Is multilateral enforcement vulnerable to bilateral renegotiation?

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Abstract

In a multilateral enforcement regime, a player who cheats on one partner is punished by many partners. But if partners can renegotiate in private, they can subvert the power of the multilateral punishment. We introduce a new notion of “bilateral renegotiation proofness” that applies to multilateral games with private monitoring. For symmetric networked environments, we characterize an optimal bilateral renegotiation proof equilibrium. While players’ ability to renegotiate bilaterally is indeed socially costly, it is perhaps not as costly as one might expect. In densely connected communities, the proportional cost imposed by bilateral renegotiation declines as the number of participants grows, and vanishes in the limit.

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1 Introduction

Multilateral enforcement is a powerful guarantor of cooperation: deviation is much less attractive if it is met with many punishments rather than one. To deter deviations, threatened punishments should be credible. Perfect Bayesian equilibrium, for instance, requires that no player—neither the deviator being punished nor another imposing the punishment—should believe he or she could profit by deviating unilaterally. This is a reasonable necessary condition for a voluntary multilateral enforcement regime.

But in a multilateral environment, pairs of agents may be able to subvert an intended punishment by jointly renegotiating to outcomes that are better for them ex post. Such joint deviations may be particularly easy to arrange—and keep secret—if productive interactions arise primarily within bilateral relationships. Without directly observing what happens in each relationship, how can all parties be sure that someone who is meant to be punished is actually being punished?

In this paper we show that while players’ ability to renegotiate bilaterally is socially costly, it is perhaps not as costly as one might expect. Ali and Miller (2013) study social networks made up of persistent bilateral relationships, and show that “contagion” imposes the harshest punishments, and thereby maximizes cooperation when renegotiation is ruled out. In a contagion equilibrium, as soon as one agent shirks in one interaction, all agents who becomes aware of the departure from the equilibrium path thereafter shirk in all their interactions, and the contagion of non-cooperative behavior spreads throughout the community. But contagion is, of course, vulnerable to bilateral renegotiation: two players who have common knowledge that they are supposed to shirk on each other forever can do better for themselves by agreeing to cooperate.

We construct a “contagion-like” equilibrium that is invulnerable to bilateral renegotiation, and optimal among bilateral renegotiation proof equilibria (Theorem 2). In this equilibrium, players minimize the cost of bilateral renegotiation by employing asymmetric play in their bilateral relationships: off the equilibrium path, whoever shirks “strictly first” in each relationship is punished, while the other partner is rewarded. A player who deviates from the equilibrium path shirks “weakly first” in each relationship, and is therefore never rewarded. Bilateral renegotiation is still costly because there are off-path histories at which both partners first shirk simultaneously, but both cannot be punished because they would then renegotiate.

Nonetheless, we show that in a multilateral enforcement environment with many players, the proportional cost imposed by bilateral renegotiation (relative to an optimal contagion equilibrium) decreases as the network of relationships grows denser, and vanishes at the limit (Theorem 3). Specifically, the network “grows denser” in this sense if, as each player gains more and
more partners, his partners expect to learn more quickly—and immediately in the limit—if he
deviates. This is the case for complete networks as the population goes to infinity, and is true
more generally as long as the cycles in the network do not become longer and less interconnected
even faster than each player’s number of partners grows. The proportional cost of renegotiation
decreases in the density of the network of relationships because, even though the fraction of off-
path histories in which both partners first shirk simultaneously increases, this negative effect is
swamped by the growing ratio between the levels of cooperation supported by multilateral en-
forcement (which increases) and by bilateral enforcement (which stays constant).

Though the problem of renegotiation in games is well studied,¹ there is no well-established
notion of what it means for partners to secretly renegotiate in infinite-horizon environments. We
propose a notion of bilateral renegotiation proofness that rules out equilibria in which any pair of
players meeting within the game could bilaterally renegotiate, in secret, to gain a strict improve-
ment for both of them. In our private monitoring environment, it is important to stipulate that
the possibility for them both to gain a strict improvement must be common knowledge between
them.

We study a particular class of environments in which all interactions occur in long-term bi-
lateral relationships. Each pair of players engages in a repeated joint production game, where a
player’s costly effort benefits his partner. Monitoring in such an environment is naturally private:
each player’s effort in a relationship is observed only his partner. Therefore, when a player who has
shirked on one partner meets another partner for the first time, it cannot be common knowledge
between them that they are off the equilibrium path. Instead, at least one of them must shirk be-
fore they have common knowledge that they can bilaterally renegotiate to make themselves both
better off.

In addition to our results on multilateral enforcement, we contribute directly to the literature
on renegotiation in two-player games by identifying conditions under which there exists an ef-
ficient renegotiation-proof equilibrium in an infinitely repeated, variable-effort generalization of
the Prisoners’ Dilemma (Theorem 1).

In a later draft of this work in progress, we will address the possibility that players could rene-
gotiate in subsets larger than pairs, and elaborate on the connection between our work and the
existing literature.

¹Key early papers: Rubinstein (1980); van Damme (1989); Farrell and Maskin (1989); Bernheim and Ray (1989).
2 The model

Bilateral interactions To model bilateral relationships with moral hazard, we model players engaged in repeated joint production opportunities. A pair of players (“partners”) meets at random times generated by a Poisson process of rate $\lambda > 0$. Each time partners $i$ and $j$ meet, they first have the opportunity to renegotiate (we discuss renegotiation below in the context of our solution concept); then they play a joint production stage game in which they simultaneously choose their effort levels $\phi_i, \phi_j \in \mathbb{R}_+$. The continuous action spaces allow partners to endogenously adjust the terms of their relationship as play unfolds. Ghosh and Ray (1996) introduced variable effort stage games to the social networks literature.2 The partners discount their payoffs over time at a common discount rate $r > 0$, and have access to arbitrary public randomization devices.

Player $i$’s stage game payoff function when meeting his partner $j$ is $b(\phi_j) - c(\phi_i)$; where $b(\phi_i)$ is the be from his partner’s effort but incurs costs $c(\phi_i)$ from his own. The benefit function $b$ and the cost function $c$ are smooth functions satisfying $b(0) = c(0) = 0$.

To illustrate the payoff functions, suppose partners $i$ and $j$ are supposed to choose strictly positive efforts $(\phi_i, \phi_j)$, but if one of them deviates he deviates to zero effort. Then the payoff matrix is a Prisoners’ Dilemma:

<table>
<thead>
<tr>
<th>$\phi_i$</th>
<th>$\phi_j$</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi_j$</td>
<td>$b(\phi_j) - c(\phi_i), b(\phi_i) - c(\phi_j)$</td>
<td>$-c(\phi_i), b(\phi_i)$</td>
</tr>
<tr>
<td>0</td>
<td>$b(\phi_i), -c(\phi_j)$</td>
<td>0, 0</td>
</tr>
</tbody>
</table>

For ease of notation, we hereafter write the net social benefit of effort $\phi$ as $(b - c)(\phi) \equiv b(\phi) - c(\phi)$; throughout the paper we assume that it grows in the following manner.

Assumption 1 (Increasing benefit from cooperation). The net social benefit of effort $(b - c)(\phi)$ is strictly increasing and weakly concave, with $(b - c)(0) = 0$. Moreover, there exists an arbitrarily small $\psi > 0$ such that $(b - c)'(\phi) > \psi$ for all $\phi \geq 0$.

Monotonicity means higher effort is always socially beneficial; concavity means it is better for partners to exert similar effort, holding their average effort constant.

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2See also Kranton (1996). Also similar is the variable stakes framework of Ali and Miller (2013, 2015), but their extensive form stage games force partners to exert symmetric effort, whereas the present model allows for asymmetric efforts.
**Assumption 2** (Increasing cost of effort). The cost \( c \) is strictly increasing and strictly convex, with \( c(0) = c'(0) = 0 \) and \( \lim_{\phi \to \infty} c'(\phi) = \infty \). Moreover the “relative cost” \( c(\phi)/(b - c)(\phi) \) is strictly increasing.

Strict convexity with the limit condition guarantees that in equilibrium effort is bounded (as long as continuation payoffs are bounded, which we assume below). Increasing relative cost means a player requires proportionally stronger incentives to exert higher effort.\(^3\)

Observe that when partners \( i \) and \( j \) meet, player \( i \)'s best deviation in the stage game is to exert zero effort, benefiting from player \( j \)'s effort at \( b(\phi_j) \) but incurring no cost of his own effort. We say the partners “cooperate” if the equilibrium calls for them to choose the same equilibrium effort \( \phi \) for some \( \phi > 0 \), and they do so. We say that partner \( i \) “works” if he chooses effort \( \phi_i > 0 \); we say that he “shirks” if he chooses \( \phi_i = 0 \).

**Social network** Bilateral interactions between partners are embedded in a social environment described by a network. Let \( N \equiv \{1, \ldots, n\} \) be the set of players, connected by an undirected network \( G \), which is a set of cardinality-2 subsets of \( N \). The network is commonly known by the players, and is fixed throughout the game. We use \( \{ij\} \) to indicate a link, and define \( |G| \) to be the number of links in \( G \). Much of our analysis concerns the incentives of each player on the link: accordingly, we use \( ij \) to signify “player \( i \) on link \( \{ij\} \)” as distinct from \( ji \) (“player \( j \) on link \( \{ij\} \)”). We focus on symmetric networks:

**Definition 1.** A network \( G \) is symmetric if for any two links \( \{ij\}, \{kl\} \in G \), there exists a graph-automorphism\(^4\) \( f: N \to N \) such that \( f(i) = k \) and \( f(j) = l \).

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\(^3\)One might wonder whether increasing relative cost is compatible with \( c'(0) = 0 < \psi < (b - c)'(0) \). In fact, increasing relative cost in a neighborhood of \( \phi = 0 \) is guaranteed by our other assumptions:

\[
\lim_{\phi \to 0} \left( \frac{c}{b-c} \right)'(\phi) = \lim_{\phi \to 0} \left( \frac{c'(b-c) - c(b-c)'}{2(b-c)^2} \right)(\phi) = \lim_{\phi \to 0} \left( \frac{c''(b-c) - c'(b-c)''}{2(b-c)(b-c)'} \right)(\phi) = \lim_{\phi \to 0} \left( \frac{c''(b-c)'}{2(b-c)'} \right)(\phi) > 0.
\]

\(^4\)A graph automorphism is a bijection \( f: N \to N \) such that \( \{f(i)f(j)\} \in G \) for all \( \{ij\} \in G \).
Equilibrium  We study perfect extended-Bayesian equilibrium, imposing the restriction that all effort choices are uniformly bounded across histories; we refer to these as *equilibria*. We restrict attention to equilibria that are symmetric: play does not depend on player names at any history, and is stationary on the equilibrium path.

