# Crash Course on Toric Geometry

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## The Kähler cone

- If $X_\Sigma$ is a simplicial projective toric variety, then
$$A_{n-1}(X_\Sigma) \otimes \mathbb{R} \cong H^2(X_\Sigma; \mathbb{R}),$$
so
$$H^{1,1}(X_\Sigma; \mathbb{R}) \cong H^2(X_\Sigma; \mathbb{R}) \cong A_{n-1}(X_\Sigma) \otimes \mathbb{R}.$$
- The **Kähler cone** in $A_{n-1}(X_\Sigma) \otimes \mathbb{R}$ is the subset consisting of all classes that correspond under the above isomorphism to Kähler classes in $H^{1,1}(X_\Sigma; \mathbb{R})$.
- Here is a concrete description of the Kähler cone:
- For any effective divisor class $a = \sum a_\rho [D_\rho]$, and each $\sigma \in \Sigma$, we can find $m_\sigma \in M_\mathbb{R}$ such that
$$\langle m_\sigma, v_\rho \rangle = -a_\rho$$
for all $\rho \in \sigma(1)$, where $v_\rho$ is the integral generator of $\rho$. (This is simply because the vectors $v_\rho$ are linearly independent.)
- **Definition**: An effective divisor class $a$ as above is **convex** if for each $\sigma \in \Sigma$, we have
$$\langle m_\sigma, v_\rho \rangle \geq -a_\rho$$
for *all* $\rho \in \Sigma(1)$.
- Let
$$\mathrm{cpl}(\Sigma) = \{a \in A^+(\Xi) \mid a \text{ is convex}\}.$$
- **Fact**: The Kähler cone of $X_\Sigma$ is the interior of $\mathrm{cpl}(\Sigma)$.

## The moment map

- Let $X_\Sigma$ be a toric variety, determined by a fan $\Sigma$ in $N_\mathbb{R} \cong \mathbb{R}^n$. Let $r = |\Sigma(1)|$.
- Let
$$G = \mathrm{Hom}_\mathbb{Z}(A_{n-1}(X), \mathbb{C}^*),$$
$$G_\mathbb{R} = \mathrm{Hom}_\mathbb{Z}(A_{n-1}(X), U(1)).$$
- Applying $\mathrm{Hom}_\mathbb{Z}(\cdot, \mathbb{C}^*)$ to the exact sequence
$$0 \to M \to \mathbb{Z}^{\Sigma(1)} \to A_{n-1}(X) \to 0$$
gives an inclusion $G \hookrightarrow (\mathbb{C}^*)^r$, and hence $G_\mathbb{R} \hookrightarrow U(1)^r$.
- The Hamiltonian action of $U(1)^r$ on $\mathbb{C}^r$ thus gives a Hamiltonian action of $G_\mathbb{R}$ on $\mathbb{C}^r$.
  - Recall, to say that the action of a Lie group $G$ on a symplectic manifold $M$ is **Hamiltonian** is to say that there exists a **moment map** $\mu : M \to \mathfrak{g}^*$ such that for all $\xi \in \mathfrak{g}$, if $\mu_\xi : M \to \mathbb{R}$ is the function $p \mapsto \mu(p)(\xi)$, we have
  $$d\mu_\xi = \rho(\xi)$$
  under the bijection between vector fields and 1-forms induced by the symplectic form. Here, $\rho(\xi)$ is the vector field describing the infinitesimal action of $\xi$, that is:
  $$\rho(\xi)_x = \left.\frac{d}{dt}\right|_{t=0} \exp(t\xi) \cdot x.$$
- Note that
$$\mathfrak{g}_\mathbb{R} = \mathrm{Hom}_\mathbb{Z}(A_{n-1}(X), \mathbb{R}),$$
$$\mathfrak{g}_\mathbb{R}^* = A_{n-1}(X) \otimes \mathbb{R}.$$
- Using this, it is possible to get an explicit description of the moment map $\mu_\Sigma$ for the action of $G_\mathbb{R}$ on $\mathbb{C}^r$.
- **Theorem**: If $X_\Sigma$ is projective and simplicial and $a \in A_{n-1}(X) \otimes \mathbb{R} = \mathfrak{g}_\mathbb{R}^*$ is a Kähler class, then
$$\mu_\Sigma^{-1}(a)/G_\mathbb{R} \cong X_\Sigma.$$
- **Idea of Proof**: We know from the fact stated previously that $a$ is strictly convex. Define a polytope $\Delta \subset M_\mathbb{R}$ by
$$\Delta = \{m \in M_\mathbb{R} \mid \langle m, v_i \rangle \geq -a_i\}.$$
Strict convexity of $a$ implies that $\Delta$ is combinatorially dual to $\Sigma$, which implies the claim.

## The secondary fan

- **Idea**: The secondary fan is a way of fitting together the Kähler cones of all the toric varieties with a fixed 1-skeleton.
- Why do this?
  - Suppose $\Delta$ is a reflexive polytope in $M$ and $\mathbb{P}_\Delta$ is the associated toric variety. Recall that if $\Sigma$ is the normal fan in $N$ of $\Delta$, then $\mathbb{P}_\Delta = X_\Sigma$.
  - Any hypersurface of $\mathbb{P}_\Delta$ whose divisor class is the anticanonical class is Calabi-Yau by the adjunction formula.
  - Try to desingularize as much as possible while remaining Calabi-Yau.
  - Desingularization is given by adding cone generators to $\Sigma$ until every cone has generators that extend to a $\mathbb{Z}$-basis for $N$.
  - **Fact**: If $Z$ is the toric variety obtained by adding a cone generator $\rho$ to $\Sigma$, then there is a birational map $f : Z \to \mathbb{P}_\Delta$ and

