Irreversible Investment: A Simplified Approach and an Economic Interpretation*

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Abstract

This paper provides two contributions to the economics of irreversibility. First, it demonstrates that the optimal investment policy in the presence of costly reversibility can be derived using the familiar priniciples of discrete time dynamic programming as an alternative to the less familiar methods of smooth pasting. In doing so, it opens the analysis of irreversible investment to a wider audience of economists. Second, it shows that the discrete time approach to the optimal investment problem developed in the paper naturally admits a clean economic interpretation of the optimal investment policy of a firm that has not been recognized in previous literature.

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Applications of irreversible investment abound in economics. Models of irreversibility have been applied to diverse economic environments, from models of capital investment (McDonald & Siegel, 1986; Abel & Eberly, 1996) to the analysis of firm entry and exit (Dixit, 1989) to optimal expenditures on consumer durables (Lam, 1989) to models of dynamic labor demand in the presence of firing costs (Bentolila & Bertola, 1990). More recent applications have ventured even farther afield to study the optimal reallocation of workers across labor markets in a search environment (Alvarez & Shimer, 2008).

This paper provides two contributions to the economics of irreversibility. First, it presents a simplified approach to the derivation of the optimal investment policy in the presence of costly reversibility. Previous literature has traditionally solved such models using the techniques of continuous time smooth pasting.[1] Since smooth pasting is not a technique commonly used in other domains of economics, it can seem complicated to the uninitiated.[2] This paper shows that the optimal investment policy of a firm can be derived using familiar principles of discrete time dynamic programming. Since such techniques are used in a broad array of economic applications, the approach adopted here opens the analysis of irreversible investment to a wider audience of economists. The second contribution of the paper is to show that this discrete time approach naturally admits a clean economic interpretation of the optimal investment policy, something that is difficult to obtain from a smooth pasting approach.

Section 1 describes the solution to a discrete time analogue to the canonical model of optimal investment in the presence of costly reversibility studied by Abel & Eberly (1996). Irreversibility is generated by the existence of a wedge between the purchase and resale prices of capital (see Ramey & Shapiro, 2001, for empirical evidence for such a wedge). I show that, in the presence of costly reversibility, the marginal effects of current investment decisions persist into the future: When investment is costly to reverse, the firm will freeze investment (neither invest nor disinvest) in future periods with positive probability, and in this event the future capital stock is entirely determined by current investment decisions. I show that a useful insight into the optimal investment problem is that these future marginal effects persist in a *recursive* fashion; in particular, they can be summarized in the form

[1] For a lucid survey of these techniques, see Dixit (1993) or Stokey (2008).

[2] For example, in his discussion of the application of smooth pasting to exchange rate target zones, Svensson (1992) states that:

> "The intuition for the smooth-pasting result is far from easy, and another frequent stumbling block for new students of the target zone literature. The reader should not feel put off if the explanation here seems difficult to follow."

of a contraction mapping. This recursive property allows one to solve for the optimal investment policy in a straightforward manner. Under conventional parameterizations of the model, optimal investment can be characterized by simple application of the method of undetermined coefficients.

The optimal investment policy derived mirrors the form derived using smooth pasting techniques in the pioneering works of Bentolila & Bertola (1990), and Abel & Eberly (1996). As in Bentolila & Bertola, optimal investment follows a "trigger policy" whereby the firm invests to keep the marginal revenue product of capital from rising above an upper trigger, disinvests to prevent the marginal revenue product from falling below a lower trigger, and otherwise freezes investment. In addition, as in Abel & Eberly, these trigger values, and hence the optimal investment policy, can be determined simply by solving a single non–linear equation in the ratio of the triggers.

Section 2 of this paper, however, shows that the discrete time solution developed here naturally admits an economic interpretation of the optimal investment policy. Under conventional assumptions on the revenue function and the evolution of shocks, the recursive property of the dynamic effects of current investment decisions has the simple implication that a currently investing or disinvesting firm will subsequently freeze investment with constant probability. The simplicity of this result admits a particularly clean economic interpretation of the investment policy. A firm that invests in a given period does so in a way that sets the marginal benefits equal to the marginal costs of an additional unit of capital. Under the optimal investment policy, I show that the former is equal to the present value of the marginal revenue product of capital, discounted by the (constant) probability that a currently investing firm subsequently freezes investment. Likewise, I show that the marginal costs of investment are equal to a similarly discounted sum of two components. The first is exactly analogous to the Jorgensonian (1963) user cost of capital, and thus reflects the implicit rental rate on a purchased unit of capital. The second component represents the marginal costs associated with the reversal of investment decisions, and reflects the probability that a currently investing firm sells capital next period multiplied by the difference between the resale price of capital and its purchase price. A symmetric logic holds for a disinvesting firm.

1 A Simplified Approach to Irreversible Investment

Consider a firm which uses capital, K, to produce output. The firm's per period revenue function is given by $F(K, X)$, where K is the capital stock, and X is a shock variable that summarizes the productivity and demand conditions of the firm, and is the source of uncertainty in the model. We assume that revenue is increasing and concave in the capital stock $F_K > 0$, $F_{KK} < 0$, increasing in X, $F_X > 0$, and supermodular, $F_{KX} > 0$. We assume that the evolution of next period's shock, X', given this period's realization, X, is described by the c.d.f. $G(X'|X)$. We also assume that a constant fraction, δ, of the firm's capital stock depreciates each period, so that the change in the capital stock across periods is given by:

$$\Delta K = I - \delta K_{-1}, \tag{1}$$

where I represents the firm's gross investment. Given this, the firm chooses its capital stock, K, or equivalently investment, I, each period to maximize the expected discounted value of its profits. However, such investment decisions are costly to reverse: the purchase price of capital, b_U, exceeds its resale price, b_L, $b_U > b_L$. It follows that we can express the value of the firm recursively as:

$$V(K_{-1}, X) = \max_K \left\{ F(K, X) - b_L I^- - b_U I^+ + \beta \int V(K, X') \, dG(X'|X) \right\}, \tag{2}$$

where I^+ and I^- respectively denote positive and negative investment. In the event that the firm decides to either invest or disinvest, optimal investment must satisfy the first–order condition:

$$F_K(K, X) + \beta D(K, X) = \begin{cases} b_U & \text{if} \quad I > 0, \\ b_L & \text{if} \quad I < 0, \end{cases} \tag{3}$$

where $D(K, X) \equiv \int V_K(K, X') \, dG(X'|X)$ represents the marginal effect of current investment decisions on the future profitability of the firm. Intuitively, the marginal revenue product of capital $F_K(K, X)$ plus any discounted expected future marginal benefits of an additional unit of capital $\beta D(K, X)$, "marginal q" in the investment literature, must be set equal to the purchase price of capital b_U in the event that the firm invests, or the resale price of capital b_L in the event that the firm disinvests.