3  Bilateral renegotiation proofness

Many equilibria have the troubling property that two partners may commonly know that the payoff vector they are supposed to receive after some history is Pareto dominated by what they could earn in their best bilateral equilibrium. If the partners can communicate, they should be able to discuss and agree to a self-enforcing arrangement in which they are both strictly better off. Such behavior off the equilibrium path raises the concern of renegotiation: why would players shirk forever when they can coordinate on continuation behavior that makes them all better off?

We adapt existing notions of renegotiation to our setting by imposing two natural restrictions: (i) renegotiation occurs only bilaterally between linked partners, and only when their link is recognized; (ii) partners renegotiate on the basis only of information that they commonly know, namely the history of interaction within their partnership. This notion of renegotiation sidesteps challenges of bargaining with private information and private monitoring (for which standard renegotiation-proofness concepts do not apply), and adheres to our formulation in which all interactions take place in pairwise meetings.

Histories and strategies  An *interaction* between players *i* and *j* at time *t* comprises the time *t* at which the pair meets, their names, and their effort choices. The interaction of a partnership \{*ij*\} is perfectly observed by partners *i* and *j*, but not observed at all by any other player. Player *i*’s private history, denoted *h*<sub>*i*</sub>, is the set of all interactions in which she has participated up to a certain point.

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5 Like sequential equilibrium, perfect extended-Bayesian equilibrium rules out various pathologies possible under perfect Bayesian equilibrium that may arise from inconsistent beliefs. Compared to sequential equilibrium, perfect extended-Bayesian equilibrium has the advantage of being well defined even for our environment with uncountably many actions and information sets. See Battigalli (1996) for details.

6 The restriction eliminates unreasonable equilibria in which effort grows with further cooperation, eventually exploding to infinity.

7 The literature on bilateral recontracting in settings with a principal and multiple agents (Crémer and Riordan 1987; McAfee and Schwartz 1995, 1994) shares aspects of our motivation, but finds results that depend crucially on what an agent is assumed to infer about whether he should accept when receiving a deviant offer to renegotiate from the principal. In contrast, our BRP notion would disqualify an equilibrium only if there existed a deviant bilateral contract such that both partners had common knowledge that it would make them better off. In this sense we allow, but do not require, partners to play as if they had renegotiated on the basis of information that is not common knowledge between them.
in the game. When we say that players $i$ and $j$ interact at a pair of private histories $(h_i, h_j)$, we specifically exclude their current interaction from $h_i$ and $h_j$. We define their partnership history as the intersection of their private histories: $h_{ij} = h_i \cap h_j$. Since players observe only what happens along their own links, partnership history $h_{ij}$ is simply a list of all the interactions on link $\{ij\}$ prior to their meeting at $h_{ij}$.

We say that a strategy for player $i$ is bilateral on link $\{ij\}$ if for every $h \in H_{ij}$, her behavior is measurable with respect to the partnership history on link $\{ij\}$. A strategy for player $i$ is bilateral if it is bilateral on each link involving player $i$; a strategy profile is bilateral if all players’ strategies are bilateral. Even in a strategy profile $\sigma$ that is not bilateral, if after some partnership history $h_{ij}$ the behavior of partners $i$ and $j$ is measurable with respect to the partnership history at all interactions on $\{ij\}$ whose partnership history contain $h_{ij}$, then we say that the partners play a continuation strategy profile $\sigma(h_{ij})$ that is bilateral on link $\{ij\}$. We denote by $U_{ij}(\sigma(h_{ij}))$ the expected continuation payoff that player $i$ obtains on link $\{ij\}$ from a bilateral continuation strategy profile $\sigma(h_{ij})$ after partnership history $h_{ij}$. We nominate $\mathcal{U}$ in flow terms; i.e., $U_{ij}(\sigma(h_{ij}))$ is the constant payoff arriving at Poisson rate $\lambda$ such that player $i$ is indifferent between this constant flow and continuation strategy profile $\sigma(h_{ij})$.

We focus on equilibria in which players do not want to renegotiate. Our innovation for the renegotiation proofness literature is that we allow the players to negotiate only bilaterally, and only in a way that is measurable with respect to events that are common knowledge between them. First we define a weak notion of bilateral renegotiation proofness, based on the idea that two partners can always renegotiate to behave as if they are at a different partnership history in the same equilibrium.

**Definition 2.** An equilibrium $\sigma$ is **weak bilateral renegotiation proof (WBRP)** if it is bilateral, and for every link $\{ij\}$ and feasible partnership history $h_{ij}$, there does not exist an alternative feasible partnership history $h'_{ij}$ such that $\sigma(h'_{ij})$ yields a strict Pareto improvement for players $i$ and $j$—i.e., $U_{ij}(\sigma(h'_{ij})) > U_{ij}(\sigma(h_{ij}))$ and $U_{ji}(\sigma(h'_{ij})) > U_{ji}(\sigma(h_{ij}))$—or any convex combination of alternative feasible partnership histories that yields such an improvement in expectation.

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8 Throughout, we consider only histories that Al and Miller (2013) term regular—those histories in which no two links have ever been recognized simultaneously. The set of histories that are not regular has zero probability, regardless of the players’ strategies.

9 We assume that their only opportunity to renegotiate during their interaction is at the start of their interaction; they cannot renegotiate after observing the realization of their public randomization device. Another way to conceptualize this restriction is to view the selection of randomization devices as arising from the negotiated agreement.
Even in an equilibrium that is not bilateral, partners always have the option to bilaterally renegotiate to play a "better" WBRP equilibrium along their link, if one exists. We focus on a demanding notion of what it means for one equilibrium to be "better" than another. The \(ij\)-joint payoff set (or simply joint payoff set when the pair of partners is implicit) of a strategy profile \(\sigma\) at partnership history \(h_{\{ij\}}\) is the union over all expected payoff vectors arising from activity on link \{\(ij\)\} given \(\sigma\) at all pairs of private histories that contain \(h_{\{ij\}}\).

We strengthen a definition from Miller and Watson (2013) to incorporate the “weak Pareto principle,” as follows.\(^\text{10}\) Say that a joint payoff set \(W \subset \mathbb{R}^2\) fully Pareto dominates another joint payoff set \(W' \neq W\) if

1. For every \(a \in W \setminus W'\) there does not exist any \(b \in W'\) for which \(b_1 > a_1\) and \(b_2 > a_2\);
2. For every \(b \in W' \setminus W\) there exists \(a \in W\) such that \(b_1 > a_1\) and \(b_2 > a_2\).

In any equilibrium, bilateral or not, no two partners should ever have common knowledge that they could gain by renegotiating to a WBRP equilibrium in their relationship.

**Definition 3.** An equilibrium \(\sigma\) is bilateral renegotiation proof (BRP) if for every link \{\(ij\)\} and feasible pair of private histories \((h_i, h_j)\), there does not exist any alternative WBRP equilibrium \(\sigma'\) such that it is common knowledge between \(i\) and \(j\) that the \(ij\)-joint payoff set of \(\sigma\) at \(h_{\{ij\}}\) is fully Pareto dominated by the \(ij\)-joint payoff set of \(\sigma'\).

In order for partners to renegotiate in a way that invalidates their equilibrium, they must have common knowledge that there exists an alternative WBRP equilibrium that fully Pareto dominates their joint payoff set at some history. Full Pareto dominance means the alternative WBRP equilibrium must be not merely better for them at the outset, but also in a sense not worse for them at any possible future history. This requirement embodies the idea that their renegotiation is forward-looking: if they could renegotiate in a way that would make them worse off at some future history, they recognize that they could also renegotiate again at that history.\(^\text{11}\)

**Remark 1.** The common knowledge requirement rules out the possibility of revealing information by offering to renegotiate. For instance, without the common knowledge requirement, consider a player \(i\) who knows he is off the equilibrium path and meets a partner \(j\) with whom he does not have

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\(^{10}\)Miller and Watson (2013) used a weaker definition to select the largest equilibrium payoff set among appropriate nested sets. We use the stronger definition so as not to force partners to renegotiate when one of them could reasonably refuse to.

\(^{11}\)Due to the full Pareto dominance requirement, in two-player games BRP is closest, among the variety of renegotiation proofness notions in the literature, to the “stationary Pareto perfection” notion of Asheim (1991).
common knowledge of being off the equilibrium path. Before they choose their efforts, they have a chance to talk via a finite-round sequential cheap talk communication protocol. If either partner sends a message that reveals being off the equilibrium path, then it becomes common knowledge between them and they renegotiate immediately; otherwise they do not renegotiate. But if each player would prefer to shirk rather than reveal if the other is still on the equilibrium path, then there cannot be an equilibrium in which they renegotiate without common knowledge. If to the contrary they would renegotiate, then in the last round of communication, if neither partner has revealed, the partner who speaks last would not reveal. By backward induction—even if both partners are off the equilibrium path—neither of them would reveal in any round.

4 Bilateral renegotiation proofness in two-player games

In a two-player game, a BRP equilibrium is simply a WBRP equilibrium that is maximal in the full Pareto dominance ordering. With the variable-effort stage game we model, BRP equilibria between two players have several useful properties that we leverage in our later results on many-player games. We show that in two-player games in our environment, a BRP equilibrium exists, is efficient among all equilibria, and implements minimax punishments. As a constructive characterization of a BRP equilibrium is elusive, we prove these results indirectly by constructing a WBRP equilibrium with these properties and then arguing that a BRP equilibrium must exist and must also display these properties.

Let $\phi^B$ be the largest solution to a player’s incentive constraint when effort is constant and symmetric on the equilibrium path, but zero off the equilibrium path:

$$c(\phi) \leq \lambda (b - c)(\phi).$$

By Assumption 2, a solution exists and satisfies $\phi^B < \infty$; it is the highest level of symmetric effort that can be supported by a punishment in which the deviator earns a continuation payoff of zero. Since zero is the stage game minimax payoff, this is the highest level of symmetric effort that can be supported in any equilibrium.

In this section we show that renegotiation imposes no cost at all in the class of bilateral games we study: the highest symmetric payoff attainable in any equilibrium is attainable in a BRP equilibrium. In addition, all lower symmetric payoffs are ruled out by BRP.

**Theorem 1.** For any two-player game under Assumptions 1 and 2, the sum of payoffs in any equilibrium $\sigma$ is bounded by $U_1(\sigma) + U_2(\sigma) \leq 2(b - c)(\phi^B)$. 
There exists a symmetric BRP equilibrium that attains this bound, and in which a player earns a continuation payoff of zero after deviating unilaterally; moreover, no other BRP equilibrium payoff is Pareto dominated by this equilibrium.

First we prove the first part of the theorem: the sum of the two players’ flow payoffs in any equilibrium (including those that are not WBRP) are bounded by $2(b - c)(\phi^B)$.

**Lemma 1.** For any two-player game under Assumptions 1 and 2, the sum of payoffs in any equilibrium $\sigma$ is bounded by $U_1(\sigma) + U_2(\sigma) \leq 2(b - c)(\phi^B)$.

**Proof.** Suppose not. Then there exists a bilateral equilibrium $\sigma$ in which $U_1(\sigma) + U_2(\sigma) > 2\phi^B$. Let $\sigma$ be a bilateral equilibrium that maximizes the LHS of this inequality. Let $\hat{\phi} = (b - c)^{-1}\left(\frac{1}{2}(U_1(\sigma) + U_2(\sigma))\right)$; this is the symmetric effort level that, if played forever, would yield the same sum of flow payoffs. Since the minimax payoff for each player is zero, the punishments that the two partners face must sum to no more than $2\frac{\lambda}{r}(b - c)(\hat{\phi})$ after deviating at any history.

To discourage shirking when intended to exert effort $\phi_i$ on the equilibrium path, partner $i$ must face a potential punishment of magnitude at least $c(\phi_i)$. Therefore even if $\phi_i$ is a random variable in partner $i$’s mixed action, adding the partners’ expected incentive constraints yields the middle inequality in

$$2c\left(\frac{E(\phi_1) + E(\phi_2)}{2}\right) \leq E(c(\phi_1) + c(\phi_2)) \leq 2\frac{\lambda}{r}(b - c)(\hat{\phi}) < 2c(\hat{\phi}).$$

The leftmost inequality is by convexity of $c$, while the rightmost inequality is by $\hat{\phi} > \phi^B$ and the definition of $\phi^B$. Since $c$ is strictly increasing, it follows that $E(\phi_1 + \phi_2) < 2\hat{\phi}$. It then follows that

$$E((b - c)(\phi_1) + (b - c)(\phi_2)) \leq 2(b - c)\left(\frac{E(\phi_1) + E(\phi_2)}{2}\right) < 2(b - c)(\hat{\phi}),$$

where the first inequality is by concavity of $(b - c)$ and the second is by strict monotonicity of $(b - c)$. It follows that $(b - c)^{-1}\left(\frac{1}{2}(U_1(\sigma) + U_2(\sigma))\right) < \hat{\phi}$, a contradiction. $\square$

Next, we construct a WBRP equilibrium $\sigma^*$ with the desired properties.

**Lemma 2.** For any two-player game under Assumptions 1 and 2, there exists a symmetric WBRP equilibrium such that both players exert effort $\phi^B$ along the equilibrium path, and such that partner $i$ earns a continuation payoff of zero after deviating unilaterally.

**Proof.** Let each player $i$’s strategy in $\sigma^*$ be given by the following a three-phase automaton.
1. Bilateral Cooperation: Work at $\phi^B$. Expect player $j$ to be in the Bilateral cooperation phase.
2. Punishment: Work at $\phi^B$. Expect player $j$ to be in the reward phase and exert effort 0.
3. Reward: Shirk. Expect player $j$ to be in the punishment phase exert effort $\phi^B$.

In any phase, if neither player deviates from expected play or both players deviate simultaneously, transition to the bilateral cooperation phase. In any phase, if player $i$ deviates from expected play, transition to the punishment phase, while if player $j$ deviates then transition to the reward phase.

This automaton is illustrated in Fig. 1.

Player $i$'s expected payoff is $\frac{1}{2} (b-c)(\phi^B)$ in the bilateral cooperation phase. In the punishment phase, because $c(\phi^B) = \frac{1}{2} (b-c)(\phi^B)$, her expected payoff is 0. In the reward phase, she expects to be rewarded in the next interaction and then return to bilateral cooperation, so her payoff is $z^* = \frac{1}{3} b(\phi^B) + \frac{1}{2} (b-c)(\phi^B)$.

Next we verify that this is an equilibrium. Observe that in the punishment phase it is a best response for a player to work at $\phi^B$, since he earns an average payoff of no greater than zero for any other effort he may choose. In the reward phase he is supposed to shirk (choose effort 0), which maximizes both his stage game payoff and his continuation payoff. Cooperating at $\phi^B$ in the bilateral cooperation phase is supported by the threat of being punished with a continuation payoff of zero, just as in a bilateral enforcement equilibrium.

Finally we verify that $\sigma^*$ meets the WBRP criterion. In the reward phase a player expects to earn $b(\phi^B)$ at the next interaction, plus $(b-c)(\phi^B)$ in all subsequent interactions. Therefore his average payoff is strictly greater in the reward phase than in the bilateral cooperation phase. It follows that the average payoff vector when one player is in the reward phase and the other player is in the punishment phase is Pareto-incomparable to the average payoff vector when both players are in the bilateral cooperation phase. Hence $\sigma^*$ is WBRP. □
Let $W^*$ be the payoff set of $\sigma^*$; i.e., $W^*$ is the Pareto frontier of the convex hull of the three points $((b-c)(\phi^B), (b-c)(\phi^B)), (z^*, 0)$, and $(0, z^*)$. Next we show that if $\sigma^*$ itself is not BRP, then there exists a BRP equilibrium with similar properties.

**Lemma 3.** Consider any sequence of joint payoff sets $\{W_k\}$ of WBRP equilibria, starting with $W_1 = W^*$, such that each $W_{k+1}$ fully Pareto dominates $W_k$ for each $k = 1, \ldots, \infty$. Under Assumptions 1 and 2, there exists a pointwise limit set $W = \lim_{k\to\infty} W_k$ that is the joint payoff set of a WBRP equilibrium such that both players exert effort $\phi^B$ along the equilibrium path, and such that partner $i$ earns a continuation payoff of zero after deviating unilaterally.

The proof is in the appendix. Intuitively, there could exist a WBRP equilibrium that fully Pareto dominates $\sigma^*$, but by Lemma 1 it cannot involve cooperation at efforts above $\phi^B$. What it could involve is a higher payoff than $z^*$ for player $i$ when her partner $j$ is being punished with a zero payoff. To find a BRP equilibrium we should look for a WBRP equilibrium that is maximal in the full Pareto dominance ordering. Though the stage game is continuous and unbounded, nonetheless we show that any improving sequence of WBRP equilibria starting with $\sigma^*$ converges to a WBRP equilibrium. To complete the proof of the theorem, we show that this implies that there exists a BRP equilibrium with the claimed properties.

**Proof of Theorem 1.** Any BRP equilibrium must have a joint payoff set that fully Pareto dominates $W^*$, and therefore by such an equilibrium Lemma 3 must satisfy the properties claimed in the theorem. It remains only to show that a BRP equilibrium exists. Suppose not; then there must exist an infinite sequence of distinct WBRP equilibria, starting with $\sigma^*$, such that the joint payoff set of each fully Pareto dominates its predecessors, but such that there does not exist any WBRP equilibrium whose joint payoff set fully Pareto dominates every member of the sequence. By Lemma 3, however, the limit $W$ of this sequence itself is associated with a WBRP equilibrium that fully Pareto dominates every member of the sequence, a contradiction. □

5 Bilateral renegotiation proof contagion

We now turn to the networked environment. There are $n$ players linked together in a symmetric network. Let $d$ be the degree of the network; i.e., each player has $d$ partners. Although monitoring is not public—each player observes only the interactions she participates in herself—community enforcement via “contagion” behavior can still support high levels of cooperation. In a contagion equilibrium, as soon as a player deviates or is deviated on by any partner, she starts shirking
on all her partners. So an initial deviation by any player induces a contagion of shirking that spreads through society. Ali and Miller (2013) show that there exists a contagion equilibrium that is optimal among all equilibria that are stationary on the equilibrium path. In this optimal contagion equilibrium, effort on the equilibrium path raised to the point at which players are just indifferent between working and shirking.

Contagion supports high levels of cooperation by threatening the harshest possible equilibrium punishment in this environment. But contagion fails bilateral renegotiation proofness rather spectacularly: under BRP, given the existence of the bilateral BRP equilibrium characterized in Theorem 1, no two players can have common knowledge that they will shirk on each other for even one period, much less forever. In this section we characterize a class of contagion-like equilibria that are optimal among symmetric BRP equilibria (Theorem 2). These BRP contagion equilibria involve lower levels of cooperation than ordinary contagion, but for densely connected societies the cost of renegotiation is proportionally small (Theorem 3), and vanishes in the limit.

BRP contagion equilibria start off like contagion equilibria, in that a player becomes contagious and starts shirking on others at effort 0 once someone deviates on him. But he shirks on each partner just once—upon the first time he shirks on a particular partner, it becomes common knowledge between them that they are off the equilibrium path. Then the two of them renegotiate to playing the two-player BRP equilibrium from Theorem 1 within their relationship, without affecting what happens in other relationships. Note that in this monitoring environment it can never become common knowledge among more than two players that they are off the equilibrium path.

The key to optimizing punishments even under bilateral renegotiation is that the payoff vector in the two-player BRP equilibrium set to which two partners renegotiate off the equilibrium should be chosen to maximally punish whoever shirked first in their relationship. By Theorem 1, one partner, but not both, can be held to a zero expected payoff. So if only one partner shirked first, the punishment is just as harsh as under true contagion. However, if both partners simultaneously shirked first then they should renegotiate to the best symmetric bilateral payoff. Observe that just because they both simultaneously shirked first does not mean they have both deviated. In a contagion-like equilibrium, suppose that player $i$ deviates from the equilibrium path by shirking on his partner $j$. If next player $j$ meets her partner $k$, equilibrium calls for her to shirk; if then players $k$ and $i$ meet their equilibrium strategies call for them to shirk on each other. In this situation, players $k$ and $i$ cannot renegotiate before they have shirked on each other, because neither one knows that the other is about to shirk. Only after they have shirked on each other does it become
common knowledge between them that they are off the equilibrium path.\footnote{This is where mere perfect Bayesian equilibrium could allow anomalies. If players’ beliefs off the equilibrium path could violate strategic independence (Battigalli 1996), then a player could believe that seeing one partner shirk implies that all his other partners are already off the equilibrium path. Incorrectly believing it is common knowledge, he could propose to renegotiate at his next meeting with each partner. Perfect extended-Bayesian equilibrium forces players’ beliefs to be generated by a common prior conditional probability system satisfying strategic independence, ruling out false belief of common knowledge.}

The cost of renegotiation is the difference between what players can attain in an ordinary contagion equilibrium and what they attain in a BRP contagion equilibrium. The cost arises entirely from the off-path events in which both partners first shirk on each other at the same time, in which case they both receive continuation rewards of \( \frac{r}{b}(b - c)(\phi^B) \). This means if one of the partners happens to be the original deviator, in this relationship he is not being punished as harshly as he would be under contagion. Since the original deviator shirks on all partners with whom he has not yet renegotiated, in any relationship in which he and his partner do not first shirk at the same time, he is strictly the first shirker and thereafter receives the same payoff within the relationship as he would under contagion.

5.1 Benchmark: Contagion

Ali and Miller (2013) characterize contagion on a symmetric network of degree \( d \) as follows. On the equilibrium path, players always exert the same effort \( \phi^C_d \) (where “C” is for “contagion”). Any player who observes any deviation becomes permanently contagious, and thereafter shirks at effort 0 in every relationship. The equilibrium path effort \( \phi^C_d \) is calibrated to bind the equilibrium path incentive constraint:

\[
(b - c)(\phi) + r(d - 1)(b - c)(\phi) \geq b(\phi) + (d - 1)b(\phi)X_d,
\]  

(4)

where \( X_d \) is the viscosity factor that describes how quickly player \( i \)'s other partners become contagious after player \( i \) deviates from the equilibrium path in his relationship some partner \( j \). The lefthand side of Eq. (4) is what player \( i \) earns by shirking in an equilibrium-path interaction with partner \( j \): \( b(\phi) \) immediately, and then, with each of \( (d - 1) \) other partners, \( b(\phi) \) from shirking if that partner is contagious. The viscosity factor of a given network of degree \( d \) is defined as

\[
X_d \equiv \int_0^\infty e^{-rt} \lambda x_d(t) \, dt,
\]  

(5)

where \( x_d(t) \) is the probability that player \( k \notin \{i, j\} \) is not contagious at time \( t \), after player \( i \) shirks on
player \( j \) at time 0. (The identities of \( i, j, \) and \( k \) do not matter because the network is symmetric.) We rearrange the binding contagion IC constraint to a more convenient form:

\[
\frac{b(\phi)}{(b - c)(\phi)} = \frac{1 + d\lambda}{1 + (d - 1)X_d}. 
\]

(6)

Since the lefthand side is strictly increasing by Assumption 2, there is a unique solution, which is denoted by \( \phi^C_d \).

While the equilibrium-path incentive constraints are satisfied by construction, the off-path constraints are still an issue. In the contagion literature that considers models with fixed-effort stage games, the off-path constraints are particularly problematic and typically bind (see, e.g., Kandori 1992; Ellison 1994). However, Ali and Miller (2013) show that in this environment with variable efforts (or in their setting, variable stakes), the off-path constraints are guaranteed to be satisfied—regardless of the players’ beliefs—if the equilibrium-path constraints bind. Hence to characterize a “binding contagion” equilibrium it suffices to solve the binding contagion IC constraint.

5.2 Contagion with renegotiation

In this section we characterize a BRP contagion equilibrium that is optimal among symmetric BRP equilibria. This equilibrium employs multilateral enforcement to sustain more cooperation than can bilateral enforcement, although less than ordinary contagion.

**Theorem 2.** For any symmetric network of degree \( d \geq 2 \) under Assumptions 1 and 2, there exists a BRP contagion equilibrium in which players always exert effort \( \phi^R_d \) that solves

\[
\frac{b(\phi)}{(b - c)(\phi)} = \frac{r + d\lambda - \lambda(d - 1)(\frac{1}{\lambda}\phi - X_d)}{r + rX_d(d - 1)} \phi^B/\phi \quad \text{for } \phi^B < \phi^R_d < \phi^C_d, \text{ and no other symmetric BRP equilibrium attains higher equilibrium payoffs.}
\]

(7)

Moreover, \( \phi^B < \phi^R_d < \phi^C_d \), and no other symmetric BRP equilibrium attains higher equilibrium payoffs.

To prove this theorem, we construct a WBRP contagion equilibrium \( \sigma^{**} \) with the desired properties, and show that it implies the existence of a BRP equilibrium with similar properties. Formally, each player \( i \)'s strategy \( \sigma^{**}_i \) is defined by a collection of “partial automata,” one for each of

\[13\text{As an example, Ali and Miller (2013) show that the viscosity factor of a complete network (one with with } d + 1 \text{ players) is } X_d^* = \frac{1}{d-1} \sum_{m=2}^{d-1} \left( \prod_{m'=2}^{m-1} \frac{\lambda(m-d-m+1)}{rX_d(d-m+1)} \right) \]
his links. A partial automaton for link $ij$ is much like an automaton, except that, because player $i$’s various links are strategically interdependent, transitions between phases on link $ij$ may be driven by interactions that occur on other links. Let each player $i$’s behavior and beliefs on link $ij$ be governed by the following five-phase partial automaton, illustrated in Fig. 2.

1. **Global Cooperation**: Exert effort $\phi_R^j$. Expect player $j$ to be in the Global Cooperation phase.
2. **Contagion**: Exert effort 0. Expect player $j$ to be in the Global Cooperation phase with some probability, and the Contagion phase with complementary probability.
3. **Bilateral cooperation**: Exert effort $\phi_B^j$. Expect player $j$ to be in the Bilateral Cooperation phase.
4. **Punishment**: Exert effort $\phi_B^j$. Expect player $j$ to be in the Reward.
5. **Reward**: Exert effort 0. Expect player $j$ to be in the Punishment phase.

The Bilateral Cooperation, Punishment, and Reward phases in $\sigma^{**}$ are identical to the corresponding phases described in the proof of Lemma 2. Transitions from the other two phases are a bit more complicated:

- In the Global Cooperation phase, if both partners exert effort $\phi_R$, remain in the Global Cooperation phase. If neither partner exerts effort $\phi_R$, transition to the Bilateral Cooperation phase. If player $i$ exerts effort $\phi_R$ but partner $j$ does not, transition to the Reward phase; if partner $j$ exerts effort $\phi_R$ but player $j$ does not, transition to the Punishment phase. If any player $k \in N_i \setminus \{j\}$ exerts effort other than $\phi_R$ along link $\{ik\}$, transition to the Contagion phase on link $ij$.
- In the Contagion phase, if neither partner exerts effort $\phi_R$, transition to the Bilateral Cooperation phase. If player $i$ exerts effort $\phi_R$ but partner $j$ does not, transition to the Reward phase; if partner $j$ exerts effort $\phi_R$ but player $j$ does not, transition to the Punishment phase. If both partners exert effort $\phi_R$, transition to the Global Cooperation phase.

The global cooperation phase is the initial phase along every link.

The proof, in the Appendix, first confirms that $\sigma^{**}$ is an equilibrium. The equilibrium-path incentive constraints bind by construction; convexity of $b$ guarantees that it binds at efforts greater than $\phi_B$. As in ordinary contagion, because the global cooperation phase incentives bind, a player has strict incentives to shirk on any partner that she believes to be in the global cooperation phase. Renegotiation however raises a different challenge: by working at $\phi_R$ in the contagion phase, a player may be able to collect a reward in the future when a partner who she believes to be in the contagion phase shirks on her. For example, on a triangle network, after Alice shirks on Bob, if Bob does not meet Carol for a long time he will assign high probability to Carol being contagious.
Figure 2. Bilateral renegotiation proof contagion partial automaton for player $i$ along link $ij$. The initial phase is outlined in bold, and transitions driven by player $i$’s equilibrium behavior are shown with bold arrows. Not all transitions are shown.
If so, then, by working at $\phi^R$ the next time he meets Carol, he will subsequently be allowed to shirk while Carol works at $\phi^R$. We show that when the equilibrium path incentive constraints bind, the contagion phase incentives are satisfied regardless of Bob’s beliefs—even if he is certain that Carol is contagious. As in the case of Theorem 1, although the WBRP contagion equilibrium may not be BRP, it implies the existence of an BRP equilibrium that attains the same payoffs on the equilibrium path. Finally, we show that for any any symmetric network with degree $d$, the WBRP contagion equilibrium attains the maximum symmetric level of cooperation under BRP.

**Remark 2.** From the optimality of our BRP contagion equilibrium among all symmetric BRP equilibria, ours is the only symmetric equilibrium that can arise from successful negotiation among the “grand coalition” if it negotiates symmetrically at and only at time 0. We could extend our notion of renegotiation proofness to allow for grand coalition renegotiation at some Poisson rate $\pi > 0$. Since only common knowledge Pareto improvements can be renegotiated, the grand coalition renegotiation cannot depend on the history of private interactions. Therefore the grand coalition should always renegotiate to the optimal BRP contagion equilibrium—at the adjusted discount rate of $r + \pi$ to account for expected future grand coalition renegotiation events.