    $$K_Z = f^*(K_{\mathbb{P}_\Delta}) - (\langle \rho, m_F \rangle + 1)D$$

    if $\rho$ lies in the facet $\langle u, m_F \rangle = -1$ of $\Delta^\circ$.
  - So if we don't want to change the canonical class, we can add all the elements of $(N \cap \Delta^\circ) \setminus \{0\}$ but no more.
  - All the Calabi-Yaus obtained in this way will have 1-skeleton $(N \cap \Delta^\circ) \setminus \{0\}$.
  - They will together form one side of the mirror. They have the same complex moduli (since related by flops), but their Kähler moduli may differ, so we want a way to talk about their Kähler moduli collectively.
- Fix a finite set of strongly convex rational 1-dimensional cones $\Xi$ in $N_\mathbb{R}$.
- Want to consider all fans $\Sigma$ with $\Sigma(1) = \Xi$.
- $A_{n-1}(X_\Sigma)$ depends only on $\Xi$, because it is determined by the exact sequence

  $$0 \to M \to \mathbb{Z}^{\Sigma(1)} \to A_{n-1}(X_\Sigma) \to 0.$$

  So denote it $A(\Xi)$. Similarly for effective classes, so denote them $A^+(\Xi)$.
- Recall that the Kähler cone of $X_\Sigma$ is the interior of $\mathrm{cpl}(\Sigma)$, and this *does* depend on $\Sigma$.
- We'd like to say that the $\mathrm{cpl}(\Sigma)$ form a fan that fills up all of $A^+(\Xi)$ as $\Sigma$ varies of fans with 1-skeleton $\Xi$, but this is not quite true yet.
- To make it true, we need to vary over all projective simplicial fans with $\Sigma(1) \subset \Xi$.

- We need to modify the definition of convexity slightly in order to make sense for such fans, but can basically define $\mathrm{cpl}(\Sigma)$ as above.

- **Theorem**: Let $\Xi$ be a finite set of strongly convex rational 1-dimensional cones in $N_{\mathbb{R}}$. As $\Sigma$ ranges over all projective simplicial fans with $\Sigma(1) \subset \Xi$, the cones $\mathrm{cpl}(\Sigma)$ and their faces form a fan in $A(\Xi)$ whose support is $A^+(\Xi)$.

- This is called the **secondary fan** or **GKZ decomposition**.

- There is also an **enlarged secondary fan** (sometimes just called the secondary fan) that fills up all of $A(\Xi)$ and contains the above as a subfan.

### Relation to the Moment Map

- The moment map $\mu_\Sigma : \mathbb{C}^{|\Sigma(1)|} \to \mathbb{R}^{|\Sigma(1)|-n}$ depends only on $\Sigma(1) = \Xi$ and its image is $A^+(\Xi)$, so we can write
$$\mu_\Xi : \mathbb{C}^\Xi \to A^+(\Xi).$$

- Let $G(\Xi) = \mathrm{Hom}_{\mathbb{Z}}(\Xi, \mathbb{C}^*)$.

- We saw before that the moment map determines $X_\Sigma$. In fact, the same statement holds even for toric varieties whose 1-skeleton is a proper subset of $\Xi$.

- **Theorem**: If $\Sigma$ is a projective simplicial fan with $\Sigma(1) \subset \Xi$ and $a \in A^+(\Xi)$ is in the interior of $\mathrm{cpl}(\Sigma)$, then there is a natural orbifold diffeomorphism $\mu_\Xi^{-1}(a)/G(\Xi)_{\mathbb{R}} \cong X_\Sigma$.

- Thus, once we have the secondary fan, we can use the moment map to reconstruct all of the toric varieties whose Kähler cones are represented.

## Example

- Consider the fan $\Sigma'$ in $N = \mathbb{Z}^2$ whose cones are generated by all the proper subsets of the set
$$v_0 = (-1,-2),\ v_1 = (1,0),\ v_2 = (0,1).$$
The corresponding toric variety is $\mathbb{P}(1,1,2)$.

- This is an orbifold. (Combinatorially, this is because $v_0$ and $v_1$ do not form a $\mathbb{Z}$-basis for $\mathbb{Z}^2$, but nevertheless any proper subset of the $v_i$'s extends to a $\mathbb{Q}$-basis.)

- Can desingularize by adding the cone generator $v_3 = (0,-1)$.

- Call the resulting fan $\Sigma$, and let $\Xi = \Sigma(1)$.

- There are two complete fans whose 1-skeleton is contained in $\Xi$, namely $\Sigma$ and $\Sigma'$.

- The usual exact sequence is

$$0 \to \mathbb{Z}^2 \to \mathbb{Z}^4 \to A_1(X_\Sigma) \to 0,$$

so $A_1(X_\Sigma) = \mathbb{Z}^2$. Explicitly, the first map is given by pairing the standard basis vectors with the integral generators of the 1-dimensional cones, which is

$$(x, y) \mapsto (-x - 2y, x, y, -y),$$

and the second map then must be

$$(a, b, c, d) \mapsto (c + d, a + b - 2d).$$

- So $A^+(\Xi)$ is the cone on $(0, 1)$, $(1, 0)$, and $(1, -2)$ in $\mathbb{R}^2$.
- **Check**: $\mathrm{cpl}(\Sigma) = \{a_0 D_0 + \cdots + a_3 D_3 \mid a_2 + a_3 \geq 0, a_0 + a_1 + 2a_2 \geq 0\}$, which is the first quadrant.
- So the secondary fan is: