ECON 642, Problem Set 1

September 29, 2008

by

Andrew McCallum

1 A warm-up closed economy question

1.1 Find the first order conditions for consumption and capital

The social planner solves the following problem:

$$\max_{C_t,N_t,K_{t+1}} \sum_{i=0}^{\infty} \beta^t \left(\frac{1}{1-\sigma}\right) \left[\left(C_t - \frac{N_t^{\omega}}{\omega}\right)^{1-\sigma} - 1 \right] \tag{1}$$

subject to:

$$A_t K_t^{\alpha} N_t^{1-\alpha} = C_t + I_t \tag{2}$$

$$K_{t+1} = I_t + (1-\delta)K_t \tag{3}$$

$$A_{t+1} = (1-\rho)\bar{A} + \rho A_t + \eta_{t+1} \tag{4}$$

$$\eta_t \sim \textit{white noise}$$

The present value lagrangian is then:

$$\begin{aligned} \mathcal{L} \quad = \quad & E_0 \left[\sum_{i=0}^{\infty} \beta^t \left(\frac{1}{1-\sigma}\right) \left[\left(C_t - \frac{N_t^{\omega}}{\omega}\right)^{1-\sigma} - 1 \right] \right. \\ & \left. + \lambda_t \left(A_t K_t^{\alpha} N_t^{1-\alpha} - C_t - I_t\right) + q_t \left(I_t + (1-\delta)K_t - K_{t+1}\right) \right] \end{aligned} \tag{5}$$

and the first order conditions for consumption, capital, labor and the relationship between the lagrange multipliers are:

$$\frac{\partial \mathcal{L}}{\partial C_t} : \; \beta^t \left(C_t - \frac{N_t^{\omega}}{\omega}\right)^{-\sigma} = \lambda_t \tag{6}$$

$$\frac{\partial \mathcal{L}}{\partial N_t} : \; \beta^t \left(C_t - \frac{N_t^{\omega}}{\omega}\right)^{-\sigma} N_t^{\omega-1} = \lambda_t (1-\alpha) A_t K_t^{\alpha} N_t^{-\alpha} \tag{7}$$

$$\frac{\partial \mathcal{L}}{\partial K_{t+1}} : \; E_t \left[\lambda_{t+1} \alpha A_{t+1} K_{t+1}^{\alpha-1} N_{t+1}^{1-\alpha} + q_{t+1}(1-\delta)\right] = q_t \tag{8}$$

$$\frac{\partial \mathcal{L}}{\partial I_t} : \; \lambda_t = q_t \tag{9}$$

Iterating the $\frac{\partial \mathcal{L}}{\partial C_t}$ forward and using the fact that multipliers are equal we can derive the social planner's Euler equation. For ease of reading define $R_t \equiv \alpha A_t K_t^{\alpha-1} N_t^{1-\alpha}$:

$$\beta E_t \left[\left(C_{t+1} - \tfrac{N_{t+1}^{\omega}}{\omega} \right)^{-\sigma} (R_{t+1} + (1-\delta)) \right] \quad = \left(C_t - \frac{N_t^{\omega}}{\omega} \right)^{-\sigma} \tag{10}$$

Combining $\frac{\partial \mathcal{L}}{\partial C_t}$ and $\frac{\partial \mathcal{L}}{\partial N_t}$ gives the optimal labor allocation equation:

$$N_t^{\omega-1} \quad = \quad (1-\alpha) A_t K_t^{\alpha} N_t^{-\alpha} = W_t \tag{11}$$

and solving for N_t:

$$N_t = [(1-\alpha) A_t K_t^{\alpha}]^{\frac{1}{\omega+\alpha-1}} \tag{12}$$

Combining $\frac{\partial \mathcal{L}}{\partial \lambda_t}$ and $\frac{\partial \mathcal{L}}{\partial q_t}$ also gives:

$$C_t = A_t K_t^{\alpha} N_t^{1-\alpha} - K_{t+1} + (1-\delta) K_t \tag{13}$$

1.2 Define the competitive equilibrium for this model

1.2.1 The representative firm maximizes profits (normalizing the price of the consumption good to 1):

$$\max_{K_t, N_t} A_t K_t^{\alpha} N_t^{1-\alpha} - R_t^f K - W_t^f N_t \tag{14}$$

The first order conditions of which are:

$$\frac{\partial \Pi}{\partial K_t} : \; \alpha A_t K_t^{\alpha-1} N_t^{1-\alpha} = R_t^f \tag{15}$$

$$\frac{\partial \Pi}{\partial N_t} : \; (1-\alpha) A_t K_t^{\alpha} N_t^{-\alpha} = W_t^f \tag{16}$$

where the superscript denotes the wage and rental rates paid by the firm.