**Remark 3.** There can also exist various types of suboptimal BRP equilibria, perhaps with additional restrictions on the viscosity factor. For instance, one can construct a “naive” BRP contagion equilibrium in which partners simply renegotiate to the Bilateral Cooperation phase after one or both of them shirk. This naive BRP contagion equilibrium requires the viscosity factor to be somewhat small: $X_d < \frac{\lambda}{(r + \lambda)^2} \frac{(d - 2)r - \lambda}{d - 1}$. For the case of complete networks, this condition is satisfied when there are sufficiently many players. Such an equilibrium is described in detail in the appendix.

### 5.3 The cost of renegotiation

As we consider symmetric networks of larger and larger degree, our next result shows that the proportional cost of renegotiation eventually becomes negligible. The proof is in the Appendix.

**Theorem 3.** Suppose there exists $\epsilon > 0$ such that $\frac{1}{\epsilon} > b''(\phi) > \epsilon$ for all $\phi \geq 0$. For any sequence of symmetric networks of degrees $d = 2, \ldots, \infty$, if $\lim_{d \to \infty} X_d = 0$ then

$$\lim_{d \to \infty} \frac{\phi^R_d}{\phi_C^d} = 1. \quad (8)$$

After player $i$ deviates on player $j$, $\lim_{d \to \infty} X_d = 0$ means that other players become contagious more quickly on networks of higher degree. This increases the chance that each player $k \notin \{i,j\}$
will be contagious when player \( i \) first meets \( k \); since this is the case in which the both shirk on each other at the same time, it is this case in which the punishment under renegotiation-proof contagion is less severe than under contagion. However, as the network scales up, the number of partners increases faster than the reduction in punishment power per partner. In the limit, the reduction becomes proportionally negligible.

Although it is possible to construct sequences of networks of degree \( d = 2, \ldots, \infty \) that violate \( \lim_{d \to \infty} X_d = 0 \), the condition holds for complete networks and for many other sensible classes of networks. The condition can be violated if the networks grow less dense (i.e., their cycles grow longer and less interconnected) even more quickly than their degrees increase.

**Remark 4.** The conclusion of Theorem 3 also holds for various suboptimal BRP equilibria, such as the naive BRP contagion equilibrium described in Remark 3. The proof of Theorem 3 in the appendix accounts for these cases.

### 6 Work in progress

This draft is a work in progress. Our next step is to address renegotiation among subsets of players larger than pairs. The logic of our proof of Theorem 3 implies, for example, that the proportional cost of \( k \)-lateral renegotiation, for any fixed \( k \), also vanishes as the network grows dense. Our goal is to identify a joint bound on the number of players who may renegotiate together and the rate at which they may do so, such that the conclusion of Theorem 3 still holds. It is clear that this bound will diverge as the degree of the network diverges, but at what relative rate is not yet clear.

### References


## Appendix A Additional Proofs

For ease of notation, we write \((b - c)(\phi_1, \phi_2) \equiv (b(\phi_2) - c(\phi_1), b(\phi_1) - c(\phi_2))\).
Proof of Lemma 3 on page 11. Consider a sequence of joint payoff sets \( W_k \) of WBRP equilibria, starting with \( W_1 = W^* \), such that each \( W_{k+1} \) fully Pareto dominates \( W_k \) for each \( k = 1, \ldots, \infty \). First we establish some facts that are true for each \( W_k \).

1. Each \( W_k \) contains the point \((b-c)(\phi^B, \phi^B)\).
   
   Proof: Each \( W_k \) must contain a point that Pareto dominates \((b-c)(\phi^B, \phi^B) \in W_1 \). All such points must respect the bound identified in Lemma 1; but \((b-c)(\phi^B, \phi^B)\) is already at this bound.

2. The endpoints of each \( W_k \) are \((z_k, 0)\) and \((0, z_k)\), for some \( z_k, z_k \in ((b-c)(\phi^B), 2(b-c)(\phi^B)) \).
   
   Proof: By Assumption 2 point \((b-c)(\phi^B, \phi^B)\) is obtained only by both partners cooperating at \( \phi^B \), which requires that any deviator receive a continuation payoff of no more than zero. Since zero is the stage game minimax for each player, \( W_k \) must contain some point \((z_k, 0)\) in order to deter deviations by partner 2 (and similarly for partner 1). But if \( z_k < z^* \) then \((z_k, 0)\) would be strictly Pareto dominated by a convex combination of \((b-c)(\phi^B, \phi^B)\) and \((z^*, 0)\), contrary to the supposition that \( W_k \) fully Pareto dominates \( W_1 \). Since \( z^* > (b-c)(\phi^B) \), it follows that \( z_k > (b-c)(\phi^B) \), and similarly \( z_k > (b-c)(\phi^B) \). Finally, by Lemma 1 we must have \( z_k, z_k \leq 2(b-c)(\phi^B) \).

3. For each \( W_k \), there exists a WBRP equilibrium with payoff set \( W_k \) such that partner \( i \) earns a continuation payoff of zero after deviating unilaterally.
   
   Proof: We have already shown that \( W_k \) contains a point for each player that delivers her a zero payoff, which is also her stage-game minimax payoff. Any unilateral deviation from an equilibrium can be deterred by a minimax punishment.

Next we show that the pointwise limit \( W = \lim_{k \to \infty} W_k \) exists and is the joint payoff set of a WBRP equilibrium. Using public randomization, without loss of generality we take each joint payoff set \( W_k \) to be the weak-Pareto frontier of the convex hull of the closure of its actual joint payoff set. It then follows from Theorem 4.1 of Warburton (1983) that \( W_k \) is connected and closed.

1. The pointwise limit \( W = \lim_{k \to \infty} W_k \) exists, and is connected and internally weak-Pareto incomparable.
   
   Proof: Since each \( W_k \subset \mathbb{R}^2_+ \) is internally weak-Pareto incomparable, we identify each \( W_k \) by a function \( F_k(x) = \max\{y : (x, y) \in W_k \cup \{0\}\} \) on the on the domain \([0, 2(b-c)(\phi^B)]\). Each \( F_k \) is evidently decreasing and integrable, and is concave on the subdomain \([0, z_k]\). Since the sequence is increasing in the full Pareto dominance ordering, \( F_k(x) \) is increasing in \( k \) for all \( x \). By Lemmas 1 and 3, \( F_k(0) = ((b-c)(\phi^B), 2(b-c)(\phi^B)] \), \( F_k(2(b-c)(\phi^B)) = 0 \), and \( \int_0^{2(b-c)(\phi^B)} F_k dx \leq 2(b-c)(\phi^B) \).
   
   Now the Monotone Convergence Theorem (e.g., Aliprantis and Border 1999, Theorem 11.17) implies that there exists a decreasing and integrable function \( F \) that is the pointwise limit of \( \{F_k\} \), with \( \int_0^{2(b-c)(\phi^B)} F dx = \lim_{k \to \infty} \int_0^{2(b-c)(\phi^B)} F_k dx \). Since \( F \) is the pointwise limit of \( \{F_k\} \), there exists \( z = \inf\{x : F(x) \leq 0\} \in ((b-c)(\phi^B), 2(b-c)(\phi^B)] \) such that \( F \) is concave on \([0, z]\).
Define \( W \equiv \{(x, F(x)) : x < z\} \cup \{(z, y) : y \in [0, F(z)]\} \). We have already shown that each point \((x, F(x))\) for which \(x \leq z\) is a limit point of \(\{W_k\}\), as is the point \((z, 0)\). Finally, each point \((z, y)\) such that \(y \in (0, F(z)]\) is a limit point of \(\{W_k\}\) since each \(W_k\) is connected and concave. Therefore \(W\) as defined is in fact the pointwise limit of \(\{W_k\}\).

2. There exists a WBRP equilibrium whose joint payoff set is \(W\).

Proof: Fix a compact, metrizable state space \(\Omega\) for the public randomization device. Since each \(W_k\) is the joint payoff set of an equilibrium, for each \(v_k \in W_k\) there exists a public distribution \(\zeta_{v_k}\) on \(\Omega\), efforts \(\{\phi_i;\Omega \rightarrow \mathbb{R}_+\}_{i=1,2}\), and a promised utility function \(w_{v_k} : \Omega \rightarrow W_k\) (simplified to deliver only the promised utility expected if neither player deviates) such that

\[
v_k = \mathbb{E}_{\zeta_{v_k}} \left( \frac{r}{r + \lambda}(b - c)(\phi_{1;v_k}, \phi_{2;v_k}) + \frac{\lambda}{r + \lambda}w_{v_k}(\omega) \right)
\]

(9)

and, for all \(\omega\) and \(i = 1, 2\),

\[
\frac{r}{r + \lambda} (b(\phi_{i;v_k}) - c(\phi_{-i;v_k})) + \frac{\lambda}{r + \lambda}w_{i;v_k}(\omega) \geq \max_{\phi \in [0, 2\phi^\#]} \frac{r}{r + \lambda} (b(\phi) - c(\phi_{-i;v_k}(\omega)))
\]

(10)

Now although \(\{\zeta_{v_k}, \phi_{v_k}, w_{v_k}\}\) need not converge, since all effort profiles used in equilibrium are bounded, they lie in the product of compact spaces with natural metrics defined for each space.\(^{14}\) Since a finite product of compact metric spaces is compact and metrizable, the sequence lies in a sequential compact space (see Munkres 2000, Theorems 26.7 and 28.2 and p. 219). Therefore there exists a subsequence \(\{\zeta_j, \phi_j, w_j\}\), where \(\{j\} \subset \{k\}\), that converges to a limit point \((\zeta, \phi, w)\) that satisfies Eqs. (9) and (10).

Since this can be done for every \(v \in W\), one can construct an equilibrium whose joint payoff set is \(W\), such that partner \(i\) earns a continuation payoff of zero after deviating unilaterally. Because \(W\) is internally weak-Pareto incomparable, this equilibrium is WBRP.

\(\square\)

**Proof of Theorem 2 on p. 14.** First we confirm that this strategy profile \(\sigma^{**}\) constitutes an equilibrium. By Lemma 2, incentives are satisfied in the Bilateral Cooperation, Punishment, and Reward phases. So it suffices to check only the incentive constraints in the Global Cooperation and Contagion phases.