Because the firm's problem is concave in the capital stock, K, and continuous in the shock parameter, X, it follows that the firm's optimal investment policy will be continuous in X and can therefore be summarized as follows:[3]

[3] This follows from the Theorem of the Maximum. See Stokey & Lucas (1989), pp. 62–63.

Proposition 1 *The firm's optimal investment policy is of the form:*

$$K = \begin{cases} U^{-1}(X) & \text{if} \quad X > U(\mathcal{K}_{-1}) & \textit{Invest,} \\ \mathcal{K}_{-1} & \text{if} \quad X \in [L(\mathcal{K}_{-1}), U(\mathcal{K}_{-1})] & \textit{Freeze,} \\ L^{-1}(X) & \text{if} \quad X < L(\mathcal{K}_{-1}) & \textit{Disinvest,} \end{cases} \tag{4}$$

where $\mathcal{K}_{-1} \equiv (1-\delta) K_{-1}$ *is the capital stock inherited from last period, and the functions* $U(\cdot)$ *and* $L(\cdot)$ *satisfy the first–order conditions:*

$$F_K(K, U(K)) + \beta D(K, U(K)) \equiv b_U, \tag{5}$$

$$F_K(K, L(K)) + \beta D(K, L(K)) \equiv b_L. \tag{6}$$

Thus, the firm's optimal investment policy will be similar to that depicted in Figure 1. It is characterized by two reservation values for the firm's shock, $L(\mathcal{K}_{-1})$ and $U(\mathcal{K}_{-1})$. Specifically, for sufficiently bad shocks ($X < L(\mathcal{K}_{-1})$ in the figure), firms will sell capital until $X = L(K)$, and the first-order condition in the disinvestment regime, (6), is satisfied. Moreover, for sufficiently good realizations of X ($X > U(\mathcal{K}_{-1})$ in the figure), firms will purchase capital until $X = U(K)$, and the first-order condition in the investment regime, (5), is satisfied. Finally, for intermediate values of X, firms neither invest nor disinvest, and so $K = \mathcal{K}_{-1}$. This, of course, occurs as a result of the kink in the firm's profits at $K = \mathcal{K}_{-1}$, which arises because the purchase price of capital exceeds its resale price, $b_U > b_L$.

Clearly, in order to solve explicitly for the functions $L(\cdot)$ and $U(\cdot)$, we must characterize the marginal effect of current investment decisions on the future profitability of the firm, $D(K, X) \equiv \int V_K(K, X')\, dG(X'|X)$. To this end, first note that, given the form of the firm's optimal investment policy, (4), we can split the expected future value of the firm next period into three regimes as follows:[4]

$$\int V(K, X')\, dG = \int_0^{L(\mathcal{K})} V^-(K, X')\, dG + \int_{L(\mathcal{K})}^{U(\mathcal{K})} V^0(K, X')\, dG + \int_{U(\mathcal{K})}^{\infty} V^+(K, X')\, dG, \tag{7}$$

where $V^{-/0/+}$ denotes the value of the firm in the disinvest/freeze/invest regimes respectively. Taking the derivative with respect to the capital stock, K, we obtain the following result for

[4] Hereafter, "dG" without further elaboration will be taken to mean "$dG(X'|X)$."

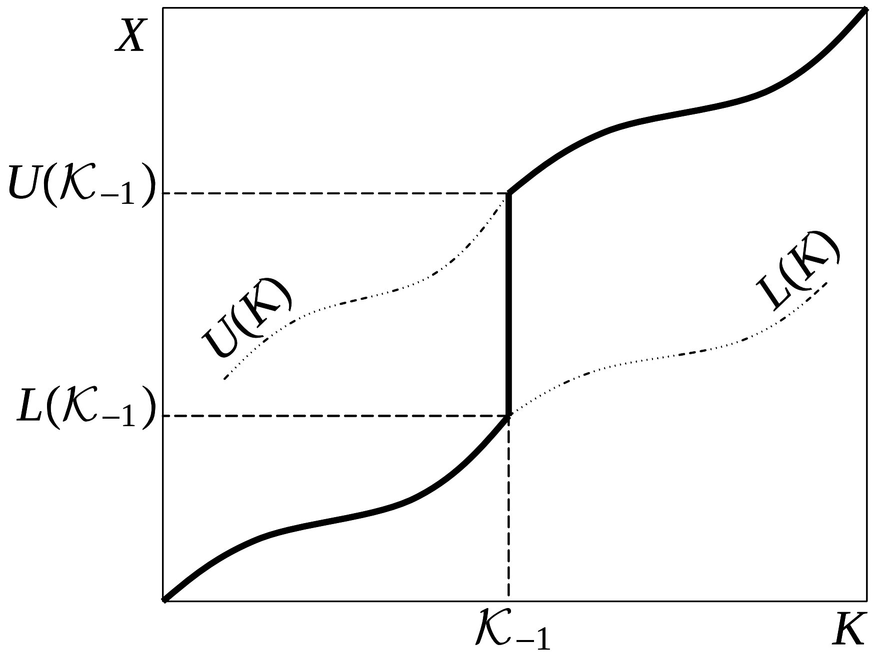

Figure 1: Optimal Investment Policy of a Firm

the marginal effect of current investment decisions on future profits:

$$\begin{aligned} D(K,X) &= \int_0^{L(\mathcal{K})} V_K^-(K,X')\,dG + \int_{L(\mathcal{K})}^{U(\mathcal{K})} V_K^0(K,X')\,dG + \int_{U(\mathcal{K})}^{\infty} V_K^+(K,X')\,dG \\ &\quad + V^-(K,L(\mathcal{K}))\frac{\partial L(\mathcal{K})}{\partial K} - V^0(K,L(\mathcal{K}))\frac{\partial L(\mathcal{K})}{\partial K} \\ &\quad + V^0(K,U(\mathcal{K}))\frac{\partial U(\mathcal{K})}{\partial K} - V^+(K,U(\mathcal{K}))\frac{\partial U(\mathcal{K})}{\partial K}. \end{aligned} \tag{8}$$

Fortunately, this result can be simplified using three simple observations. First, the optimized value function must be continuous in the shock variable X, which implies that the latter four terms in (8) cancel:

$$\begin{aligned} V^-(K,L(\mathcal{K})) &= V^0(K,L(\mathcal{K})), \\ V^0(K,U(\mathcal{K})) &= V^+(K,U(\mathcal{K})). \end{aligned} \tag{9}$$

Intuitively, as the firm's shock realization approaches the (dis)investment trigger, the firm must by definition be indifferent between (dis)investing and freezing investment, and thus these value matching conditions must hold.

Second, in the case where the firm either invests or disinvests next period (regimes $^+$ and $^-$), the first–order conditions for optimal (dis)investment, (5) and (6), will hold. Thus, noting that $I' = K' - (1-\delta) K$, we can use the envelope theorem to pin down two of the forward derivatives in equation (8):

$$\begin{aligned} V_K^- (K, X') &= (1-\delta) b_L, \\ V_K^+ (K, X') &= (1-\delta) b_U. \end{aligned} \tag{10}$$

These are analogous to the "smooth pasting" conditions imposed in continuous time models of irreversible investment (see e.g. Abel & Eberly, 1996, p.584).

The third and final term in (8) that we need to characterize is the marginal effect of the current investment decision on the future value of the firm in the event that it neither invests nor disinvests, $V_K^0 (K, X')$. In this case, however, it is simple to show that:

$$\begin{aligned} V_K^0 (K, X') &= \frac{\partial}{\partial K} \left\{ F(\mathcal{K}, X') + \beta \int V(\mathcal{K}, X'') \, dG(X''|X') \right\} \\ &= (1-\delta) F_K (\mathcal{K}, X') + \beta (1-\delta) D(\mathcal{K}, X'). \end{aligned} \tag{11}$$

Piecing all these components together, we obtain the following result:

Proposition 2 *The marginal effect of current investment decisions on future profits is given recursively by:*

$$\begin{aligned} D(K, X) &= (1-\delta) \left[\int_0^{L(\mathcal{K})} b_L dG + \int_{L(\mathcal{K})}^{U(\mathcal{K})} F_K (\mathcal{K}, X') \, dG + \int_{U(\mathcal{K})}^{\infty} b_U dG \right] \\ &\quad + \beta (1-\delta) \int_{L(\mathcal{K})}^{U(\mathcal{K})} D(\mathcal{K}, X') \, dG. \end{aligned} \tag{12}$$

This is a contraction mapping in $D(\cdot)$ and therefore has a unique fixed point.

The intuition for this result is as follows. Because of the existence of kinked adjustment costs (the purchase price of capital exceeding its resale price) the firm will freeze investment with positive probability. In this event (when $X' \in [L(\mathcal{K}), U(\mathcal{K})]$), the future capital stock is entirely determined by the current capital stock, and the marginal effects of the firm's current investment choice persist into the future. Proposition 2 shows that these marginal effects persist into the future in a *recursive* fashion. We will see that this recursive property will help us obtain an explicit solution for the firm's optimal investment policy.

1.1 A Digression: The Reversible Investment Case

A useful point of contrast in what follows is the special case where investment is fully reversible, i.e. when the purchase price of capital is equal to its resale price, $b_U = b_L \equiv b$. In this case, it is straightforward to show that the marginal effect of current investment decisions on the future profitability of the firm, D, is given by the simple envelope condition, $D = (1-\delta)\, b$. Thus, the firm's first–order condition for optimal investment, (3), becomes:

$$\text{marginal } q = F_K(K, X) + \beta(1-\delta)\, b = b, \tag{13}$$

and the firm's optimal investment policy is given simply by:

$$F_K(K, X) = [1 - \beta(1-\delta)]\, b \approx (r+\delta)\, b, \tag{14}$$

where r is the real rate of interest. Thus, the firm sets the marginal revenue product of capital, $F_K(K, X)$, equal to the Jorgenson (1963) user cost of capital, $(r+\delta)\, b$.

1.2 The Revenue Function and the Evolution of Shocks

In order to solve for an individual firm's investment policy in the case with costly reversibility, we must first specify the forms of the revenue function, $F(K)$, and the evolution of shocks, $G(X'|X)$. To this end, we assume that the revenue function is of the following log–linear form:

$$F(K, X) = XK^{1-\gamma}, \tag{15}$$

where $\gamma < 1$. It is well–known that such a revenue function can be derived from the combination of an isoelastic product demand curve and a constant returns to scale Cobb–Douglas production function when other (flexible) inputs have been pre–optimized.[5] In addition, we assume that the shocks, X, evolve according to the geometric random walk:[6]

$$\log X' = \log X - \frac{1}{2}\sigma^2 + \varepsilon', \text{ where } \varepsilon' \sim N\left(0, \sigma^2\right). \tag{16}$$

The latter is, of course, the discrete time analogue to the geometric Brownian motion assumption employed in continuous time models of optimal investment under uncertainty, including Abel & Eberly (1996) and Bentolila & Bertola (1990). It will become clear in what follows

[5] See, for example, Abel & Eberly (1996).

[6] We consider a driftless random walk for reasons of notional simplicity. It is trivial to add a drift term to (16) and solve the model in an analogous fashion to that described below.

that a very useful implication of (16) is the following result:

Lemma 1 *If shocks evolve according to the geometric random walk (16), then:*

$$\int_{aX}^{bX} (X')^k \, dG(X'|X) \propto X^k. \tag{17}$$

In particular, for the case where $k = 1$, $\int_{aX}^{bX} X' dG(X'|X) = \lambda \mathbf{p}(a,b) X$, *where* $\mathbf{p}(a,b) = \Pr(X' \in [aX, bX] | X)$, *and* λ *is a selection correction such that* $\lambda \gtreqless 1$ *as* $ab \gtreqless 1$.

Thus the random walk assumption on shocks, (16), implies that the partial expectation of $(X')^k$ is proportional to the current realization X^k when the limits of integration are also proportional to X. In the linear case where $k = 1$, the factor of proportionality is the product of the probability that X' lies within the limits of integration $\mathbf{p}(a,b)$, and a selection correction λ. The term λ accounts for the position of the truncation points in the distribution of X'.[7] Intuitively, if the truncation is geometrically symmetric around the current realization, $(ab = 1)$, then there is no selection correction $(\lambda = 1)$. If the truncation is skewed toward values above (below) the current realization Y $(ab > (<) 1)$, then the partial expectation of future shocks is scaled up (down) accordingly.

It will become clear that this result is very helpful when it comes to solving for the firm's optimal investment policy. In particular, substitution of the revenue function (15) into the recursion for the marginal effect of current investment decisions on future profits, $D(K, X)$, in (12) reveals that this involves the calculation of such partial expectations.

1.3 The Optimal Investment Policy

From the previous analysis, it is clear that solution of the optimal investment policy in the presence of costly irreversibility boils down to solving for three functions. First, inspection of the first–order conditions for a (dis)investing firm, (5) and (6), reveals that we must solve for the marginal effect of current investment decisions on the future profitability of a firm for an investing firm $D(K, U(K))$, and for a disinvesting firm $D(K, L(K))$. Given these, we can then solve for the functions $L(K)$ and $U(K)$ and thereby characterize the investment policy, (4). Note, however, that this exercise is not trivial because the functions, D, L, and U are mutually interdependent. That is, it is necessary to solve for these functions simultaneously.

[7] In particular, λ is the log–normal analogue to the inverse Mills ratio correction that arises in truncated means of normal random variables.

It turns out that, given the assumptions on the functional forms of the production technology and the process of shocks, we are able to obtain a very simple solution to this system of functions. In particular, we first conjecture that the functions are of the form:

$$U(K) = uK^{\gamma},\ L(K) = lK^{\gamma},\ D(K,X) = \sum_{k=0}^{\infty} d_k(K)\,X^k. \tag{18}$$

Thus, we anticipate that the trigger values $U(K)$ and $L(K)$ take a simple log–linear form, which is verified in what follows. However, we allow for a very flexible form for the future marginal effects of a current investment decision, $D(K,X)$. We do this to remain agnostic about the form of $D(K,X)$ for values of X away from the triggers $U(K)$ and $L(K)$, noting from (5) and (6) that the solution for the optimal investment policy requires only knowledge of $D(K,X)$ at these triggers. It is shown below that, under the conjecture, $D(K,X)$ takes a very simple form for X equal to $U(K)$ or $L(K)$.

To get a sense for why the conjecture works, consider as an example the case of an investing firm ($X = U(K)$). To solve for the future marginal effects of the firm's current investment decision, $D(K, U(K))$, substitution of the conjecture (18) into the recursion (12) reveals that one must solve terms in the partial expectation $\int_{L(\mathcal{K})}^{U(\mathcal{K})} (X')^k\, dG(X'|U(K))$. Under the conjecture in (18), and given that $X = U(K)$ for an investing firm, we know that the upper limit of integration $U(\mathcal{K})$ is equal to $(1-\delta)^{\gamma} X$, and the lower limit $L(\mathcal{K})$ is equal to $(l/u)(1-\delta)^{\gamma} X$. Thus, the limits of integration are proportional to the current realization X. The results of the Lemma therefore imply that the partial expectation $\int_{L(\mathcal{K})}^{U(\mathcal{K})} (X')^k\, dG(X'|U(K))$ also will be proportional to X^k. A symmetric reasoning holds for the case of a disinvesting firm fow which $X = L(K)$.

Due to the simplicity of this result, it is straightforward to derive a solution for $D(K,X)$ for $X = U(K)$ or $L(K)$ by simply substituting the conjecture for $D(K,X)$ in (18) into the recursion (12), using the results of the Lemma, and applying the method of undetermined coefficients. The Appendix confirms that application of this method yields solutions of the form:

$$\begin{aligned} D(K, U(K)) &= c_u + d_u U(K)\,\mathcal{K}^{-\gamma}, \\ D(K, L(K)) &= c_l + d_l U(K)\,\mathcal{K}^{-\gamma}, \end{aligned} \tag{19}$$

for some coefficients $\{c_u, d_u, c_l, d_l\}$.

Substitution of these solutions into the first-order conditions for the functions $U(K)$ and $L(K)$ in (5) and (6) reveals that the conjecture in (18) is therefore satisfied. The Appendix provides the following exact solution for the optimal investment policy:

Proposition 3 *The optimal investment policy is of the form in (18), where the constants u and l satisfy:*

$$u = \frac{1}{1-\gamma}\frac{1-\beta(1-\delta)^{1-\gamma}\lambda_+\mathbf{p}_+^0}{1-\beta(1-\delta)\mathbf{p}_+^0}\left\{b_U\left[1-\beta(1-\delta)\right]+\beta(1-\delta)\mathbf{p}_+^-(b_U-b_L)\right\}, \qquad (20)$$

$$l = \frac{1}{1-\gamma}\frac{1-\beta(1-\delta)^{1-\gamma}\lambda_-\mathbf{p}_-^0}{1-\beta(1-\delta)\mathbf{p}_-^0}\left\{b_L\left[1-\beta(1-\delta)\right]-\beta(1-\delta)\mathbf{p}_-^+(b_U-b_L)\right\}, \qquad (21)$$

where, if $\{-,0,+\}$ respectively denotes the disinvestment, freeze, and investment regimes:

- $\mathbf{p}_i^j$ *denotes the probability of moving from regime i in the current period to regime j next period, and is a function solely of $G \equiv u/l$; and*
- $\lambda_+ < 1$ *and $\lambda_- > 1$ are selection corrections for future expected productivity, X', in the event that a currently investing or disinvesting firm freezes investment in the future.*

Equations (20) and (21) are exactly analogous to the smooth pasting solutions derived in Bentolila & Bertola (1990) and in Abel & Eberly (1996), which prescribe that the firm invests in order to keep the marginal revenue product of capital from rising above an upper trigger, $U(K)K^{-\gamma} = u$ here, disinvests to prevent the marginal revenue product from falling below a lower trigger, $L(K)K^{-\gamma} = l$, and otherwise freezes investment. At first blush, equations (20) and (21) may appear complicated. We will see in the next section that simple rearrangement of these conditions yields a surprisingly intuitive economic interpretation of the optimal investment policy.

2 An Economic Interpretation

To see the value added of Proposition 3, however, a particularly useful point of contrast is the solution derived by Abel & Eberly (1996). In particular, the analogous solution presented in their analysis (see their equations (19a) and (19b)) is given by:

$$u = \frac{b_U}{\phi(1/G)}, \text{ and } l = \frac{b_L}{\phi(G)}, \qquad (22)$$

where $G \equiv u/l$ is the ratio of the upper and lower triggers, and where $\phi(\cdot)$ is a given non–linear function. As Abel & Eberly state, a key benefit of this insight is that it reduces a seemingly difficult optimization problem down to the solution of a single variable, the ratio G. Inspection of (20) and (21) reveals that the same is also true of the solution derived here, since the probabilities that determine the optimal investment policy in Proposition 3 are solely functions of G.

However, gaining intuition for the optimal investment policy from (22) is not straightforward because the $\phi(\cdot)$ function does not have an obvious economic interpretation. A useful contribution of the solution in equations (20) and (21) is that it provides such an economic interpretation of the optimal investment policy, and thereby complements the analyses of Bentolila & Bertola and Abel & Eberly. To see this, consider first the optimal policy for an investing firm, (20). A particularly intuitive way of writing this solution is as follows:

$$\overbrace{\sum_{s=0}^{\infty}\left[\beta(1-\delta)^{1-\gamma}\lambda_{+}\mathbf{p}_{+}^{0}\right]^{s}(1-\gamma)XK^{-\gamma}}^{\text{Discounted Marginal Benefit of Capital}}$$
$$=\sum_{s=0}^{\infty}\left[\beta(1-\delta)\mathbf{p}_{+}^{0}\right]^{s}\left\{\underbrace{b_U\left[1-\beta(1-\delta)\right]}_{\text{Jorgensonian Effect}}+\underbrace{\beta(1-\delta)\mathbf{p}_{+}^{-}(b_U-b_L)}_{\text{Irreversibility Effect}}\right\}. \quad (23)$$

As indicated by the annotation, the left hand side of (23) represents the expected discounted marginal benefit from purchasing a unit of capital today. To see this, note first that the current period's marginal revenue product of capital is equal to $(1-\gamma)XK^{-\gamma}$. Thus, the left hand side of (23) states that the expected discounted marginal benefit from investing a unit of capital today is equal to the present value of the current period's marginal revenue product of capital, discounted by a factor $\beta(1-\delta)^{1-\gamma}\lambda_{+}\mathbf{p}_{+}^{0}$ each period.

To understand the intuition for the latter discount factor, recall from equation (12) that the marginal effects of a unit of capital installed today will persist into the future when investment is costly to reverse, because the firm will freeze investment in the future with positive probability. It turns out that, under the optimal investment policy, an investing firm discounts the future marginal effects of its current investment by the (constant) probability of subsequently freezing investment $\mathbf{p}_{+}^{0}$, thereby giving rise to the $\mathbf{p}_{+}^{0}$ component of the discount factor.

The $(1-\delta)^{1-\gamma}$ component of the the discount factor arises for the simple reason that, in the event that the firm freezes investment, the marginal revenue product of capital decays by a fraction $(1-\delta)^{1-\gamma}$ each period, because the marginal unit of capital depreciates by a factor δ each period. Finally, the $\lambda_{+}<1$ component accounts for the fact that, when the firm invests, it purchases capital until the marginal revenue product of capital is equal to its upper bound, the upper trigger u in (20). Thus, in the event that the firm freezes investment in subsequent periods, the firm expects the marginal revenue product of capital to fall below this upper bound. It turns out under the optimal policy that the firm expects the marginal revenue product to decay by a factor $\lambda_+ < 1$ each period.

The right hand side of (23), which represents the expected discounted marginal costs of installing an additional unit of capital today, is also very intuitive. First consider the terms in braces, which reflect the per period marginal costs of purchasing an additional unit of capital today. The first term, $b_U [1 - \beta(1-\delta)]$, is simply the Jorgensonian user cost of a unit of purchased capital, analogous to the case of fully reversible investment, (14).

The second term in braces reflects the effects of investment irreversibility on the marginal costs of investment. Intuitively, if the firm changes its mind next period, and sells the additional unit of capital it purchased today, it can only obtain the resale price $b_L < b_U$. Under the optimal policy, an investing firm will do this with probability $\mathbf{p}_+^-$ next period, and discounts next period's marginal costs by a factor $\beta(1-\delta)$.

The final component of the right hand side of (23) states that the total marginal cost of an additional unit of capital is equal to the present value of the latter per period marginal costs, discounted by a factor $\beta(1-\delta)\,\mathbf{p}_+^0$ each period. This has a similar interpretation to the left hand side of (23). Because the firm subsequently freezes investment with probability $\mathbf{p}_+^0$, the marginal costs of an additional unit of capital persist into the future with constant probability $\mathbf{p}_+^0$. Moreover, because a fraction δ of the additional unit of capital depreciates each period, the firm further discounts the future marginal costs by a factor $\beta(1-\delta)$.

A similar approach can be used to interpret the optimal policy of a disinvesting firm. Specifically, we can rewrite (21) as:

$$\overbrace{\sum_{s=0}^{\infty}\left[\beta(1-\delta)^{1-\gamma}\lambda_-\mathbf{p}_-^0\right]^s(1-\gamma)XK^{-\gamma}}^{\text{Discounted Marginal Benefit of Capital}}$$
$$=\sum_{s=0}^{\infty}\left[\beta(1-\delta)\,\mathbf{p}_-^0\right]^s\left\{\underbrace{b_L[1-\beta(1-\delta)]}_{\text{Jorgensonian Effect}}-\underbrace{\beta(1-\delta)\,\mathbf{p}_-^+(b_U-b_L)}_{\text{Irreversibility Effect}}\right\} \tag{24}$$

In this case, the left hand side has an analogous interpretation to (23): The current marginal revenue product of capital, $(1-\gamma)XK^{-\gamma}$, persists into the future with probability $\mathbf{p}_-^0$, and depreciates by a factor $(1-\delta)^{1-\gamma}$ each period. The selection correction in this case is given by $\lambda_- > 1$: When the firm disinvests, it sells capital until the marginal revenue product of capital is equal to its lower bound, the lower trigger l in (20). Thus, in the event that the firm freezes investment in subsequent periods, the firm expects the marginal revenue product of capital to appreciate by a factor $\lambda_- > 1$ each period.

Similarly, the right hand side of (24) can be interpreted symmetrically. The terms in braces reflect the Jorgensonian return on sold capital, $b_L [1 - \beta (1 - \delta)]$, and the marginal costs of irreversibility from selling a unit of capital: Under the optimal policy, the firm changes its mind with probability $\mathbf{p}_-^+$ next period and purchases back the unit of capital it sold this period at a higher price, $b_U > b_L$. Because the current capital stock persists into the future with probability $\mathbf{p}_-^0$, and because the capital stock depreciates by a factor δ each period, the firm discounts these marginal effects by a factor $\beta (1 - \delta) \mathbf{p}_-^0$ each period.

3 Conclusion

The theory of irreversible investment has been the subject of a vast literature. The explicit solution to optimal investment problems in the presence of costly reversibility has been derived in previous literature using the techniques of smooth pasting. This paper seeks to complement these findings with two additional contributions. First, while the techniques of smooth pasting have enabled much theoretical progress in the solution of models of irreversible investment, such techniques are less widely used in other domains of economics. The first contribution of this paper is to show that the solution for optimal investment can be derived using the more familiar techniques of discrete time dynamic programming. In doing so, this paper opens the analysis of irreversibility to a wider audience of economists.

Second, while the techniques of smooth pasting have yielded explicit solutions for optimal investment in the presence of costly reversibility, the economic interpretation of the optimal investment policy has been less clear. I show that the discrete time approach developed in this paper provides a solution to the investment problem that admits a particularly intuitive interpretation of the investment policy. Specifically, I show that under conventional assumptions on the revenue function and the evolution of shocks, the marginal effects of a firm's current (dis)investment decision persist into future periods with constant probability. As a result, I show that an investing firm sets the present value of the marginal revenue product of capital equal to the present value of the Jorgensonian user cost of purchased capital plus the expected costs of investment reversals, where these present values are discounted by the constant probability that an investing firm freezes investment in subsequent periods. An analogous result holds for a disinvesting firm. My hope is that this interpretation will deepen and elucidate our understanding of the implications of investment irreversibility in future work.

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5 Appendix

A Proofs

Proof of Proposition 2. Define the mapping in equation (12) as $(\mathbf{C}D)(K,X)$. To verify that $\mathbf{C}$ is a contraction mapping, we confirm that Blackwell's sufficient conditions for a contraction hold here (see Stokey & Lucas, 1989, p.54). To verify monotonicity, fix $(\mathcal{K},X)=\left(\bar{\mathcal{K}},\bar{X}\right)$, and take $\hat{D}\geq D$. Then note that:

$$\begin{aligned}\int_{L(\bar{\mathcal{K}})}^{U(\bar{\mathcal{K}})}\hat{D}\left(\bar{\mathcal{K}},X'\right)dG\left(X'|\bar{X}\right)- & \int_{L(\bar{\mathcal{K}})}^{U(\bar{\mathcal{K}})}D\left(\bar{\mathcal{K}},X'\right)dG\left(X'|\bar{X}\right)\\ = & \int_{L(\bar{\mathcal{K}})}^{U(\bar{\mathcal{K}})}\left[\hat{D}\left(\bar{\mathcal{K}},X'\right)-D\left(\bar{\mathcal{K}},X'\right)\right]dG\left(X'|\bar{X}\right)\geq 0\end{aligned} \tag{25}$$

Since $\left(\bar{\mathcal{K}},\bar{X}\right)$ were arbitrary, it thus follows that $\mathbf{C}$ is monotonic in D. To verify discounting, note that:

$$[\mathbf{C}(D+a)](K,X)=(\mathbf{C}D)(K,X)+\beta a\left[G\left(U\left(\mathcal{K}\right)|X\right)-G\left(L\left(\mathcal{K}\right)|X\right)\right]\leq(\mathbf{C}D)(K,X)+\beta a \tag{26}$$

Since $\beta<1$ it follows that $\mathbf{C}$ is a contraction. It therefore follows from the Contraction Mapping Theorem that $\mathbf{C}$ has a unique fixed point. ■

Proof of Lemma. It is useful for the remainder of the proofs to consider a slightly more general case to that stated in the Lemma. Consider a random variable y where $\log y\sim N\left(m,s^2\right)$. Since y is log-Normally distributed, the p.d.f. of y is given by $g(y)=\frac{1}{sy}\phi\left(\frac{\log y-m}{s}\right)$, where $\phi(\cdot)$ is the p.d.f. of the standard Normal. It follows that:

$$\begin{aligned}\int_{\underline{y}}^{\overline{y}}y^k dG(y) &= \int_{\underline{y}}^{\overline{y}}y^k\frac{1}{sy\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{\log y-m}{s}\right)^2\right]dy\\ &= \int_{\underline{y}}^{\overline{y}}y^{k-1}\frac{1}{s\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{\log y-m}{s}\right)^2\right]dy\end{aligned} \tag{27}$$

Defining $z\equiv\log y-m\Longrightarrow y=\exp(z+m)\implies dy=\exp(m+z)\,dz$, we obtain:

$$\begin{aligned}\int_{\underline{y}}^{\overline{y}}y^k dF(y) &= \int_{\log\underline{y}-m}^{\log\overline{y}-m}\exp\left((k-1)(z+m)\right)\frac{1}{s\sqrt{2\pi}}\exp\left[m+z-\frac{1}{2s^2}z^2\right]dz\\ &= \int_{\log\underline{y}-m}^{\log\overline{y}-m}\exp\left((k-1)m\right)\frac{1}{s\sqrt{2\pi}}\exp\left[m+kz-\frac{1}{2s^2}z^2\right]dz\end{aligned} \tag{28}$$

Completing the square for the term in brackets:

$$\frac{1}{2s^2}z^2 - kz = \frac{1}{2s^2}\left(z^2 - 2ks^2 z\right) = \frac{1}{2s^2}\left(z - ks^2\right)^2 - \frac{1}{2}k^2 s^2$$

Substituting back into the former expression, we obtain:

$$\begin{aligned} \int_{\underline{y}}^{\overline{y}} y^k dF(y) &= \exp\left((k-1)m\right) \int_{\log \underline{y} - m}^{\log \overline{y} - m} \frac{1}{s\sqrt{2\pi}} \exp\left[m + \frac{1}{2}k^2 s^2 - \frac{1}{2}\left(\frac{z - ks^2}{s}\right)^2\right] dz \quad (29) \\ &= \exp\left[km + \frac{1}{2}k^2 s^2\right] \left\{\Phi\left[\frac{\log \overline{y} - m}{s} - ks\right] - \Phi\left[\frac{\log \underline{y} - m}{s} - ks\right]\right\} \end{aligned}$$

Setting $y = X'$, $\underline{y} = aX$, $\overline{y} = bX$, and noting from equation (16) that $m = \log X - \frac{1}{2}\sigma^2$ and $s = \sigma$, we can write:

$$\int_{aX}^{bX} \left(X'\right)^k dG\left(X'|X\right) = X^k p_k(a,b), \quad (30)$$

where $p_k(a,b) \equiv e^{\frac{1}{2}k(k-1)\sigma^2}\left\{\Phi\left[\frac{1}{\sigma}\left(\log b - \left(k - \frac{1}{2}\right)\sigma^2\right)\right] - \Phi\left[\frac{1}{\sigma}\left(\log a - \left(k - \frac{1}{2}\right)\sigma^2\right)\right]\right\}$. To obtain the result stated in the Lemma, setting $k = 1$ yields $\int_{aX}^{bX} X' dG\left(X'|X\right) = X p_1(a,b)$. Then note that:

$$\mathbf{p}(a,b) \equiv \Pr\left(X' \in [aX, bX]\,|X\right) = \Phi\left[\frac{1}{\sigma}\left(\log b + \frac{1}{2}\sigma^2\right)\right] - \Phi\left[\frac{1}{\sigma}\left(\log a + \frac{1}{2}\sigma^2\right)\right] = p_0(a,b). \quad (31)$$

Thus, we can write:

$$\int_{aX}^{bX} X' dG\left(X'|X\right) = \lambda \mathbf{p}(a,b) X, \quad (32)$$

where $\lambda \equiv p_1(a,b)/p_0(a,b)$ is a selection correction. Finally, note that $\lambda \gtrless 1$ as:

$$\Phi\left[\frac{1}{\sigma}\left(\log a + \frac{1}{2}\sigma^2\right)\right] - \Phi\left[\frac{1}{\sigma}\left(\log a - \frac{1}{2}\sigma^2\right)\right] \gtrless \Phi\left[\frac{1}{\sigma}\left(\log b + \frac{1}{2}\sigma^2\right)\right] - \Phi\left[\frac{1}{\sigma}\left(\log b - \frac{1}{2}\sigma^2\right)\right]. \quad (33)$$

From the symmetry of the Normal distribution, it follows that if $ab = 1$ ($\log a = -\log b$) then $\lambda = 1$. Moreover, it follows from this that $\lambda \gtrless 1$ as $ab \gtrless 1$. ■

Proof of Proposition 3. To solve for the optimal policy function, we need to specify forms for the functions $L(K)$, $U(K)$, and $D(K,X)$. We conjecture that these are of the form:

$$L(K) = lK^\gamma, U(K) = uK^\gamma, \text{ and } D(K,X) = \sum_{k=0}^{\infty} d_k(K) X^k.$$

We confirm in what follows that the conjecture is verified. Recall from the first–order conditions that we need to solve for $D(K,X)$ for $X = L(K)$ and $X = U(K)$. We deal with each of these cases in turn.

(*i*) *A Disinvesting Firm,* $X = L(K)$: Substituting the conjecture into (12), and applying the results of equation (30) yields

$$
\begin{aligned}
\textstyle\sum_{k=0}^{\infty} d_k(K)\, X^k &= (1-\delta)\, b_L p_0\left[0,(1-\delta)^{\gamma}\right] \\
&\quad + (1-\delta)^{1-\gamma}(1-\gamma)\, K^{-\gamma} p_1\left[(1-\delta)^{\gamma},(1-\delta)^{\gamma}(u/l)\right] X \\
&\quad + (1-\delta)\, b_U p_0\left[(1-\delta)^{\gamma}(u/l),\infty\right] \\
&\quad + \beta(1-\delta) \textstyle\sum_{k=0}^{\infty} d_k(\mathcal{K})\, p_k\left[(1-\delta)^{\gamma},(1-\delta)^{\gamma}(u/l)\right] X^k, \qquad (34)
\end{aligned}
$$

where $X = L(K) = lK^{\gamma}$. Equating coefficients on X^k implies that $d_k(K) \equiv 0$ for all $k \geq 2$. For the remaining coefficients, conjecture that $d_0(K) = d_0$ for all K, and that $d_1(K) = d_1 K^{-\gamma}$. Substituting these conjectures and solving for d_0 and d_1 yields the following solution for $D(K, L(K))$:

$$
\begin{aligned}
D(K,L(K)) &= (1-\delta)\frac{b_L\kappa_2 + b_U(1-\Phi_2)}{1-\beta(1-\delta)(\Phi_2-\kappa_2)} \\
&\quad + (1-\delta)\frac{(\Phi_1-\kappa_1)}{1-\beta(1-\delta)^{1-\gamma}(\Phi_1-\kappa_1)}(1-\gamma)\, L(K)\, \mathcal{K}^{-\gamma}, \qquad (35)
\end{aligned}
$$

where:

$$
\begin{aligned}
\kappa_1 &\equiv \Phi\left(\tfrac{1}{\sigma}\left[\gamma\log(1-\delta)-\tfrac{1}{2}\sigma^2\right]\right), \quad \Phi_1 \equiv \Phi\left(\tfrac{1}{\sigma}\left[\log G+\gamma\log(1-\delta)-\tfrac{1}{2}\sigma^2\right]\right), \\
\kappa_2 &\equiv \Phi\left(\tfrac{1}{\sigma}\left[\gamma\log(1-\delta)+\tfrac{1}{2}\sigma^2\right]\right), \quad \Phi_2 \equiv \Phi\left(\tfrac{1}{\sigma}\left[\log G+\gamma\log(1-\delta)+\tfrac{1}{2}\sigma^2\right]\right),
\end{aligned} \qquad (36)
$$

and where $G \equiv u/l$. Substituting this back into the first-order condition for a disinvesting firm:

$$
\frac{(1-\gamma)\, L(K)\, K^{-\gamma}}{1-\beta(1-\delta)^{1-\gamma}(\Phi_1-\kappa_1)} = \frac{b_L\left[1-\beta(1-\delta)\right]-\beta(1-\delta)(1-\Phi_2)(b_U-b_L)}{1-\beta(1-\delta)(\Phi_2-\kappa_2)} \qquad (37)
$$

Finally, note that $(\Phi_2-\kappa_2) = \Pr(K' = \mathcal{K}|X = L(K)) \equiv \mathbf{p}_{-}^{0}$ is the probability that a currently disinvesting firm freezes investment next period; $1-\Phi_2 = \Pr(K' > \mathcal{K}|X = L(K)) \equiv \mathbf{p}_{-}^{+}$ is the probability that a currently disinvesting firm subsequently invests next period; and $(\Phi_1-\kappa_1) = \lambda_{-}\mathbf{p}_{-}^{0}$ where $\lambda_{-} \equiv \frac{\Phi_1-\kappa_1}{\Phi_2-\kappa_2}$ is a selection correction. The results of the Lemma imply that $\lambda_{-} > 1$.

(*ii*) *An Investing Firm*, $X = U(K)$: Following an analogous method to that above yields the following solution for $D(K, U(K))$:

$$
\begin{aligned}
D(K,U(K)) &= (1-\delta)\frac{b_L\Phi_4 + b_U(1-\kappa_2)}{1-\beta(1-\delta)(\kappa_2-\Phi_4)} \\
&\quad + (1-\delta)\frac{(\kappa_1-\Phi_3)}{1-\beta(1-\delta)^{1-\gamma}(\kappa_1-\Phi_3)}(1-\gamma)\, U(K)\, \mathcal{K}^{-\gamma} \qquad (38)
\end{aligned}
$$

where:

$$
\begin{aligned}
\Phi_3 &\equiv \Phi\left(\tfrac{1}{\sigma}\left[-\log G+\gamma\log(1-\delta)-\tfrac{1}{2}\sigma^2\right]\right), \\
\Phi_4 &\equiv \Phi\left(\tfrac{1}{\sigma}\left[-\log G+\gamma\log(1-\delta)+\tfrac{1}{2}\sigma^2\right]\right),
\end{aligned} \qquad (39)
$$

and where κ_1, κ_2, and G are defined as above. Substituting this solution into the first-order condition for an investing firm, (5), we obtain:

$$\frac{(1-\gamma)\,U(K)\,K^{-\gamma}}{1-\beta(1-\delta)^{1-\gamma}(\kappa_1-\Phi_3)}=\frac{b_U\left[1-\beta(1-\delta)\right]+\beta(1-\delta)\,\Phi_4\,(b_U-b_L)}{1-\beta(1-\delta)(\kappa_2-\Phi_4)} \tag{40}$$

Finally, note that $(\kappa_2-\Phi_4)=\Pr(K'=\mathcal{K}|X=U(K))\equiv\mathbf{p}_+^0$ is the probability that a currently investing firm freezes investment next period; $\Phi_4=\Pr(K'<\mathcal{K}|X=U(K))\equiv\mathbf{p}_+^-$ is the probability that a currently investing firm subsequently disinvests next period; and $(\kappa_1-\Phi_3)=\lambda_+\mathbf{p}_+^0$ where $\lambda_+\equiv\frac{\kappa_1-\Phi_3}{\kappa_2-\Phi_4}$ is a selection correction. Moreover, the results of the Lemma imply that $\lambda_+<1$.

(*iii*) *Solving for G:* Note that the probabilities Φ_1,Φ_2,Φ_3, and Φ_4 all depend on the ratio of the upper investment trigger to the lower disinvestment trigger, G. Using the definition of G and the solution for the investment triggers, we obtain the mapping:

$$G\equiv\frac{u}{l}=\frac{A(\kappa_1-\Phi_3)}{A(\Phi_1-\kappa_1)}\frac{B(\Phi_2-\kappa_2)}{B(\kappa_2-\Phi_4)}\frac{C(\Phi_4,R)}{C(\Phi_2,R)-(R-1)}\equiv T(G,R) \tag{41}$$

where $R\equiv b_U/b_L$ is the ratio of the purchase price of capital to its resale price, and:

$$\begin{aligned}A(x)&\equiv 1-\beta(1-\delta)^{1-\gamma}x\\ B(x)&\equiv 1-\beta(1-\delta)x\\ C(x,R)&\equiv R\left[1-\beta(1-\delta)\right]+(R-1)\beta(1-\delta)x\end{aligned} \tag{42}$$

Given the solution for G from this non–linear mapping, it follows that Φ_1,Φ_2,Φ_3, and Φ_4 are all given constants, and thus that the probabilities $\mathbf{p}_i^j$, $i\in\{-,+\}$, $j\in\{-,0,+\}$ are also given constants, as required. ■