C}} \int u \circ f dP$ for all $f \in \cup^{k+1}_{n=1} S^l_n$. This completes the induction argument.

To complete the proof of the lemma it must be shown that if $f \in \overline{S}^l \setminus \cup^\infty_{n=1} S^l_n$, then $P^l \in \arg\min_{P\in\mathcal{C}} \int u \circ f dP$.

We will first show the following: (i) $\cup^\infty_{n=1} S^l_n \supseteq S^l_0$, (ii) $\cup^\infty_{n=1} S^l_n$ contains all statewise combinations over A, and (iii) if $S \supseteq S^l_0$ and S contains all statewise combinations over A then $\cup^\infty_{n=1} S^l_n \subseteq S$.

To see (i), note that $S^l_0 = \left\{f^1 \in F : f^1(\omega) \sim f^0_1(\omega)_A f^0_2(\omega) \text{ where } f^0_1 = f^0_2 \in S^l\right\} \subseteq S^l_1$. To see (ii), consider $f, g \in \cup^\infty_{n=1} S^l_n$. We need to show that the statewise combination over A of any such f and g is also in $\cup^\infty_{n=1} S^l_n$. Since $S^l_n \subseteq S^l_{n+1}$, there exists a N such that for all $n \geq N$, $f, g \in S^l_n$. But then h such that $h(\omega) \sim f(\omega)_A g(\omega)$ for all $\omega \in \Omega$, must be an element of S^l_{N+1}.

Finally, to prove (iii), consider a set S as in (iii). Fix $f \in \cup^\infty_{n=1} S^l_n$. If $f \notin S$ then S cannot contain all statewise combinations over A since f can be reached in a finite number of statewise combination operations starting from elements of S^l_0.

By (iii) and the definition of $\overline{S}^l$, we have that the closure of $\cup^\infty_{n=1} S^l_n$ is $\overline{S}^l$. Fix $f \in \overline{S}^l \setminus \cup^\infty_{n=1} S^l_n$. There must exist a sequence of acts $\{f_i\}$ converging to f in the supnorm with $f_i \in \cup^\infty_{n=1} S^l_n$. Since u is continuous, for all $P \in \mathcal{C}$, $\left\{\int u \circ f_i dP\right\}$ converges to $\int u \circ f dP$. Let $P^f \in \arg\min_{P\in\mathcal{C}} \int u \circ f dP$. As $\int u \circ f_i dP^l \leq \int u \circ f_i dP^f$ for all i, $\int u \circ f dP^l > \int u \circ f dP^f$ would contradict continuity of preferences. This proves $P^l \in \arg\min_{P\in\mathcal{C}} \int u \circ f dP$ for all $f \in \overline{S}^l \setminus \cup^\infty_{n=1} S^l_n$. ■

Now we complete the proof of Lemma 6.25. Assume we have f, g, $h \in F$ such that there exist l, $k \in F$ for which f, $h \in \overline{S}^l$ and g, $h \in \overline{S}^k$. If f', $g' \in F$ are such that

$$f'(\omega) \sim h(\omega)_A f(\omega) \text{ for all } \omega \in \Omega$$

and

$$g'(\omega) \sim h(\omega)_A g(\omega) \text{ for all } \omega \in \Omega,$$

we have that for all $P \in \mathcal{C}$

$$\int u \circ f' dP = \rho \int u \circ h dP + (1-\rho) \int u \circ f dP$$

and

$$\int u \circ g' dP = \rho \int u \circ h dP + (1-\rho) \int u \circ g dP.$$

Since $f,\ h,\ f' \in \overline{S}^{l}$, $P^l \in \arg\min_{P\in\mathcal{C}} \int u \circ f dP$, $P^l \in \arg\min_{P\in\mathcal{C}} \int u \circ h dP$ and $P^l \in \arg\min_{P\in\mathcal{C}} \int u \circ f' dP$. Similarly, since $g,\ h,\ g' \in \overline{S}^{k}$, $P^k \in \arg\min_{P\in\mathcal{C}} \int u \circ g dP$, $P^k \in \arg\min_{P\in\mathcal{C}} \int u \circ h dP$ and $P^k \in \arg\min_{P\in\mathcal{C}} \int u \circ g' dP$. Therefore,

$$\min_{P\in\mathcal{C}} \int u \circ f' dP = \rho \int u \circ h dP^l + (1-\rho) \int u \circ f dP^l$$

and

$$\min_{P\in\mathcal{C}} \int u \circ g' dP = \rho \int u \circ h dP^k + (1-\rho) \int u \circ g dP^k.$$