In the Global Cooperation phase, suppose that player \(i\) shirks on link \(\{ij\}\). Then player \(i\) transitions to the Punishment phase on link \(\{ij\}\) and receives a continuation payoff of zero, just as in an ordinary contagion profile. Now consider what happens when player \(i\) subsequently meets player \(k \neq j\). If player \(k\) is in the Global Cooperation phase, then along link \(\{ik\}\) player \(i\) earns \(b(\phi)\) immediately and zero thereafter, also as in an ordinary contagion profile. If instead player \(k\) is in the Contagion phase, both of them shirk simultaneously, and they both transition to the Bilateral Cooperation phase to earn continuation payoffs.

\(^{14}\) \(\{\zeta_{v_k}\}\) is compact and metrizable by Theorem 14.11 of Aliprantis and Border (1999); each \(\{\phi_{i;v_k}\}\) lies in the set of probability measures over \([0, 2\phi^\#]\) by Lemma 1; \(\{w_{v_k}\}\) may also be identified by 2-dimensional vectors in \([0, 2\phi^\#]^2\) by Lemma 1. It is now clear that each sequence lies in a compact metrizable space.
of \( \frac{1}{r}(b - c)(\phi^B) \) rather than zero. Accordingly, if \( d \) is the degree of the symmetric network and \( X_d \) is the viscosity factor, the incentive constraint in the Global Cooperation phase is

\[
(b - c)(\phi) + d \frac{\lambda}{r}(b - c)(\phi) \\
\geq b(\phi) + (d - 1) \int_0^\infty e^{-rt} \lambda \left( x_d(t)b(\phi) + \left( e^{-\lambda t} - x_d(t) \right) \frac{\lambda}{r}(b - c)(\phi^B) \right) dt \\
= b(\phi) + (d - 1) \left( b(\phi)X_d + \left( \frac{\lambda}{r + \lambda} - X_d \right) \frac{\lambda}{r}(b - c)(\phi^B) \right).
\]

(11)

Most of these terms coincide with Eq. (4); it is the \( \left( e^{-\lambda t} - x_d(t) \right) \frac{\lambda}{r}(b - c)(\phi^B) \) term inside the integral that differs and highlights the cost of renegotiation: when player \( i \) meets a partner who is also in the contagion phase, after they shirk on each other they can renegotiate to bilateral cooperation.\(^{15}\)

We show that Eq. (11) is slack at \( \phi = \phi^B \). Since \( \phi^B \) is set to make the bilateral Nash reversion incentive constraint bind, \( b(\phi^B) = \frac{\lambda}{r}(b - c)(\phi^B) \). So Eq. (11) is slack at \( \phi^B \) if and only if

\[
\left( \frac{r + \lambda}{r} + (d - 1) \left( \frac{r + \lambda}{r}X_d + \left( \frac{\lambda}{r + \lambda} - X_d \right) \frac{\lambda}{r} \right) \right) (b - c)(\phi^B) < \left( 1 + d \frac{\lambda}{r} \right)(b - c)(\phi^B)
\]

\[
\iff \quad (d - 1) \left( X_d + \frac{\lambda}{r + \lambda} \right) < (d - 1) \frac{\lambda}{r} \iff X_d < \frac{\lambda}{r + \lambda},
\]

which is implied by Lemma 2 of Ali and Miller (2013). Because \( c \) is continuous and strictly convex (Assumption 2), there exists \( \phi^* > \phi^B \) that binds Eq. (11) (i.e., that solves Eq. (7)), at which each player is indifferent between working at \( \phi^*_d \) and shirking in the Global Cooperation phase.

To establish that \( \sigma^{*\ast} \) is an equilibrium, it remains to verify the contagion phase incentives. First, Bob’s contagion phase incentive constraint if he believes Carol is contagious is

\[
-c(\phi) + \int_0^\infty e^{-rt} e^{-\lambda t} \lambda \left( b(\phi^B) + \frac{\lambda}{r}(b - c)(\phi^B) \right) dt \leq \frac{\lambda}{r}(b - c)(\phi^B).
\]

(13)

Since \( \phi^*_d > \phi^B \), to show that Eq. (13) is satisfied at \( \phi = \phi^*_d \), it suffices to show that Eq. (13) is slack at \( \phi = \phi^B \):

\[
-c(\phi^B) + \frac{\lambda}{\lambda + r} \left( b(\phi^B) + \frac{\lambda}{r}(b - c)(\phi^B) \right) < \frac{\lambda}{r}(b - c)(\phi^B).
\]

(14)

\(^{15}\)This expression bears further explanation. For the entirety of this footnote, condition on the event that player \( i \) deviates from the equilibrium path by shirking on link \( \{i, j\} \) at time 0 and assume that all players otherwise follow \( \sigma^{*\ast} \). Let event \( P_i \) be that at time \( t \) players \( i \) and \( k \) have not met since since time 0, and let event \( Q_k \) be that at time \( t \) player \( k \) has seen any partner other than player \( i \) shirk. Note that \( P_i \) and \( Q_k \) are independent. Then we claim that \( e^{-\lambda t} - x_d(t) \) is the probability of event \( P_i \cap Q_k \). To see this, note that the viscosity factor can be rewritten as \( X_d = \int_0^\infty e^{-rt} \lambda e^{-\lambda t} \hat{x}_d(t) dt \), where \( e^{-\lambda t} = \Pr(P_i) \) and \( \hat{x}_d(t) = 1 - \Pr(Q_k) \). Then \( \Pr(P_i \cap Q_k) = e^{-\lambda t}(1 - \hat{x}_d(t)) = e^{-\lambda t} - x_d(t) \).
Recall that \( c(\phi^B) = \frac{1}{2}(b - c)(\phi^B) \) by definition of \( \phi^B \). Therefore Eq. (13) is slack at \( \phi = \phi^B \) if

\[
\frac{\lambda}{r} > -\frac{r}{\lambda + r} \frac{\lambda}{r} + \frac{\lambda}{\lambda + r} \left( 1 + \frac{\lambda}{r} \right),
\]

which is guaranteed. Next, we address Bob’s Contagion phase incentive constraint if he believes Carol is not contagious by considering its opposite. Suppose Bob instead of shirking is supposed to cooperate with Carol. But by construction his equilibrium-path incentive constraint binds, meaning he is indifferent over whether to shirk when nobody is contagious. Now he knows Alice is contagious, so he faces a weaker punishment;\(^{16}\) therefore he strictly prefers to shirk. Now we have shown that Contagion phase incentives of \( \sigma^{**} \) are satisfied for extremal beliefs. Since expected payoffs are linear in player \( i \)'s belief about his current partner, the Contagion phase incentive constraints are satisfied for all intermediate beliefs. Evidently \( \sigma^{**} \) is an equilibrium.

Finally, we show that if \( \sigma^{**} \) is not BRP, then there must exist some BRP equilibrium that attains the same equilibrium path payoffs. Observe that two partners cannot bilaterally renegotiate when one or both of them is in the Contagion phase—they cannot arrive in the Contagion phase by any transition that is common knowledge between them, and they stay in the Contagion phase only for one interaction.\(^{17}\) Moreover, according to \( \sigma^{++} \), they bilaterally renegotiate to the bilateral equilibrium \( \sigma^* \) described in Lemma 2 as soon as it becomes common knowledge between them that they are off the equilibrium path. Though \( \sigma^* \) may not be BRP, Theorem 1 establishes that there exists some BRP equilibrium in bilateral strategies that supports cooperation at efforts \( \phi^B \); to do so, it must employ minimax punishments. In \( \sigma^{**} \), the first player to depart from Global Cooperation behavior along any link is punished with the minimax continuation payoff; the only possible way to relax Eq. (11) would be to punish both partners when they simultaneously depart from Global Cooperation behavior. However, the reason that each player departed from Global Cooperation behavior is not common knowledge information and therefore cannot be used to assign blame. In order to be measurable with respect to their common knowledge, their continuation payoff must be symmetric. Due to WBRP, they therefore cannot receive continuation payoffs less than \( \frac{1}{2}(b - c)(\phi^B) \). That is, no BRP equilibrium can impose stronger incentives in the Global Cooperation phase. Since \( \sigma^{**} \) maximizes equilibrium path payoffs subject to these constraints, there must exist an BRP equilibrium that attains the same payoffs.

Optimality of this equilibrium among all symmetric WBRP equilibria is established by Lemma 4, below.

\[\square\]

**Lemma 4.** For any symmetric network with degree \( d \) and viscosity factor \( X_d \), no symmetric BRP equilibrium supports cooperation at effort greater than \( \phi^{B*}_d \).

\(^{16}\) We just showed that he will not deviate by cooperating with a partner who he expects to shirk on him.

\(^{17}\) They could stay in the Contagion phase for more than one interaction only if the contagious player deviates (or both contagious players deviate, as the case may be) by working at \( \phi^{B*}_d \). However, such a deviation does not generate common knowledge of the Contagion phase, because it is indistinguishable from the behavior specified for the Global Cooperation phase.
Proof. Suppose that \((b - c)(\phi)\) is the average payoff of a symmetric BRP equilibrium in which everyone cooperates along the equilibrium path. On the equilibrium path, suppose that player \(i\) shirks on link \(ij\), and that thereafter player \(i\)'s punishment on link \(ij\) is to receive an average payoff of \(P\). In our BRP contagion equilibrium \(\sigma^{**}\), \(P = 0\); more generally \(P \geq 0\). Now consider what happens when player \(i\) subsequently meets player \(k \neq j\) for the first time after his initial deviation. If player \(k\) still believes that nobody has deviated or is otherwise supposed to cooperate, then along link \(ik\) player \(i\) earns at least \(b(\phi) + \frac{1}{2}P\), by shirking immediately and earning \(P\) thereafter. If instead player \(k\) is off the equilibrium path, then at worst for player \(i\) they may both shirk and then continue with each earning an average continuation payoff of \((b - c)(\phi^{B})\), which we will prove below.

Accordingly, if \(d\) is the degree of the symmetric network and \(X_d\) is the viscosity factor, the incentive constraint to cooperate on the equilibrium path is

\[
b(\phi) + \frac{\lambda}{r}P + (d - 1) \left( X_d \left( b(\phi) + \frac{\lambda}{r}P \right) + \left( \frac{\lambda}{r + \lambda} - X_d \right) \frac{\lambda}{r} (b - c)(\phi^{B}) \right) \leq d \frac{\lambda}{r} (b - c)(\phi) + (b - c)(\phi). \tag{16}
\]

Notice if \(P = 0\) (as in \(\sigma^{**}\)) then Eq. (16) reduces to

\[
b(\phi) + (d - 1) \left( X_d b(\phi) + \left( \frac{\lambda}{r + \lambda} - X_d \right) \frac{\lambda}{r} (b - c)(\phi^{B}) \right) \leq d \frac{\lambda}{r} (b - c)(\phi) + (b - c)(\phi), \tag{17}
\]

in which case cooperating at \(\phi^{B}_d\) is supported on the equilibrium path. If instead \(P > 0\), then

\[
b(\phi) + \frac{\lambda}{r}P + (d - 1) \left( X_d \left( b(\phi) + \frac{\lambda}{r}P \right) + \left( \frac{\lambda}{r + \lambda} - X_d \right) \frac{\lambda}{r} (b - c)(\phi^{B}) \right)
\]

\[
> b(\phi) + (d - 1) \left( X_d b(\phi^{B}_d) + \left( \frac{\lambda}{r + \lambda} - X_d \right) \frac{\lambda}{r} (b - c)(\phi^{B}) \right) = d \frac{\lambda}{r} (b - c)(\phi^{B}_d) + (b - c)(\phi^{B}_d). \tag{18}
\]

Thus the equilibrium effort supported in a symmetric equilibrium in which \(P > 0\) must be less than \(\phi^{B}_d\).

We still need to show that if player \(i\) and \(k\) first meet when both are already off the equilibrium path, then letting them first shirk on each other and then renegotiate to the Bilateral Cooperation phase at efforts \(\phi^{B}\) is the harshest possible punishment that can be imposed on whichever (if either) of them was the original deviator. BRP requires that their renegotiated play must be measurable with respect to their partnership history, which unfortunately does not identify the original deviator. We already showed that no BRP equilibrium payoff is Pareto dominated by the average payoff vector \(((b - c)\phi^{B}, (b - c)\phi^{B})\) (Theorem 1). Now suppose player \(i\) and \(k\) negotiate to a bilateral equilibrium strategy with average payoff vector \((U_i, U_k) \neq ((b - c)\phi^{B}, (b - c)\phi^{B})\). BRP implies that \((U_i, U_k)\) must also be on the Pareto frontier of bilateral WBRP equilibria, so without loss of generality assume that \(U_k < (b - c)(\phi^{B}) < U_i\). Since the equilibrium must be symmetric, on first deviating player \(i\) must expect the same payoff in each of his \((d - 1)\) other relationships. Therefore a necessary condition for his incentive constraint on the equilibrium path is

\[
b(\phi) + \frac{\lambda}{r}P + (d - 1) \left( X_d \left( b(\phi) + \frac{\lambda}{r}P \right) + \left( \frac{\lambda}{r + \lambda} - X_d \right) \frac{\lambda}{r} U_i \right) \leq d \frac{\lambda}{r} (b - c)(\phi) + (b - c)(\phi). \tag{19}
\]
Since the LHS of Eq. (19) is greater than LHS of Eq. (16), \( \phi_d^B \) will not satisfy Eq. (19). It follows that cooperating can be supported on the equilibrium path at efforts no greater than \( \phi_d^R \).

\[ \square \]

**Details about Remark 3 on page 17.** The “naive” BRP contagion equilibrium is similar to our BRP contagion equilibrium. The difference is that in the Global Cooperation phase, if player \( i \) shirks on link \( \{ij\} \), there is no Punishment phase for player \( i \). Instead players \( i \) and \( j \) transition immediately to the Bilateral Cooperation phase on link \( \{ij\} \) and each receives continuation payoff of \( \frac{1}{r}(b-c)(\phi^B) \). Now consider what happens when player \( i \) subsequently meets player \( k \neq j \). If player \( k \) is in the Global Cooperation phase, then along link \( \{ik\} \) player \( i \) earns \( b(\phi) \) immediately and a continuation payoff of \( \frac{1}{r}(b-c)(\phi^B) \) thereafter rather than zero. If instead player \( k \) is in the contagion phase, both of them shirk simultaneously, and they both transition to the Bilateral Cooperation phase to earn continuation payoffs of \( \frac{1}{r}(b-c)(\phi^B) \). Accordingly, the incentive constraint in the Global Cooperation phase is

\[
(1 + d^\frac{\lambda}{r})(b-c)(\phi) \\
\geq b(\phi) + \frac{\lambda}{r}(b-c)(\phi^B) \\
+ (d-1) \int_0^\infty e^{-t\lambda} \left(x_d(t)(b(\phi) + \frac{\lambda}{r}(b-c)(\phi^B)) + (e^{-t\lambda} - x_d(t))\frac{\lambda}{r}(b-c)(\phi^B)\right) dt \\
= b(\phi) + \frac{\lambda}{r}(b-c)(\phi^B) \\
+ (d-1) \left(x_d b(\phi) + x_d \frac{\lambda}{r}(b-c)(\phi^B) + \left(\frac{\lambda}{r+\lambda} x_d \frac{\lambda}{r}(b-c)(\phi^B)\right) \right).
\]

Since \( b(\phi^B) = \frac{r+\lambda}{r}(b-c)(\phi^B) \), Eq. (20) is slack at \( \phi = \phi^B \) under the extra constraint:

\[
(1 + d^\frac{\lambda}{r})(b-c)(\phi^B) > \left(1 + \frac{\lambda}{r} + (d-1) \left(\frac{\lambda}{r+\lambda} x_d + \frac{\lambda}{r} x_d + \left(\frac{\lambda}{r+\lambda} x_d \frac{\lambda}{r}\right)\right)\right)(b-c)(\phi^B) \\
\iff \quad X_d < \frac{\lambda}{\left(r + \frac{\lambda}{r}\right)^2 - \frac{\lambda}{r+\lambda}}.
\]

If \( X_d \to 0 \) as \( d \to \infty \), then the condition Eq. (21) is eventually satisfied. Suppose this is the case and \( d \) is sufficiently large, and let \( \phi_d^{NR} \) bind Eq. (20). Then Eqs. (19) and (21) imply that \( \phi_d^B < \phi_d^{NR} < \phi_d^R \). The rest of proof is the same as the proof for Theorem 2.

\[ \square \]

**Proof of Theorem 3 on page 17.** We begin with several straightforward claims.

**Claim 1.** For all \( d > 0 \) and all symmetric networks with degree \( d, d(X_d - \frac{\lambda}{r}) - X_d < 0 \).

This is a straightforward consequence of Lemma 2 of Ali and Miller (2013), which establishes \( X_d < \frac{\lambda}{r+\lambda} \).

**Claim 2.** If \( \lim_{d \to \infty} X_d = 0 \) then \( \lim_{d \to \infty} \frac{d^{\frac{\lambda+1}{\sigma-1}}}{\Gamma(\sigma-1)X_d} = \infty \).
Note that the claim gives a necessary and sufficient condition for \( \lim_{d \to \infty} \phi_d^C = \infty \), by Eq. (6). Also note that \( \frac{b(\phi)}{b(\phi) - c(\phi)} \) is monotone, given Assumption 2.

Define \( \phi_d^{GR} \) to be the equilibrium stage payoff for any suboptimal BRP equilibrium path that binds the equilibrium-path incentive constraints (see details below). Let \( P \) be the average continuation payoff off equilibrium for players who deviated, as defined in the proof of Lemma 4. Letting \( \tilde{A} = \frac{\lambda}{r}(b - c)(\phi^B) - \frac{\lambda}{r}P \), \( A = \frac{\lambda}{r}(b - c)(\phi^B) \) and \( B = \frac{\lambda}{r} \tilde{A} \), we find that, by Eqs. (4), (7) and (16), \( \phi_d^{GR}, \phi_d^B \) and \( \phi_d^C \) are the largest solutions to the following equations, respectively:

\[
\begin{align*}
\phi_d^{GR} = 1X_d\phi_d^{GR} - \left( d\left( \frac{1}{r} + 1 \right) (b - c)(\phi_d^{GR}) = (d - 1)X_d\tilde{A} - (d - 1)B - \frac{\lambda}{r}P, \tag{22} \\
\phi_d^B = (d - 1)X_d\phi_d^B - \left( d\left( \frac{1}{r} + 1 \right) (b - c)(\phi_d^B) = (d - 1)X_dA - (d - 1)B, \tag{23} \\
\phi_d^C = (d - 1)X_d\phi_d^C - \left( d\left( \frac{1}{r} + 1 \right) (b - c)(\phi_d^C) = 0. \tag{24} 
\end{align*}
\]

Note that

\[
(d - 1)X_d\tilde{A} - (d - 1)B - \frac{\lambda}{r}P \leq (d - 1)X_dA - (d - 1)B < 0, \tag{25}
\]

where the last inequality is equivalent to \( X_d < \frac{\lambda}{r\tilde{A}} \), implied by Lemma 2 of Ali and Miller (2013).

Define \( F_d(\phi) = b(\phi) + (d - 1)X_d\phi - \left( d\left( \frac{1}{r} + 1 \right) (b - c)(\phi) \right) \). Note that \( F_d \) inherits convexity from \( b \) and \(-(b - c)\). Now \( F'_d(\phi) = b'(\phi) + (d - 1)X_d\phi' - \left( d\left( \frac{1}{r} + 1 \right) (b - c)'(\phi) \right) \) and

\[
\begin{align*}
F'_d(0) &= b'(0) + (d - 1)X_d\phi'(0) - \left( d\left( \frac{1}{r} + 1 \right)(b - c)'(0) \\
&= \left( d\left( X_d - \frac{\lambda}{r} \right) - X_d \right) \phi'(0) < 0,
\end{align*}
\]

where the second equality comes from Assumption 2 \( (c'(0) = 0) \) and the inequality comes from Claim 1. By the convexity of \( F_d \), it is easy to see that for any fixed \( d > 0 \), there exists a \( \phi_{\text{large}} > 0 \) such that \( F_d(\phi_{\text{large}}) > 0 \). Also, since \( F'_d > 0 \), there exists one unique critical point \( \phi_d^{\text{min}} \) that solves \( F'_d(\phi) = 0 \), i.e.,

\[
\begin{align*}
b'(\phi_d^{\text{min}}) + (d - 1)X_d\phi'(\phi_d^{\text{min}}) - \left( d\left( \frac{1}{r} + 1 \right)(b - c)'(\phi_d^{\text{min}}) = 0. \tag{28} 
\end{align*}
\]

Then

\[
b'(\phi_d^{\text{min}}) = \frac{d\left( \frac{1}{r} + 1 \right)(b - c)'(\phi_d^{\text{min}})}{1 + (d - 1)X_d} > \frac{d\left( \frac{1}{r} + 1 \right)\psi}{1 + (d - 1)X_d}, \tag{29}
\]

where \( \psi > 0 \) is the lower bound on \( (b - c)' \) from Assumption 1. Note that \( F_d(\phi) < 0 \) for all \( \phi < \phi_d^{\text{min}} \), and \( F'_d(\phi) > 0 \) for all \( \phi > \phi_d^{\text{min}} \).