1.2.2 The representative agent maximizes utility subject to her budget constraint:

$$\max_{C_t, N_t, K_{t+1}} \sum_{i=0}^{\infty} \beta^t \left(\frac{1}{1-\sigma} \right) \left[\left(C_t - \frac{N_t^{\omega}}{\omega} \right)^{1-\sigma} - 1 \right] \tag{17}$$

subject to:

$$C_t + I_t = W_t^c N_t + R_t^c K_t \tag{18}$$

$$K_{t+1} = I_t + (1-\delta) K_t \tag{19}$$

$$A_{t+1} = (1-\rho)\bar{A} + \rho A_t + \eta_{t+1} \tag{20}$$

$$\eta_t \sim white\, noise$$

The consumer owns all factors of production and so recieves rental and wage payments from the firm. These amounts are denoted by W_t^c and R_t^c. As the firm's owner, the representative consumer also receives any profits made by the firm. Since the firm has a CRS production function, profits are zero.

The present value lagrangian is then:

$$\mathcal{L} = E_0\left[\sum_{i=0}^{\infty}\beta^t\left(\frac{1}{1-\sigma}\right)\left[\left(C_t - \frac{N_t^\omega}{\omega}\right)^{1-\sigma} - 1\right]\right.$$
$$\left. +\lambda_t\left(W_t^c N_t + R_t^c K_t - C_t - I_t\right) + q_t\left(I_t + (1-\delta)K_t - K_{t+1}\right)\right] \tag{21}$$

The first order conditions for the representative agent are:

$$\frac{\partial\mathcal{L}}{\partial C_t} : \beta^t\left(C_t - \frac{N_t^\omega}{\omega}\right)^{-\sigma} = \lambda_t \tag{22}$$

$$\frac{\partial\mathcal{L}}{\partial N_t} : \beta^t\left(C_t - \frac{N_t^\omega}{\omega}\right)^{-\sigma} N_t^{\omega-1} = \lambda_t W_t^c \tag{23}$$

$$\frac{\partial\mathcal{L}}{\partial K_{t+1}} : E_t\left[\lambda_{t+1}R_{t+1}^c + q_{t+1}(1-\delta)\right] = q_t \tag{24}$$

$$\frac{\partial\mathcal{L}}{\partial I_t} : \lambda_t = q_t \tag{25}$$

combining $\frac{\partial\mathcal{L}}{\partial C_t}$, $\frac{\partial\mathcal{L}}{\partial K_{t+1}}$ and $\frac{\partial\mathcal{L}}{\partial I_t}$ gives:

$$\beta E_t\left[\left(C_{t+1} - \frac{N_{t+1}^\omega}{\omega}\right)^{-\sigma}\left(R_{t+1}^c + (1-\delta)\right)\right] = \left(C_t - \frac{N_t^\omega}{\omega}\right)^{-\sigma} \tag{26}$$

combining $\frac{\partial\mathcal{L}}{\partial C_t}$ and $\frac{\partial\mathcal{L}}{\partial N_t}$ gives labor supply:

$$N_t^{\omega-1} = W_t^c \tag{27}$$

1.2.3 Factor and goods markets clear

In order for the labor markets to clear, labor demand from the firm's problem must equal labor supply from the representative agent's problem. This condition is equivalent to equating the wage paid by the firm and the wage received by the consumer:

$$W_t^f = (1-\alpha)A_t K_t^\alpha N_t^{-\alpha} = demand = supply = N_t^{\omega-1} = W_t^c \tag{28}$$

Capital markets will clear under an analogous condition:

$$R_t^f = \alpha A_t K_t^{\alpha-1} N_t^{1-\alpha} = demand = supply = \alpha A_t K_t^{\alpha-1} N_t^{1-\alpha} = R_t^c \tag{29}$$

There is no storage or savings so output produced by the firm equals income paid to the representative agent. This condition implies the goods market clears:

$$Y_t^f = A_t K_t^\alpha N_t^{1-\alpha} = W_t^c N_t + R_t^c K_t$$
$$= (1-\alpha)A_t K_t^\alpha N_t^{-\alpha} N_t + \alpha A_t K_t^{\alpha-1} N_t^{1-\alpha} K_t = Y_t^c \tag{30}$$

1.2.4 Social planner and consumer's problem are equivalent

Since the optimally conditions are identical in both problems, so are the allocations. Solving the social planner's problem is equivalent to solving the representative agent's.

We can also see the wage and rental rates in the representative consumer's problem equals the lagrange multipliers in the planner's problem from equations (XXX) and (YYY) how????????

1.3 Log-linearizing each of the first order conditions around the steady-state.

1.3.1 First find the nonstochastic steady-state value of consumption, capital and technology via the first order equations and technology shock process.

From the social planner's problem we know the nonstochastic steady-state values will be:

From the Euler equation and the definition of $\bar{R}$:

$$\frac{1}{\beta} \quad = \bar{R} + (1-\delta) = \alpha \bar{A}\bar{K}^{\alpha-1}\bar{N}^{1-\alpha} + (1-\delta) \tag{31}$$

From the accounting identity combined with the equaiton of motion for capital:

$$\bar{C} = \bar{A}\bar{K}^{\alpha}\bar{N}^{1-\alpha} - \delta\bar{K} \tag{32}$$

From the technological shock process:

$$\bar{A} = 1 \tag{33}$$

Using these three equations, we can find the steady-state values of all other variables of interest.

Solve for $\bar{K}$ by starting with 31, and substituting in 12, the optimal labor quantity, at steady-state to get:

$$\frac{1}{\beta} = \alpha\bar{K}^{\alpha-1}\left[(1-\alpha)\bar{A}\bar{K}^{\alpha}\right]^{\frac{1-\alpha}{\omega+\alpha-1}} + (1-\delta) \tag{34}$$

simplifying:

$$\frac{1}{\beta} = \alpha\bar{K}^{\alpha-1}[(1-\alpha)\bar{A}]^{\frac{1-\alpha}{\omega+\alpha-1}}\bar{K}^{\frac{\alpha(1-\alpha)}{\omega+\alpha-1}} + (1-\delta) \tag{35}$$

$$\frac{1}{\beta} = \alpha\bar{K}^{\alpha\left(1+\frac{1-\alpha}{\omega+\alpha-1}\right)-1}[(1-\alpha)\bar{A}]^{\frac{1-\alpha}{\omega+\alpha-1}} + (1-\delta) \tag{36}$$

call $\gamma \equiv \dfrac{1-\alpha}{\omega+\alpha-1}$:

$$\frac{1}{\beta} = \alpha\bar{K}^{\alpha(1+\gamma)-1}[(1-\alpha)\bar{A}]^{\gamma} + (1-\delta) \tag{37}$$

$$\bar{K} = \left[\left(\frac{1}{\beta} - (1-\delta)\right)\frac{1}{\alpha[(1-\alpha)\bar{A}]^{\gamma}}\right]^{\dfrac{1}{\alpha(1+\gamma)-1}} \tag{38}$$

which we can plug back into 12 to get $\bar{N}$ and 32to get $\bar{C}$ in terms of parameters and $\bar{A}$.

1.3.2 Log-linearize around the steady-state

Log-linearizing the left-hand side of the Euler equation gives:

$$f(\tilde{C}_{t+1},\tilde{K}_{t+1},\tilde{N}_{t+1}) \approx \beta\left(\bar{C}-\frac{\bar{N}^{\omega}}{\omega}\right)^{-\sigma}\left(\alpha\bar{A}\bar{K}^{\alpha-1}\bar{N}^{1-\alpha}+(1-\delta)\right) \quad (39)$$

$$-\beta\left[\sigma\left(\alpha\bar{A}\bar{K}^{\alpha-1}\bar{N}^{1-\alpha}+(1-\delta)\right)\left(\bar{C}-\frac{\bar{N}^{\omega}}{\omega}\right)^{-\sigma-1}\bar{C}\right]\tilde{C}_{t+1} \quad (40)$$

$$+\beta\left[\left(\bar{C}-\frac{\bar{N}^{\omega}}{\omega}\right)^{-\sigma}\left(\alpha\bar{A}\bar{K}^{\alpha-1}\bar{N}^{1-\alpha}(\alpha-1)\right)\right]\tilde{K}_{t+1} \quad (41)$$