But since $\int u \circ h dP^l = \int u \circ h dP^k$, we have $\min_{P\in\mathcal{C}} \int u \circ f' dP \geq \min_{P\in\mathcal{C}} \int u \circ g' dP$ if and only if $\int u \circ f dP^l = \min_{P\in\mathcal{C}} \int u \circ f dP \geq \min_{P\in\mathcal{C}} \int u \circ g dP = \int u \circ g dP^k$. Thus, $f \succeq g$ if and only if $f' \succeq g'$. This proves $\overline{S}$-act-independence. ■

Lemma 6.27. *The representation in Theorem 5.1 ⇒Axiom 7 (act-uncertainty aversion).*

Proof. Let $f,\ g \in F$. Suppose $f \sim g$ and

$$f'(\omega) \sim f(\omega)_A\, g(\omega) \text{ for all } \omega \in \Omega.$$

By the representation, we have that $u(f'(\omega)) = \rho u(f(\omega)) + (1-\rho) u(g(\omega))$ for all $\omega \in \Omega$. Now, $f \sim g$ implies $\min_{P\in\mathcal{C}} \int u \circ f dP = \min_{P\in\mathcal{C}} \int u \circ g dP$. Note,

$$\min_{P\in\mathcal{C}} \int u \circ f dP \leq \int u \circ f d\overline{P} \text{ for all } \overline{P} \in \mathcal{C}$$

and

$$\min_{P\in\mathcal{C}} \int u \circ g dP \leq \int u \circ g d\overline{P} \text{ for all } \overline{P} \in \mathcal{C}.$$

This implies

$$\begin{aligned}
&\min\nolimits_{P\in\mathcal{C}} \int u \circ f dP = \rho \left(\min\nolimits_{P\in\mathcal{C}} \int u \circ f dP\right) + (1-\rho)\left(\min\nolimits_{P\in\mathcal{C}} \int u \circ g dP\right) \\
&\leq \rho \left(\int u \circ f d\overline{P}\right) + (1-\rho)\left(\int u \circ g d\overline{P}\right) \\
&= \int \left(\rho (u \circ f) + (1-\rho)(u \circ g)\right) d\overline{P} \\
&= \int u \circ f' d\overline{P} \text{ for all } \overline{P} \in \mathcal{C}.
\end{aligned}$$

Hence, $\min_{P\in\mathcal{C}} \int u \circ f dP \leq \min_{P\in\mathcal{C}} \int u \circ f' dP$ and therefore, $f' \succeq f$. ■

The above lemmas together prove necessity of the axioms in Theorem 5.1. ■

6.3. Proof that $\overline{S}^f$ contains all affinely related acts

Proposition 6.28. *If $f \in F$ is not a constant act, then $\overline{S}^f$ contains all acts that are affinely related to f.*

Proof. By definition, all constant acts are in $\overline{S}^f$. Fix any non-constant act $g \in F$ that is affinely related to f. There must exist an $\alpha > 0$ and $\beta \in \mathbb{R}$ such that, for all $\omega \in \Omega$,

$$u(g(\omega)) = \alpha u(f(\omega)) + \beta.$$

There exists $\varepsilon > 0$ such that $\frac{\varepsilon}{\pi(A)}\alpha(u \circ f) \in B(K)$ and $\frac{\varepsilon}{1-\pi(A)}\beta^* \in B(K)$. Let $h \in F$ be such that $u \circ h = \frac{\varepsilon}{\pi(A)}\alpha(u \circ f)$ and h' be such that $u \circ h' = \frac{\varepsilon}{1-\pi(A)}\beta^*$. Since h' is a constant act, it is in $\overline{S}^f$. By Lemma 6.10, $h \in \overline{S}^f$ as well.

Let $g'(\omega) \sim h(\omega)_A h'(\omega)$ for all $\omega \in \Omega$. Since $\overline{S}^f$ contains all statewise combinations over A, $g' \in \overline{S}^f$. Note that by the representation in (6.1), $u \circ g' = \varepsilon(\alpha(u \circ f) + \beta^*)$. Another application of Lemma 6.10 yields $g \in \overline{S}^{g'}$. Since, by Lemma 6.26, $\overline{S}^{g'} \subseteq \overline{S}^f$, we have $g \in \overline{S}^f$. ■

References

[1] Dunford, N. and J. T. Schwartz (1957) *Linear Operators, Part I.* Wiley Interscience, New York.

[2] Gilboa, I. and D. Schmeidler (1989) "Maxmin expected utility with non-unique prior," *Journal of Mathematical Economics* **18**, 141–153.

[3] Nakamura, Y. (1990) "Subjective Expected Utility with Non-additive Probabilities on Finite State Spaces," *Journal of Economic Theory* **51**, 346–366.