From Claim 2, the RHS of Eq. (29) goes to infinity; by monotonicity of \( b'(\phi) \), \( \lim_{d \to \infty} \phi_d^{\text{min}} = \infty \). It is
easy to check that there exist $\phi_{\text{pos}} > \phi_{\text{neg}} > 0$ such that $F_d(\phi_{\text{neg}}) < 0 < F_d(\phi_{\text{pos}})$; from continuity, there exists $\phi^C_d$ which solves Eq. (24).

From Theorem 2, Eq. (25), and Eq. (6), $F_d(\phi^\text{min}_d) < F_d(\phi^\text{GR}_d) < F_d(\phi^R_d) < F_d(\phi^C_d)$. From convexity, $F''_d > 0$, thus $F'_d(\phi) > 0$ for all $\phi > \phi^\text{min}_d$. Hence, by monotonicity, $\phi^\text{min}_d < \phi^\text{GR}_d < \phi^R_d < \phi^C_d$. Since $\lim_{d \to \infty} \phi^\text{min}_d = \infty$, $\lim_{d \to \infty} \phi^R_d = \infty$, and $\lim_{d \to \infty} \phi^\text{GR}_d = \infty$.

Next, we consider linear approximation of $F_d$ near $\phi^R_d$ and $\phi^C_d$: due to the convexity of $F_d$,

$$\frac{(d-1)B - (d-1)X_dA}{F_d(\phi^C_d)} < \phi^C_d - \phi^R_d < \frac{(d-1)B - (d-1)X_dA}{F_d(\phi^R_d)}.$$  (30)

From Eq. (25), $(d-1)B - (d-1)X_dA > 0$. Dividing $\phi^R_d$ and adding 1 on all sides of the inequalities, we get:

$$\frac{(d-1)B - (d-1)X_dA}{\phi^R_d F_d(\phi^R_d)} + 1 < \frac{(d-1)B - (d-1)X_dA}{\phi^C_d F_d(\phi^C_d)} + 1.$$  (31)

Taking limits, we get:

$$1 \leq \lim_{d \to \infty} \frac{\phi^C_d}{\phi^R_d} \leq \lim_{d \to \infty} \frac{(d-1)B - (d-1)X_dA}{\phi^R_d F_d(\phi_d^R)} + 1.$$  (32)

Similarly, for the suboptimal case $\phi^\text{GR}_d$,

$$1 \leq \lim_{d \to \infty} \frac{\phi^C_d}{\phi^\text{GR}_d} \leq \lim_{d \to \infty} \frac{(d-1)X_d\hat{A} - (d-1)B - \frac{\hat{P}}{2}}{\phi^\text{GR}_d F_d(\phi^\text{GR}_d)} + 1 \leq \lim_{d \to \infty} \frac{(d-1)X_d\hat{A} - (d-1)B}{\phi^\text{GR}_d F_d(\phi^\text{GR}_d)} + 1.$$  (33)

The rest of proof goes through as long as $\hat{A}$ and $B$ are constant. If we replace $\hat{A}$ with $A$ then the rest of proof will be same for the two cases. Thus we can only focus on the case for $\phi^R_d$.

Since

$$\frac{(d-1)B - (d-1)X_dA}{\phi^R_d F_d(\phi^R_d)} + 1 < \frac{(d-1)B}{\phi^R_d F_d(\phi^R_d)} + 1,$$  (34)

given Eq. (32) it suffices to show that

$$\lim_{d \to \infty} \frac{(d-1)B}{\phi^R_d F_d(\phi^R_d)} = \lim_{d \to \infty} \frac{(1 - \frac{1}{A})B}{\phi^R_d F_d(\phi^R_d)} = 0 \iff \lim_{d \to \infty} \frac{1}{d} \phi^R_d F_d(\phi^R_d) = \infty.$$  (35)

Now, $F'(\phi) \leq F'(\phi^R_d)$ for all $\phi^\text{min}_d \leq \phi \leq \phi^R_d$. Hence

$$F_d(\phi^R_d) - F_d(\phi^\text{min}_d) = \int_{\phi^\text{min}_d}^{\phi^R_d} F'_d(\phi) d\phi \leq \int_{\phi^\text{min}_d}^{\phi^R_d} F'_d(\phi^R_d) d\phi = F'_d(\phi^R_d) (\phi^R_d - \phi^\text{min}_d).$$  (36)
where the first equality is from Fundamental Theorem of Calculus, and the inequality is from the fact that $F_d'$ is monotone. Transposing $F_d'(\phi_d^R)\phi_d^{\min} - F_d(\phi_d^{\min})$ to another side in Eq. (36), we get

$$F_d'(\phi_d^R)\phi_d^R \geq F_d'(\phi_d^R)\phi_d^{\min} + F_d(\phi_d^{\min})$$

where the first equality is derived by plugging in

$$F_d'(\phi_d^R) = (1 + (d - 1)X_d)b'(\phi_d^R) - \left(\frac{\lambda}{r} + 1\right)(b - c)'(\phi)$$

and the last inequality is from the fact that $(b - c)$ is concave:

$$(b - c)(\phi_d^{\min}) = \int_0^{\phi_d^{\min}} (b - c)'(\phi) d\phi \geq \int_0^{\phi_d^{\min}} (b - c)'(\phi_d^R) d\phi = (b - c)'(\phi_d^R)\phi_d^{\min}.$$ (41)

From Eq. (29),

$$b'(\phi_d^{\min}) = \frac{(d\lambda + 1)(b - c)'(\phi_d^{\min})}{1 + (d - 1)X_d} = \int_0^{\phi_d^{\min}} b''(\phi) d\phi \leq \int_0^{\phi_d^{\min}} \frac{1}{\epsilon} d\phi = \phi_d^{\min}/\epsilon,$$ (42)

where the second equality is from Fundamental Theorem of Calculus and the inequality is from the condition in Theorem 3 that $\frac{1}{\epsilon} > b''(\phi) > \epsilon$. Then from Assumption 1 it follows that

$$\phi_d^{\min} \geq \frac{(d\lambda + 1)(b - c)'(\phi_d^{\min})\epsilon}{1 + (d - 1)X_d} \geq \frac{(d\lambda + 1)\psi\epsilon}{1 + (d - 1)X_d}.$$ (43)
Notice

\[
b'(\phi_d^R) \phi_{d}^{\text{min}} - b(\phi_{d}^{\text{min}}) = \int_{0}^{\phi_{d}^{\text{min}}} b'(\phi_d^R) \, d\phi - \int_{0}^{\phi_{d}^{\text{min}}} b'(\phi) \, d\phi \\
= \int_{0}^{\phi_{d}^{\text{min}}} (b'(\phi_d^R) - b'(\phi)) \, d\phi \\
= \int_{0}^{\phi_{d}^{\text{min}}} \int_{\phi}^{\phi_d^R} b''(s) \, ds \, d\phi \\
\geq \int_{0}^{\phi_{d}^{\text{min}}} \int_{\phi}^{\phi_d^R} \epsilon \, ds \, d\phi \\
= \int_{0}^{\phi_{d}^{\text{min}}} \epsilon (\phi_d^R - \phi) \, d\phi \\
= \epsilon \left( \phi_d^R \phi_{d}^{\text{min}} - \frac{1}{2} (\phi_{d}^{\text{min}})^2 \right) \\
= \frac{\epsilon}{2} \phi_{d}^{\text{min}} (2\phi_d^R - \phi_{d}^{\text{min}}) \\
\geq \frac{\epsilon}{2} \frac{(d^2 + 1)\psi}{1 + (d - 1)X_d} (2\phi_d^R - \phi_{d}^{\text{min}}),
\]

where the last inequality is derived from plugging Eq. (43). Using Eq. (44) in Eq. (37), we get

\[
F_d(\phi^R_d) \phi_d^R \geq (d - 1)X_dA - (d - 1)B + \frac{\epsilon^2 \psi (d^2 + 1)}{2} (2\phi_d^R - \phi_{d}^{\text{min}}).
\]

Thus,

\[
\lim_{d \to \infty} \frac{F_d(\phi^R_d) \phi_d^R}{d} \geq \lim_{d \to \infty} \frac{(d - 1)X_dA - (d - 1)B + \frac{\epsilon^2 \psi (d^2 + 1)}{2} (2\phi_d^R - \phi_{d}^{\text{min}})}{d} \\
= -B + \frac{\epsilon^2 \psi}{2} \lim_{d \to \infty} (\phi_d^R + \phi_d^R - \phi_{d}^{\text{min}}) \\
\geq -B + \frac{\epsilon^2 \psi}{2} \lim_{d \to \infty} (\phi_d^R).
\]

Since \( \lim_{d \to \infty} (\phi_d^R) = \infty \), the RHS of Eq. (45) goes to \( \infty \), so

\[
\lim_{d \to \infty} \frac{F_d(\phi^R_d) \phi_d^R}{d} = \infty.
\]

Now Eq. (35) is proved, and plugging it into Eq. (32) gives

\[
1 \leq \lim_{d \to \infty} \frac{\phi_C}{\phi_d^R} \leq 1,
\]

which concludes our proof.
Similar arguments can be applied to also prove

\[ 1 \leq \lim_{d \to \infty} \frac{\phi_{\text{C}}^d}{\phi_{\text{CR}}^d} \leq 1, \quad (48) \]

□