$$+\beta\left[\left(\bar{C}-\frac{\bar{N}^{\omega}}{\omega}\right)^{-\sigma}\left(\alpha\bar{A}\bar{K}^{\alpha-1}\bar{N}^{1-\alpha}\right)\right]\tilde{A}_{t+1} \quad (42)$$

$$+\beta\left[\left(\bar{C}-\frac{\bar{N}^{\omega}}{\omega}\right)^{-\sigma}\left(\alpha\bar{A}\bar{K}^{\alpha-1}\bar{N}^{1-\alpha}(1-\alpha)\right)+\sigma\left(\alpha\bar{A}\bar{K}^{\alpha-1}\bar{N}^{1-\alpha}+(1-\delta)\right)\left(\bar{C}-\frac{\bar{N}^{\omega}}{\omega}\right)^{-\sigma-1}\bar{N}^{\omega}\right]\tilde{N}_{t+1} \quad (43)$$

Using the steady-state value of $\bar{R}$ this simplifies to:

$$f(\tilde{C}_{t+1},\tilde{K}_{t+1},\tilde{N}_{t+1}) \approx \left(\bar{C}-\frac{\bar{N}^{\omega}}{\omega}\right)^{-\sigma} \quad (44)$$

$$-\left[\sigma\left(\bar{C}-\frac{\bar{N}^{\omega}}{\omega}\right)^{-\sigma-1}\bar{C}\right]\tilde{C}_{t+1} \quad (45)$$

$$+\beta\left[\left(\bar{C}-\frac{\bar{N}^{\omega}}{\omega}\right)^{-\sigma}\left(\alpha\bar{A}\bar{K}^{\alpha-1}\bar{N}^{1-\alpha}(\alpha-1)\right)\right]\tilde{K}_{t+1} \quad (46)$$

$$\beta\left[\left(\bar{C}-\frac{\bar{N}^{\omega}}{\omega}\right)^{-\sigma}\left(\alpha\bar{A}\bar{K}^{\alpha-1}\bar{N}^{1-\alpha}\right)\right]\tilde{A}_{t+1} \quad (47)$$

$$+\left[\beta\left(\bar{C}-\frac{\bar{N}^{\omega}}{\omega}\right)^{-\sigma}\left(\alpha\bar{A}\bar{K}^{\alpha-1}\bar{N}^{1-\alpha}(1-\alpha)\right)+\sigma\left(\bar{C}-\frac{\bar{N}^{\omega}}{\omega}\right)^{-\sigma-1}\bar{N}^{\omega}\right]\tilde{N}_{t+1} \quad (48)$$

Log-linearizing the right-hand side gives:

$$h(\tilde{C}_t,\tilde{N}_t) \approx \left(\bar{C}-\frac{\bar{N}^{\omega}}{\omega}\right)^{-\sigma} \quad (49)$$

$$-\left[\sigma\left(\bar{C}-\frac{\bar{N}^{\omega}}{\omega}\right)^{-\sigma-1}\bar{C}\right]\tilde{C}_t \quad (50)$$

$$+\left[\sigma\left(\bar{C}-\frac{\bar{N}^{\omega}}{\omega}\right)^{-\sigma-1}\bar{N}^{\omega}\right]\tilde{N}_t \quad (51)$$

Now move all $t+1$ variables to the left-hand side and log-linearize 13:

$$\bar{K}(1+\tilde{K}_{t+1}) \approx$$
$$\bar{A}\bar{K}^{\alpha}\bar{N}^{1-\alpha}(1+\alpha\tilde{K}_t+(1-\alpha)\tilde{N}_t)-\bar{C}(1+\tilde{C}_t)+(1-\delta)\bar{K}(1+\tilde{K}_t) \quad (52)$$

(denote $\bar{Y}\equiv\bar{A}\bar{K}^{\alpha}\bar{N}^{1-\alpha}$)

$$\bar{K}+\bar{K}\tilde{K}_{t+1} \approx \quad (53)$$

$$\bar{Y}+\alpha\bar{Y}\tilde{K}_t+\bar{Y}(1-\alpha)\tilde{N}_t-\bar{C}-\bar{C}\tilde{C}_t+(1-\delta)\bar{K}+(1-\delta)\bar{K}\tilde{K}_t \quad (54)$$

$$\bar{K}\tilde{K}_{t+1} \approx \tag{55}$$
$$-\bar{K} + \bar{Y} - \bar{C} + \bar{K} - \delta\bar{K} + \alpha\bar{Y}\tilde{K}_t + \bar{Y}(1-\alpha)\tilde{N}_t - \bar{C}\tilde{C}_t + (1-\delta)\bar{K}\tilde{K}_t \tag{56}$$

$$\bar{K}\tilde{K}_{t+1} \approx \tag{57}$$
$$\bar{Y} - \bar{C} - \delta\bar{K} + \alpha\bar{Y}\tilde{K}_t + \bar{Y}(1-\alpha)\tilde{N}_t - \bar{C}\tilde{C}_t + (1-\delta)\bar{K}\tilde{K}_t \tag{58}$$

and in steady-state we know $\bar{I} = \delta\bar{K}$, so then by rearranging terms we have:

$$\bar{K}\tilde{K}_{t+1} \approx \tag{59}$$
$$-\bar{C}\tilde{C}_t + ((1-\delta)\bar{K} + \alpha\bar{Y})\tilde{K}_t + \bar{Y}(1-\alpha)\tilde{N}_t \tag{60}$$

Log-linearizing the shock process gives:

$$\bar{A}(1 + \tilde{A}_{t+1}) \approx (1-\rho)\bar{A} + \rho\bar{A}(1 + \tilde{A}_t) \tag{61}$$

$$\bar{A} + \bar{A}\tilde{A}_{t+1} \approx \bar{A} - \rho\bar{A} + \rho\bar{A} + \bar{\rho A}\tilde{A}_t \tag{62}$$

$$\tilde{A}_{t+1} \approx \rho\tilde{A}_t \tag{63}$$

We are interested in the impulse responses of Y and N, so we also need to log-linearize the production function:

$$\bar{Y}(1 + \tilde{Y}_t) \approx \bar{Y}\left(1 + \tilde{A}_t + \alpha\tilde{K}_t + (1-\alpha)\tilde{N}_t\right) \tag{64}$$

$$\bar{Y} + \bar{Y}\tilde{Y}_t \approx \bar{Y} + \bar{Y}\tilde{A}_t + \alpha\bar{Y}\tilde{K}_t + (1-\alpha)\bar{Y}\tilde{N}_t \tag{65}$$

$$0 \approx \alpha\tilde{K}_t + \tilde{A}_t + (1-\alpha)\tilde{N}_t - \tilde{Y}_t \tag{66}$$

and the optimal labor allocation equation:

$$(1-\alpha)A_t K_t^{\alpha} N_t^{-\alpha} = N_t^{\omega-1} \tag{67}$$

$$\overline{MPN}\left(1 + \tilde{A}_t + \alpha\tilde{K}_t - \alpha\tilde{N}_t\right) \approx \bar{N}^{\omega-1}(1 + (\omega-1)\tilde{N}_t) \tag{68}$$

$$\tilde{A}_t + \alpha\tilde{K}_t - \alpha\tilde{N}_t \approx (\omega-1)\tilde{N}_t \tag{69}$$

$$0 \approx -\alpha\tilde{K}_t - \tilde{A}_t + (\omega - 1 + \alpha)\,\tilde{N}_t \tag{70}$$

In order to apply the method of redundant variables, we must also log-linearize R_t , W_t and the accounting identity.

$$R_t = \alpha A_t K_t^{\alpha-1} N_t^{1-\alpha} \tag{71}$$

$$\bar{R}(1 + \tilde{R}_t) \approx \bar{R}(1 + \tilde{A}_t + (\alpha-1)\tilde{K}_t + (1-\alpha)\tilde{N}_t) \tag{72}$$

$$\approx (\alpha-1)\tilde{K}_t + \tilde{A}_t + (1-\alpha)\tilde{N}_t - \tilde{R}_t \tag{73}$$

Wage from the profit maximizing equation:

$$(1-\alpha)K_t^{\alpha}N_t^{-\alpha} = N_t^{\omega-1} \tag{74}$$

$$\bar{W}(1+\alpha\tilde{K}_t - \alpha)\tilde{N}_t) \approx \bar{N}^{\omega-1}(1+(\omega-1)\tilde{N}_t) \tag{75}$$

$$0 \approx (\omega-1)\tilde{N}_t - \tilde{W}_t \tag{76}$$

From the accounting identity:

$$Y_t = C_t + I_t \tag{77}$$

$$\bar{Y}(1+\tilde{Y}_t) \approx \bar{C}(1+\tilde{C}_t) + \bar{I}(1+\tilde{I}_t) \tag{78}$$

$$0 \approx \bar{C}\tilde{C}_t - \bar{Y}\tilde{Y}_t + \bar{I}\tilde{I}_t \tag{79}$$

where $\bar{I} = \delta K$.

