PROJECT DESCRIPTION

ERIC RAMOS

My primary research focus is in commutative algebra and its applications. I am especially interested in understanding the mechanisms underlying various well known asymptotic stability phenomena from representation theory, topology, and combinatorics using the language of commutative algebra. In the following sections I will spend time outlining much of my work in this direction. A particular focus will be put on its relevance to the literature at large, as well as the many interesting open problems which naturally arise.

1. Homological invariants of FI-modules

1.1. Background. Before we provide formal definitions, we begin with an example for motivation. For any finite set $T$, and any topological space $X$, we use $\text{Conf}_T(X)$ to denote the topological space of injections from $T$ to $X$. If $T'$ is any other finite set, and there is an injection $T \hookrightarrow T'$, then precomposition naturally defines a map of topological spaces $\text{Conf}_{T'}(X) \to \text{Conf}_T(X)$. For any fixed $i$, the cohomology groups $H^i(\text{Conf}_n(X); k)$ therefore inherit an interesting structure. Namely, for each $n$ $H^i(\text{Conf}_n(X); k)$ is a $k[\mathfrak{S}_n]$-module, and between any two values of $n$ the corresponding modules are compatible, in some sense. This can be thought of as the motivating philosophy for the study of FI-modules: a means of encoding an infinite collection of compatible symmetric group representations into a single object. Using this structure, Church, Ellenberg, and Farb were able to prove many non-trivial facts about the configuration spaces of orientable manifolds [CEF].

Write FI for the category of finite sets and injections. An FI-module over a commutative ring $k$ is a functor from FI to the category of $k$-modules. These modules, as defined here, were first introduced by Church, Ellenberg, and Farb in their seminal paper [CEF]. Since their inception in the work of Church, Ellenberg, and Farb, FI-modules, and modules over other similar categories, have seen an enormous amount of use in topology, representation theory, and other fields. In its most general form, results of this form are said to be in the field of representation stability. This has been the subject of a recent AIM workshop, as well as an upcoming AMS special session. See [CEF] [CEFN] [PS] [W] [EW-G] [JR] for a small taste of these results.

The category of FI-modules over $k$, FI-Mod, is an abelian category with abelian operations defined point-wise. We say that $V$ is finitely generated if there is a finite set $\{v_i\} \subseteq \cup_T V(T)$, which is contained in no proper submodule of $V$. This notion was first explored by Church, Ellenberg, and Farb in [CEF], although it was not fully explored until the follow-up work of Church, Ellenberg, Farb, and Nagpal [CEFN] [N]. In the paper [CEFN], it is shown that the category of finitely generated FI-modules over a Noetherian ring have a Noetherian property. That is, the category FI-mod of finitely generated FI-modules over a Noetherian ring $k$ is abelian. This fact was proven in certain specific cases by Church, Ellenberg and Farb in [CEF], by Snowden in [S], and by Church, Ellenberg, Farb and Nagpal in [CEFN].

The Noetherian property is the first step in treating finitely generated FI-modules from a homological perspective. Prior to my own work, this philosophy is most apparent in works of Church and Ellenberg [CE], Sam and Snowden [SS], and Gan and Li [GL]. In [CE], Church and Ellenberg consider homology functors $H_i$ for FI-modules. If $V$ is an FI-module, then $H_0(V)$ is defined to be the FI-module with points $H_0(V)(T) = V(T)/\cup_{T' \subseteq T} V(T')$. The homology functors $H_i$ are defined to be the (left) derived functors of $H_0$. It follows directly from the definition that the module $H_0(V)$ is finitely supported, i.e. $V(T) = 0$ for $|T| \gg 0$, whenever $V$ is finitely generated. Remarkably, it can be shown that for any finitely generated module $V$, the quantity $\max_n \{H_i(V)(n) \neq 0\} - i$ is
bounded independently of $i$. This was first proven for certain choices of $k$ by Sam and Snowden in [SS], and expanded to allow for any commutative ring $k$ by Church and Ellenberg [CE]. This bound, which we shall henceforth refer to as the *regularity* of $V$, was used by Church and Ellenberg to generalize certain stability results of Putman on congruence subgroups [Pu]. Note that the regularity in this context is exactly analogous to the notion of Castelnuovo-Mumford regularity, a concept of foundational importance to commutative algebra [E]. Understanding the mechanisms underlying the finiteness of regularity therefore becomes critical in explaining certain concrete phenomena in the study of congruence subgroups.

Another property connected to homological invariants is the existence of the *Hilbert polynomial*. If $V$ is a finitely generated FI-module over a field $k$, then there exists a polynomial $P(X) \in \mathbb{Q}[X]$ such that for all finite sets $T$ with $|T| \gg 0$,

$$\dim_k(V(T)) = P(|T|).$$

This fact was proven in the case where $k$ is a field of characteristic 0 by Church, Ellenberg, and Farb in [CEF], and was expanded to all fields by Church, Ellenberg, Farb, and Nagpal in [CEFN]. Seeing a result like this, it therefore becomes quite natural to ask when this polynomial behavior begins. A bound is given by Church, Ellenberg, and Farb in the cases wherein $k$ is a field of characteristic 0 in [CEF], although no bound is given in the cases in which $k$ is a general field. My work reveals that this question, as well as that of the regularity of a module, are deeply connected to one another through the study of local cohomology.

### 1.2. Local cohomology

Let $V$ be an FI-module over a Noetherian ring $k$. Then we say that an element $v \in V(T)$ is *torsion* if there is some finite set $T'$, and an injection $f : T \to T'$ such that $v$ is in the kernel of the map induced by $f$. We define $H^0_m(V)$ to be the maximal torsion submodule of $V$. The assignment $V \mapsto H^0_m(V)$ defines a left exact functor, and we denote its derived functors by $H^i_m$. These are the *local cohomology functors*.

In the case where $k$ is a field of characteristic 0, Sam and Snowden studied the local cohomology functors, and proved that many important invariants of FI-modules, such as regularity, were encoded by them [SS]. When $k$ is a field of characteristic 0, however, Sam and Snowden show that FI-modules can be equivalently thought of as a subclass of modules over a polynomial ring in infinitely many variables [SS]. The first challenge to generalizing the results of Sam and Snowden to general coefficient rings was in finding a consistent language which would allow us to use the commutative algebra techniques of their work. This was accomplished in my work with Li. [LR].

To begin, we show that the modules $H^i_m(V)$ can always be computed within the category $\text{FI-mod}$ of finitely generated FI-modules, so long as $V$ is finitely generated.

**Theorem 1.1** (Li and Ramos [LR]). Let $V$ be a finitely generated FI-module over a Noetherian ring $k$. Then for each $i$, the module $H^i_m(V)$ is finitely generated, and $H^i_m(V)(T) = 0$ for all $T$ with $|T| \gg 0$. Moreover, $H^i_m(V) = 0$ for $i \gg 0$.

Considering the theory of local cohomology in the context of local rings, the above might be quite surprising. Indeed, classically it is known that the top non-vanishing local cohomology module is never finitely generated [LS].

The above theorem implies that each finitely generated module $V$ has a well defined smallest, and largest, non-vanishing local cohomology module. We define the *depth* of $V$ to be the smallest index $i$ such that $H^i_m(V) \neq 0$. In [R], I introduced a definition of depth which can be used for FI-modules over any ring. It is a theorem of Li and myself these two definitions are equivalent [LR].

**Theorem 1.2** (Li and Ramos [LR]). Let $V$ be a finitely generated FI-module over a Noetherian ring $k$. Then the regularity of $V$ is at most $\max_i \{\deg(H^i_m(V)) + i\}$.

The quantity $\max_i \{\deg(H^i_m(V)) + i\}$ is very reminiscent of the Castelnuovo-Mumford regularity of a module over a polynomial ring [E]. In fact, conjecturally $\max_i \{\deg(H^i(V)) + i\}$ is equal to
the regularity of the FI-module $V$, just as is the case in the classical setting. This conjecture was proven for torsion modules by Gan and Li in [GL2], but otherwise remains open.

**Theorem 1.3** (Li and Ramos [LR]). *Let $V$ be a finitely generated FI-module over a field $k$. Then the dimension $\dim_k(V(T))$ agrees with a polynomial for $|T|$ at least $\max_i \{ \deg(H_{\text{m}}^i(V)) \}$.*

Note that the above two theorems provide an explicit connection between the regularity of a module and the obstruction to its Hilbert polynomial. Once again, such a connection exists and is classically known about modules over a polynomial ring [E].

**1.3. Future Directions.** While my work with Li lays the groundwork for a “commutative algebra” approach to the study of FI-modules, there is still much which is not well understood. To begin, the conjecture of Li and myself on the connection between regularity and the local cohomology of a module has still not been fully resolved. Proving this would greatly deepen our understanding of the structure of FI-modules, and their parallels to modules over polynomial rings. Moreover, using the computational tools of Sam and Snowden for local cohomology [SS], we would have explicit means for computing the regularity of an FI-module over a field of characteristic 0. The work of Church and Ellenberg [CE] implies that these computations have real applications in the study of congruence subgroups.

It is also an interesting question to ask how these homological invariants are interacting with the various topological examples of FI-modules. While the theorems of the previous section can be used to bound the obstruction to the Hilbert polynomial, for instance, it is often the case in practice that these bounds are non-optimal. This would suggest that “niceness” in the topology from which the example arises is being observed by the local cohomology modules. Can this be formally described? Conversely, can one deduce facts about the topology of a provided example, given its local cohomology?

**2. Coherence in the representation theory of categories**

**2.1. Background.** In the previous sections, we saw that there is a large premium put on the condition that an FI-module be finitely generated. However, for one to have a Noetherian property, it is necessary that one work over a Noetherian ring. For some applications one would like to work with FI-modules over rings which are not Noetherian. Once again pulling from commutative algebra, perhaps the next best thing after Noetherianity is coherence. We say an FI-module over a ring $k$ is **coherent** if it is generated by elements which only appear in finitely many degrees, and the module of relations between these generators is also generated in finitely many degrees. Note that a coherent module need not be finitely generated, but the (possibly infinitely many) generators must appear in only finitely many degrees.

The first time coherence was studied with respect to FI-modules was in the work of Church and Ellenberg [CE]. In this work they prove that coherent FI-modules over any ring $k$ have finite regularity.

**2.2. The coherence of FI-modules.** Much of the work described on the local cohomology of FI-modules was only doable because the Noetherian property allows us to treat FI-mod using the techniques of abelian categories. Our first goal will therefore be to prove something similar for coherent modules. The following theorem was independently proven by Li and myself.

**Theorem 2.1** (Li [L], Ramos [R2]). *The category of coherent FI-modules over a commutative ring $k$ is abelian. That is, the kernel and cokernel of any morphism of coherent FI-modules are themselves coherent.*

Perhaps the most notable fact about the above theorem is that it does not have any conditions on the ring $k$. This implies that while FI-modules may not be Noetherian, they are always coherent. The regularity theorem of Church and Ellenberg, along with the above theorem, seems to imply that the work completed in Section 1.2 should still hold for coherent FI-modules. This is indeed the case.
Theorem 2.2 (Ramos [R2]). The theorems of Section 1.2 continue to hold with finitely generated replaced in all places by coherent, and the Noetherian hypothesis removed from the ring $k$.

2.3. Future directions. While many of the various well studied concrete examples of FI-modules often turn out to be finitely generated, more recent examples have shown that coherent modules also arise naturally. For instance, the work of Church and Ellenberg provides examples of this kind in the study of congruence subgroups for general rings [CE]. What remains interesting is the question of whether finite generation is something one should expect to find in more general and nuanced examples. For instance, if one were to change FI to some more general category, whose representations are not necessarily Noetherian, can something still be said about coherent representations?

As an example, consider the category VI($\mathbb{Z}$), of finite rank free $\mathbb{Z}$ modules, with split linear injections. Representations of this category were considered by Putman and Sam in [PS], and have more recently been used in the work of Patzt [Pa]. Representations of similar categories were critical in the resolution of the Lannes-Schwartz Artinian conjecture [PS]. It was proven by Putman and Sam that representations of VI($\mathbb{Z}$) will not have the Noetherian property over any ring $k$. However, the work of Patzt [Pa] seems to imply that these representations appear in nature. In that work, Patzt specifically considers their applications to the study of Torelli groups.

I am interested in studying VI($\mathbb{Z}$) representations, and especially interested in understanding whether they satisfy any kind of coherence. A result in this direction would provide at least a framework for approaching the many natural examples of VI($\mathbb{Z}$) representations without needing to rely on any kind of Noetherian hypothesis. It would also expand upon results of Patzt [Pa].

For instance, if $k$ is a field of characteristic 0, can one prove a structure theorem similar to that of Nagpal [N] for coherent VI($\mathbb{Z}$)-modules?

3. The configuration spaces of graphs

3.1. Background. A graph will always refer to a 1-dimensional CW complex which is both connected and compact. The study of the configuration spaces of graphs has seen a recent surge in popularity due to their connection with robotics [G] [Far]. For any graph $G$, we will largely concern ourselves with unordered configurations, $\text{UConf}_n(G)$, the quotient of the usual configuration space by the action of the symmetric group.

It is a theorem of Ghrist [G], which was later reproven and expanded upon by Abrams [A], that $\text{UConf}_n(G)$ is a $K(\pi, 1)$ for all $n \geq 1$ and all graphs $G$. It follows from this that in order to understand the homotopy type of $\text{UConf}_n(G)$, it largely suffices to understand the fundamental group $B_n G := \pi_1(\text{UConf}_n(G))$. In the literature, the braid groups $B_n G$ have largely been studied from the perspective of geometric group theory. For instance, it is now fairly well understood when these groups are right-angle Artin [KP] [KKP].

Another direction of great interest are the homology groups $H_i(B_n G)$. It follows from the discussion in the previous paragraph the usual homology of the space $\text{UConf}_n(G)$ is canonically isomorphic to the group homology of $B_n G$. Using this isomorphism, authors such as Kim, Ko, and Park [KKP] [KP], as well as Farley and Sabalka [PS], have been able to prove surprising facts about the homology groups $H_i(B_n G)$. For instance, Ko and Park have shown that $H_1(B_2 G) \cong H_1(B_n G)$ for all $n$ whenever $G$ is biconnected [KP]. Farley has also used the isomorphism to provide a computational method for determining the groups $H_i(B_n G)$ whenever $G$ is a tree [Fa]. All of these results use a discrete Morse structure on $\text{UConf}_n(G)$, which was developed by Farley and Sabalka [FS]. In my work, I have used this isomorphism to prove non-trivial facts about the homology of the unordered configuration spaces of trees.

3.2. Stability phenomena in the homology of tree braid groups. The philosophy behind my work on the configuration spaces of graphs is based in the philosophy of asymptotic algebra as a whole. Namely, whenever a family of algebraic objects exhibits asymptotic stability phenomena, it is often the case that they can be encoded in a single object, which is finitely generated in the
appropriate sense. The work of Church, Ellenberg, and Farb, for instance, involved encoding the cohomology groups of configuration spaces of higher dimensional manifolds into an FI-module which is finitely generated. In my work on the configuration spaces of trees, I show that the homology groups $H_i(B_nG)$ can be encoded in a finitely generated graded module over a polynomial ring.

For a graph $G$, an **essential vertex** is any vertex of degree at least 3. An **essential edge** is a connected component of the space obtained by removing all essential vertices of $G$. We also define the quantity $\Delta^i_G$ to be the maximum number of connected components that $G$ can be broken into by removing exactly $i$ vertices.

**Theorem 3.1** (Ramos [R3]). Let $G$ be a tree, and fix $i \geq 0$. Then there is a polynomial $P_i^G \in \mathbb{Q}[t]$ of degree $\Delta^i_G - 1$ such that for all $n \geq 0$

$$P_i^G(n) = \dim_{\mathbb{Q}}(H_i(B_nG;\mathbb{Q}))$$

This was proven using the structure theorems of Farley and Sabalka [FS], as well as the computational theorems of Farley [Fa]. In fact, I compute the polynomials $P_i^G$ explicitly in terms of certain invariants of the tree $G$ in [R3]. It follows from this computation that the homology groups $H_i(B_nG)$ do not fully depend on the tree $G$.

**Corollary 3.2** (Ramos [R3]). Let $G$ be a tree. Then the homology group $H_i(B_nG)$ depends only on $i$, $n$, and the degree sequence of $G$.

The key insight to proving the above theorem, which is perhaps more significant than the result itself, is that the groups $H_i(B_nG)$ carry a natural action by a polynomial ring.

**Theorem 3.3** (Ramos [R3]). Let $G$ be a tree, and let $S_G$ denote the integral polynomial ring with variables indexed by the essential edges of $G$. Then for each $i \geq 0$, there is an action of $S_G$ on the graded $\mathbb{Z}$-module $\mathcal{H}_i := \bigoplus_n H_i(B_nG)$, turning $\mathcal{H}_i$ into a finitely generated graded $S_G$-module. Moreover, $\mathcal{H}_i \otimes_{\mathbb{Z}} \mathbb{Q}$ decomposes as a direct sum of graded twists of squarefree monomial ideals of Krull dimension at most $\Delta^i_G$.

### 3.3. Future directions.

The first obvious question arising from my work is whether it can be applied to more general graphs. Indeed, in [R3] it is shown that the action of $S_G$ on $\mathcal{H}_i$ will still be well defined, provided certain diagrams commute. My first goal will therefore be to examine the general case, and prove that my methods apply there as well. In fact, it is my belief that my methods can be applied to prove the following conjecture.

**Conjecture 3.4.** If $G$ is a graph, which is neither a line segment nor a circle, then $\dim_{\mathbb{Q}}(H_i(B_nG;\mathbb{Q}))$ is eventually equal to a polynomial of degree at most $\Delta^i_G - 1$.

Another question which one might ask is whether anything can be said about the usual configuration spaces $\text{Conf}_n(G)$. The current consensus in the literature is that these spaces are considerably more complicated than the unordered configuration spaces, and very little is known about their behavior [BP] [KP]. My work suggests, however, that the homologies of these spaces can be endowed with the structure of a representation a category, which we denote $\text{FP}$. While the definition of this category is somewhat technical, one may think of $\text{FP}$ as an extension of $\text{FI}$, where each injective morphism is decorated by an ordering on the complement of its image. It can be shown that representations of $\text{FP}$ are non-Noetherian, and seem to be largely chaotic. I believe that viewing the homologies of $\text{Conf}_n(G)$ as an $\text{FP}$-module can yield non-trivial asymptotic information about these spaces.
References


[LR] L. Li, Two homological proofs of the Noetherianity of FL1, arXiv:1603.04552