Then defining $\bar{X} \equiv \left(\bar{C} - \dfrac{\bar{N}^{\omega}}{\omega}\right)$ and using $\bar{R} = \dfrac{1}{\beta} - (1-\delta) = \alpha\bar{K}^{\alpha-1}\bar{N}^{1-\alpha}$

We can then write the problem as:

$$\begin{bmatrix} -\sigma\bar{X}^{-\sigma-1}\bar{C} & \beta\bar{X}^{-\sigma}\bar{R}(\alpha-1) & \beta\bar{X}^{-\sigma}\bar{R} & a_{14} & 0 & 0 & 0 & 0 \\ 0 & \bar{K} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} E_t \begin{bmatrix} \tilde{C}_{t+1} \\ \tilde{K}_{t+1} \\ \tilde{A}_{t+1} \\ \tilde{N}_{t+1} \\ \tilde{Y}_{t+1} \\ \tilde{R}_{t+1} \\ \tilde{W}_{t+1} \\ \tilde{I}_{t+1} \end{bmatrix} =$$

$$\begin{bmatrix} -\sigma\bar{X}^{-\sigma-1}\bar{C} & 0 & 0 & \sigma\bar{X}^{-\sigma-1}\bar{N}^{\omega} & 0 & 0 & 0 & 0 \\ -\bar{C} & b_{22} & 0 & \bar{Y}(1-\alpha) & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 1 & (1-\alpha) & -1 & 0 & 0 & 0 \\ 0 & -\alpha & -1 & (\omega-1+\alpha) & 0 & 0 & 0 & 0 \\ 0 & \alpha-1 & 1 & 1-\alpha & 0 & -1 & 0 & 0 \\ 0 & \alpha & 1 & -\alpha & 0 & 0 & -1 & 0 \\ \bar{C} & 0 & 0 & 0 & -\bar{Y} & 0 & 0 & \delta\bar{K} \end{bmatrix}$$

where $a_{14} = \beta\bar{X}^{-\sigma}\bar{R}\,(1-\alpha) + \sigma\bar{X}^{-\sigma-1}\bar{N}^{\omega}$, and $b_{22} = (1-\delta)\bar{K} + \alpha\bar{Y}$.

1.4 Model calibration and graphing the solution

Taking the parameter values from Schmitt-Grohe & Uribe (2001)

The share of income to labor: $\alpha = 0.32$

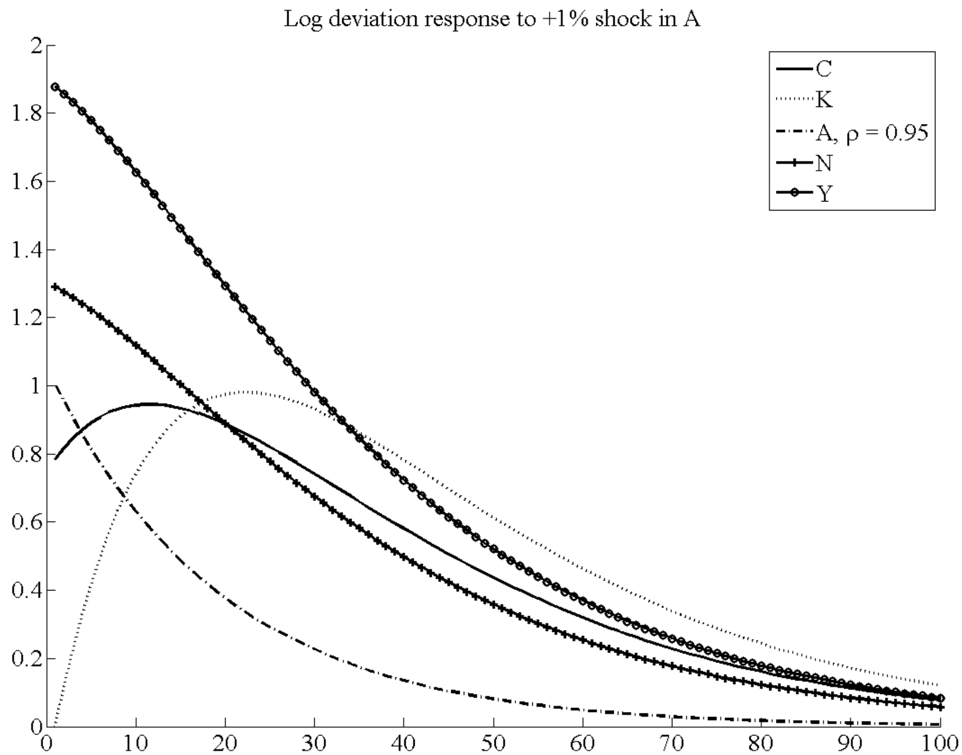
The subjective discount rate: $\beta = 0.96$

The depreciation rate: $\delta = 0.10$

The intertemporal elasticity of substitution: $\sigma = 2$

The elasticity of labor supply: $\omega = 1.455$(and is related to Frisch Waugh labor supply elasticity via $\omega = 1 + \frac{1}{\eta}$ implies $\eta = 2.20$)

Using the method of redunant variables, see the attached MatLab code, we can calculate the



following:

2 A small open production economy

2.1 Characterize the optimal equilibrium solution

The budget constraint now includes one period bonds (which can be rolled over each period):

$$A_t K_t^{\alpha} N_t^{1-\alpha} + (1 + r(b_{t-1}))b_{t-1} = C_t + I_t + b_t \tag{80}$$

the equation of motion for debt is given by:

$$b_t = (1 + r(b_{t-1}))b_{t-1} + C_t + I_t - Y_t \tag{81}$$

which says our bond purchase this period is equal to consumption, investment, and interest payments, minus income.

The specific functional form for the debt elastic interest rate is:

$$r(b_{t-1}) = r + \psi(e^{b_{t-1}-\bar{b}} - 1) \tag{82}$$

Each agent assumes their bond holding decision will not effect the interest rate even though in the aggregate this is not the case. The first order condition for bonds is then:

$$\frac{\partial \mathcal{L}}{\partial b_t} : \; E_t\left[\lambda_{t+1}\right](1 + r(b_t)) = \lambda_t \tag{83}$$

Since we have introduced debt in this model, we also need to include the no-Ponzi condition that the expected present discounted value of debt cannot be positive. Furthermore, by the optimality argument it cannot be negative and is then:

$$\lim_{j\to\infty} E_t\left[\frac{b_{t+j}}{\prod_{s=1}^{j}(1+r_s)}\right] = 0 \tag{84}$$

Combining this with $\frac{\partial\mathcal{L}}{\partial C_t}$ we get the Lucas asset pricing equation:

$$\beta E_t\left[\left(C_{t+1} - \tfrac{N_{t+1}^{\omega}}{\omega}\right)^{-\sigma}\right](1+r(b_t)) \quad = \left(C_t - \frac{N_t^{\omega}}{\omega}\right)^{-\sigma} \tag{85}$$

The Euler equation from combining $\frac{\partial\mathcal{L}}{\partial C_t}$ and $\frac{\partial\mathcal{L}}{\partial K_{t+1}}$ remains the same as the closed economy:

$$\beta E_t\left[\left(C_{t+1} - \tfrac{N_{t+1}^{\omega}}{\omega}\right)^{-\sigma}(R_{t+1}+(1-\delta))\right] \quad = \left(C_t - \frac{N_t^{\omega}}{\omega}\right)^{-\sigma} \tag{86}$$

Combining $\frac{\partial\mathcal{L}}{\partial C_t}$ and $\frac{\partial\mathcal{L}}{\partial N_t}$ also gives the same optimal labor allocation equation:

$$N_t^{\omega-1} \quad = \quad (1-\alpha)A_t K_t^{\alpha} N_t^{-\alpha} = W_t \tag{87}$$

and we also still have:

$$K_{t+1} = I_t + (1-\delta)K_t \tag{88}$$

$$A_{t+1} = (1-\rho)\bar{A} + \rho A_t + \eta_{t+1} \tag{89}$$

$$\eta_t \sim white\ noise$$

2.2 Log-linearize the new optimality conditions

We can mostly use the same equations from the closed economy case but need to log-linearize the Lucas asset pricing equation and the altered budget constraint.