Introduction to the gauged linear sigma model

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Abstract. We describe the definition of the gauged linear sigma model (GLSM), focusing specifically on Fan–Jarvis–Ruan–Witten theory and its generalization, the hybrid model. These theories are related, via the celebrated Landau–Ginzburg/Calabi–Yau (LG/CY) correspondence, to the Gromov–Witten theory of complete intersections in weighted projective space. We discuss how the LG/CY correspondence motivates the meaning of the GLSM, and conversely, how a GLSM perspective can be used to prove the correspondence in many cases.

The goal of these lectures is to motivate the definition of the gauged linear sigma model, or GLSM, a generalization of Gromov–Witten theory as well as other (perhaps less familiar) objects such as Fan–Jarvis–Ruan–Witten (FJRW) theory. In particular, we show that, from a GLSM viewpoint, the definition of FJRW theory is an entirely natural analogue of the definition of Gromov–Witten theory. The celebrated Landau–Ginzburg/Calabi–Yau correspondence relates these two theories to one another, and placing both in the more general context of the GLSM hints at a method by which this correspondence could be proven. In the last lecture, we discuss the proof of the Landau–Ginzburg/Calabi–Yau correspondence in genus zero, and we mention ongoing work that points toward a proof in higher genus.

1. Preliminaries on orbifolds and Gromov–Witten theory

Before beginning our discussion of the GLSM, we must recall some basics about orbifolds and their cohomology, as well as the definition of Gromov–Witten theory and quantum cohomology.

1.1. What is an orbifold? Roughly speaking, an orbifold is a space locally homeomorphic to the quotient of an open subset of Euclidean space by the action of a finite group. More precisely, analogously to the definition of a manifold, one can define:

Definition 1.1. Let $X$ be a topological space. An orbifold chart on $X$ consists of:

(1) an open set $U \subseteq X$;
(2) an open set $\tilde{U} \subseteq \mathbb{R}^n$ for some $n$;
(3) a smooth action of a finite group $G$ on $\tilde{U}$;

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(4) a continuous, $G$-invariant map $\tilde{\phi} : \tilde{U} \to U$ such that the induced map

$$\phi : \tilde{U}/G \to U$$

is a homeomorphism.

From here, one must prescribe what it means for two overlapping orbifold charts to be compatible. This is surprisingly subtle, and we encourage the careful reader to attempt to write down a reasonable definition for herself before reading further.

**Definition 1.2.** We say that an orbifold chart

$$\tilde{V}/H \xrightarrow{\psi} V \subseteq X$$

is a *subchart* of an orbifold chart

$$\tilde{U}/G \xrightarrow{\phi} U \subseteq X$$

if $V \subseteq U$ and there is a group homomorphism $\tau : H \to G$ and a smooth embedding $\iota : \tilde{V} \hookrightarrow \tilde{U}$ such that:

1. $\iota(h \cdot x) = \tau(h) \cdot \iota(x)$ for all $x \in \tilde{V}$ and $h \in H$, where $\cdot$ denotes the group actions of $H$ on $\tilde{V}$ and $G$ on $\tilde{Y}$;
2. $\psi(x) = \tilde{\phi}(\iota(x))$ for all $x \in V$;
3. for each $x \in V$, the stabilizer $H_x$ is isomorphic to the stabilizer $G_{\iota(x)}$.

Given two orbifold charts

$$\tilde{U}_1/G \xrightarrow{\phi_1} U_1 \subseteq X$$
and
$$\tilde{U}_2/G \xrightarrow{\phi_2} U_2 \subseteq X,$$

we say that these charts are *compatible* if every point of $U_1 \cap U_2$ is contained in an open set $U_3$ for which

$$\tilde{U}_3/G \xrightarrow{\phi_3} U_3 \subseteq X$$

is a subchart of both of the two original charts.

Having defined compatibility of charts, we can finally define:

**Definition 1.3.** A (smooth) orbifold $X$ is a topological space $X$ together with a family of compatible charts covering $X$.

The key point is that, although $U$ is homeomorphic to $\tilde{U}/G$, the orbifold chart remembers more than just the topological quotient $\tilde{U}/G$: it remembers the action of $G$, and specifically, where that action has isotropy. In particular, the definition of compatibility of orbifold charts implies that each point $x \in X$ has an isotropy group that is well-defined up to isomorphism:

**Definition 1.4.** Let $X$ be an orbifold and let $x \in X$. The *isotropy group* (or *stabilizer*) of $X$ is the stabilizer $G_{\tilde{x}}$ of any point $\tilde{x} \in \tilde{U}$ such that $\tilde{\phi}(\tilde{x}) = x$, where $\phi : \tilde{U}/G \to U$ is any orbifold chart. (Choosing a different $\tilde{x}$ in the same orbifold chart yields an isomorphic stabilizer because $\tilde{\phi}$ is $G$-invariant, while choosing a different orbifold chart yields an isomorphic stabilizer by condition (3) of Definition 1.2.)

Let us turn to some of the key examples of orbifolds.
Example 1.5. Let the cyclic group $\mathbb{Z}_n$ act on $\mathbb{C}$ via multiplication by $n$th roots of unity. Then there is an orbifold, denoted
\[ \mathcal{X} = [\mathbb{C}/\mathbb{Z}_n], \]
in which the underlying topological space is $X = \mathbb{C}$ and there is a single global chart
\[ \tilde{\phi} : \mathbb{C} \to \mathbb{C} \]
\[ \tilde{\phi}(z) = z^n. \]
In particular, $\tilde{\phi}$ descends to a continuous map $\phi : \mathbb{C}/\mathbb{Z}_n \to \mathbb{C}$ whose inverse $z \mapsto [z^{1/n}]$ is indeed well-defined and continuous.

The isotropy group of any nonzero element $x \in \mathbb{C}$ is trivial, in this example, whereas the isotropy group of $0 \in \mathbb{C}$ is $\mathbb{Z}_n$. As a result, one typically represents the orbifold $[\mathbb{C}/\mathbb{Z}_n]$ graphically as the space $\mathbb{C}$ with a copy of the group $\mathbb{Z}_n$ attached to the origin.

Example 1.6. Generalizing the above example, let $M$ be any smooth manifold and let $G$ be a finite group acting smoothly on $M$. Then there is a "global quotient" orbifold
\[ \mathcal{X} = [M/G] \]
with underlying topological space $X = M/G$. To construct $\mathcal{X}$ as an orbifold, one must use the fact that any $x \in M$ for which the $G$-action has isotropy group $G_x$ is contained in a (manifold) chart $U_x \cong \tilde{U}_x \subseteq \mathbb{R}^n$ for which $U_x$ is invariant under the action of $G_x$. The orbifold charts then consist of the subsets $U_x/G_x \subseteq M/G$ with the quotient maps $\tilde{\phi} : \tilde{U}_x \to U_x/G_x$ and the action of $G_x$ on $\tilde{U}_x$ induced by the homeomorphism $\tilde{U}_x \cong U_x$.

A special case of the above, which appears initially trivial but is surprisingly rich, is the following:

Example 1.7. If $M$ is a single point and $G$ is any finite group (acting in the only possible way on a point), we denote the resulting global quotient orbifold by $BG$.

The following example, on the other hand, is provably not (always) a global quotient:

Example 1.8. Let $\mathbb{C}^*$ act on $\mathbb{C}^{n+1}$ by
\[ \lambda \cdot (z_0, \ldots, z_n) = (\lambda^{w_0} z_0, \ldots, \lambda^{w_n} z_n), \]
where $(\lambda_0, \ldots, \lambda_n)$ are coprime natural numbers. Then there is an orbifold
\[ \mathcal{X} = \mathbb{P}(w_0, \ldots, w_n), \]
referred to as a weighted projective space, in which $X$ is the topological quotient
\[ X = (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*. \]
To form the orbifold charts, first note that, as in the case of ordinary projective space, each $x \in X$ is contained in one of the subsets
\[ U_i = \{ [z_0, \ldots, z_n] \in (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^* \mid z_i \neq 0 \} \subseteq X. \]
Then $\tilde{U}_i \cong \mathbb{C}^n$, on which the coordinates are denoted $(z_0, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n)$ and the map $\tilde{\phi}_i : \tilde{U}_i \rightarrow U_i$ is

$$\tilde{\phi}_i(z_0, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n) = [z_0 : \cdots : z_{i-1} : 1 : z_{i+1} : \cdots : z_n].$$

The group $\mathbb{Z}_{w_i}$, $w_i$ th roots of unity acts on $\tilde{U}_i$ by

$$\zeta \cdot (z_0, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n) = (\zeta^{w_i}z_0, \ldots, \zeta^{w_i-1}z_{i-1}, \zeta^{w_i+1}z_{i+1}, \ldots, \zeta^{w_n}z_n),$$

and it is straightforward to check that this action makes the above into an orbifold chart.

**Exercise 1.9.** Consider the action of $\mathbb{Z}_2$ on the torus $\mathbb{T} = S^1 \times S^1 \subseteq \mathbb{C}^* \times \mathbb{C}^*$, where the nontrivial element of $\mathbb{Z}_2$ acts by

$$(e^{it_1}, e^{it_2}) \mapsto (e^{-it_1}, e^{-it_2}).$$

Find an explicit atlas of orbifold charts on $X = \mathbb{T}/\mathbb{Z}_2$. What, topologically, is $X$?

**Exercise 1.10.** Consider the action of the symmetric group $S_3$ on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ given by permuting the three coordinates; the resulting quotient is the *symmetric product* $X = S^3(\mathbb{P}^1)$. At which points does the action of $S_3$ have nontrivial isotropy, and what is the isotropy group? Find an explicit atlas of orbifold charts on $X$, and convince yourself that, topologically, $X \cong \mathbb{P}^3$.

### 1.2. Orbifold vector bundles and a first pass at orbifold cohomology.

In general, all of the geometric constructions one might associate to a manifold can be extended appropriately to orbifolds. The philosophy, when defining the orbifold analogue of a manifold construction, is to apply the manifold construction on the charts $\tilde{U}$ and insist that it be equivariant with respect to the actions of the local groups $G$.

This idea is easiest to make precise in the case when the orbifold is a global quotient. For example, in that case, the orbifold analogue of a vector bundle can be defined as follows:

**Definition 1.11.** Let $\mathcal{X} = [M/G]$ be a global quotient orbifold. Then an *orbifold vector bundle* on $\mathcal{X}$ consists of a vector bundle $\pi : E \rightarrow M$ together with an action of $G$ on $E$ that takes the fiber over each $x \in M$ linearly to the fiber over $gx$. A *section* of an orbifold vector bundle on $\mathcal{X}$ is a section $s$ of $\pi$ that is equivariant with respect to the $G$-actions on $E$ and $M$.

By carrying out the above construction in orbifold charts, this definition can be extended to arbitrary orbifolds; we refer the reader to [1] for a careful definition.

Orbifolds also have their own version of the tangent bundle. In the global quotient case, this is:

**Definition 1.12.** Let $\mathcal{X} = [M/G]$ be a global quotient orbifold. Then the *tangent bundle* to $\mathcal{X}$, denoted $T\mathcal{X}$, is the orbifold vector bundle defined by the ordinary tangent bundle $\pi : TM \rightarrow M$, on which $G$ acts by the derivative of its action on $M$.

The typical operations on vector bundles extend to the orbifold setting, so, having defined the tangent bundle $T\mathcal{X}$, one can define the cotangent bundle, and its wedge powers $\wedge^k T^*\mathcal{X}$, and sections of $\wedge^k T^*\mathcal{X}$—that is, differential $k$-forms. This leads to a definition of orbifold de Rham cohomology $H^*_\text{dR}(\mathcal{X})$. 
Unfortunately, though, this definition of orbifold de Rham cohomology captures none of the information of the orbifold structure. More precisely, one has the following theorem:

**Theorem 1.13 (Satake).** For any orbifold $X$ with underlying topological space $X$, there is an isomorphism

\[ H_{dR}^*(X) \cong H^*(X; \mathbb{R}). \]

So, for example, the orbifold de Rham cohomology of $BG$ is simply the cohomology of a point; it sees nothing of the group $G$.

The way out of this sad state of affairs was proposed by Chen and Ruan [4]: one should first define orbifold quantum cohomology, and then restrict to its degree-zero part to obtain a definition of orbifold cohomology that truly takes the orbifold structure into account.

Before discussing Chen–Ruan’s definition of orbifold cohomology, we digress to recall the basics of quantum cohomology in the non-orbifold setting.


Let $X$ be a smooth projective variety, and for simplicity, fix an embedding $i : X \hookrightarrow \mathbb{P}^r$. For any integers $n, \beta \geq 0$, the moduli space of genus-zero stable maps to $X$ is the parameter space $\overline{M}_{0,n}(X, \beta)$ of tuples $(C; q_1, \ldots, q_n; f)$, where:

- $C$ is an algebraic curve of (arithmetic) genus zero with at worst nodal singularities;
- $q_1, \ldots, q_n \in C$ are a collection of distinct smooth points of $C$;
- $f : C \to X$ is a morphism for which $i_* f_* [C] = \beta \in H_2(\mathbb{P}^r; \mathbb{Z}) \cong \mathbb{Z}$;
- the data $(C; q_1, \ldots, q_n; f)$ is stable in the sense that it has finitely many automorphisms, or equivalently, any irreducible component of $C$ on which $f$ is constant has at least three “special points,” which are either marked points $q_i$ or nodes.

This moduli space is equipped with evaluation maps

\[ ev_i : \overline{M}_{0,n}(X, \beta) \to X \]

\[ ev_i(C; q_1, \ldots, q_n; f) = f(q_i). \]

From here, the idea of Gromov–Witten invariants is to choose $n$ cohomology classes $\alpha_1, \ldots, \alpha_n \in H^*(X)$ and study

\[ \int_{\overline{M}_{0,n}(X, \beta)} ev_1^*(\alpha_1) \cdots ev_n^*(\alpha_n). \] (1.1)

In particular, if $\alpha_1, \ldots, \alpha_n$ were Poincaré dual to subvarieties $Y_1, \ldots, Y_n$ of $X$, then (1.1) would, naively, capture the number of points in $ev_1^{-1}(Y_1) \cap \cdots ev_n^{-1}(Y_n)$—in other words, the number of degree-$\beta$, genus-zero curves passing through these $n$ subvarieties.

However, $\overline{M}_{0,n}(X, \beta)$ can be a very singular space, and it can have different components of different dimensions, so it is in general unclear how to make sense of the integral (1.1) in a meaningful way. It is a deep and difficult foundational fact that, despite its singularity, $\overline{M}_{0,n}(X, \beta)$ admits a pure-dimensional homology class $[\overline{M}_{0,n}(X, \beta)]^{vir}$ that enjoys some of the properties that a fundamental class on a smooth space would have. Integrating against this virtual fundamental class gives the precise definition of Gromov–Witten invariants:
\textbf{Definition 1.14.} Let $n, \beta \in \mathbb{Z}_{\geq 0}$, and let $\alpha_1, \ldots, \alpha_n \in H^*(X)$. Then the associated genus-zero Gromov–Witten invariant is

$$\langle \alpha_1 \cdots \alpha_n \rangle_{0, n, \beta}^X := \int_{[\overline{M}_{0, n}(X, \beta)]^{vir}} \ev_1^*(\alpha_1) \cdots \ev_n^*(\alpha_n).$$

We are now prepared to define quantum cohomology in the variety setting:

\textbf{Definition 1.15.} Let $q$ denote a formal parameter. The quantum cohomology of $X$ is the vector space $H^*(X)[[q]]$ equipped with the product $*$ defined by

$$\left(\alpha \ast \beta, \gamma\right) := \sum_{\beta \geq 0} q^\beta \langle \alpha \beta \gamma \rangle_{0, 3, \beta}^X,$$

where $\langle , \rangle$ is the Poincaré pairing

$$(\omega, \nu) := \int_X \omega \cup \nu.$$

The fundamental feature of this definition is that its degree-zero part recovers the usual cohomology:

\textbf{Exercise 1.16.} Verify that setting $q = 0$ in the quantum product recovers $H^*(X)$ with its usual cup product.

\section*{1.4. Orbifold cohomology.} We would like to generalize the definition of Gromov–Witten invariants and quantum cohomology to orbifolds, but in order to do so carefully, one must define what is meant by a morphism of orbifolds. This definition is remarkably difficult to make precise; it can be defined via equivariant morphisms in local orbifold charts, but the atlas of orbifold charts on the domain orbifold may need to be refined before doing so.

Rather than covering the careful definition of an orbifold morphism, we will content ourselves with one key example and one key fact:

\textbf{Example 1.17.} Let $\mathcal{X} = [M/G]$ and $\mathcal{Y} = [N/H]$ be global quotient orbifolds. Then one example of an orbifold morphism is a pair $(f_0, f_1)$ consisting of a continuous map $f_0 : M \to N$ and a homomorphism $f_1 : G \to H$ such that $f_0$ is equivariant with respect to $f_1$.

\textbf{Fact 1.18.} Let $\mathcal{X}$ and $\mathcal{Y}$ be any orbifolds, and let $f : \mathcal{X} \to \mathcal{Y}$ be an orbifold morphism. Then $f$ induces a map $\overline{f} : X \to Y$ between the underlying topological spaces of $\mathcal{X}$ and $\mathcal{Y}$, respectively, as well as a homomorphism of isotropy groups $\lambda_x : G_x \to G_{\overline{f}(x)}$ for each $x \in X$.

\textbf{Exercise 1.19.} Verify that, in the special case of Example 1.17, one indeed obtains maps $\overline{f}$ and $\lambda_x$ as in Fact 1.18.

The key upshot of Fact 1.18 is the following. Equipped with the notion of an orbifold morphism, one can define a moduli space $\overline{M}_{0, n}(\mathcal{X}, \beta)$ of stable maps $f : C \to \mathcal{X}$ from genus-zero orbifold curves $C$ to $\mathcal{X}$. In the orbifold setting, though, there are two pieces of local data around each marked point $q_i$:
• the image $\overline{f}(q_i) \in \mathcal{X}$, and
• the homomorphism $\lambda_{q_i}$.

Moreover, the isotropy groups of orbifold curves are in fact cyclic, and they have a canonical generator $\zeta_i$ induced by the orientation. Thus, the data of $\lambda_{q_i}$ is encoded in a single element $\lambda_{q_i}(\zeta_i)$ of the isotropy group of $\mathcal{X}$ at $f(q_i)$.

As a result of these considerations, the natural evaluation maps on $\overline{\mathcal{M}}_{0,n}(\mathcal{X}, \beta)$ land not in $\mathcal{X}$ itself but in a richer orbifold:

**Definition 1.20.** The *inertia orbifold* of $\mathcal{X}$ is

$$I\mathcal{X} := \{(x,g) \mid x \in \mathcal{X}, g \in G_x\}.$$  

As its name suggests, the inertia orbifold can be given the structure of an orbifold; the description in Definition 1.20 is merely the underlying topological space. We illustrate the orbifold structure only in a single example, which is the most important example for what follows. The reader is referred to [1] for a more careful general definition.

**Example 1.21.** Let $\mathcal{X} = [\mathbb{C}/\mathbb{Z}_r]$. Then $X = \mathbb{C}$, and any nonzero $x \in \mathbb{C}$ has trivial isotropy whereas the isotropy group of $0 \in \mathbb{C}$ is $\mathbb{Z}_r$. It follows that, on the level of the underlying space,

$$(1.2) \quad I[\mathbb{C}/\mathbb{Z}_r] = \mathbb{C} \sqcup \{0\} \sqcup \cdots \sqcup \{0\}.$$ 

To give this the structure of an orbifold, let $\mathbb{Z}_r$ act on the above by sending

$$(x,g \in G_x) \mapsto (hx, hgh^{-1} \in G_{hx})$$

for each $h \in \mathbb{Z}_r$. Because $\mathbb{Z}_r$ is abelian, this action amounts to sending $(x,g)$ to $(hx,g)$, so it preserves the $r$ components in (1.2). As an orbifold, then,

$$I[\mathbb{C}/\mathbb{Z}_r] = [\mathbb{C}/\mathbb{Z}_r] \sqcup B\mathbb{Z}_r \sqcup \cdots \sqcup B\mathbb{Z}_r.$$ 

Having defined the inertia orbifold, the evaluation maps are

$$\text{ev}_i : \overline{\mathcal{M}}_{0,n}(\mathcal{X}, \beta) \to I\mathcal{X}$$

$$\text{ev}_i(C; q_1, \ldots, q_n; f) = (\overline{f}(q_i), \lambda_{q_i}(\zeta_i)).$$

From here, the definition of Gromov–Witten invariants carries over verbatim, but the insertions $\alpha_1, \ldots, \alpha_n$ to these invariants are chosen from $H^*(I\mathcal{X})$ as opposed to $H^*(\mathcal{X})$. In particular, quantum cohomology in the orbifold setting is a product structure on $H^*(I\mathcal{X})[[q]]$.

In light of Exercise 1.16, one should expect the “ordinary” cohomology of orbifolds to be obtained from their quantum cohomology by setting $q = 0$. Rather than proving this as a theorem, one takes this result as the definition of orbifold cohomology:

**Definition 1.22.** Let $\mathcal{X}$ be an orbifold. Then the *orbifold (or Chen–Ruan) cohomology* of $\mathcal{X}$ is the vector space

$$H^*_\text{CR}(\mathcal{X}) := H^*(I\mathcal{X}),$$

where the right-hand side denotes the de Rham (or singular) cohomology of $I\mathcal{X}$, which is the same as the cohomology of its underlying topological space. The product structure on $H^*_\text{CR}(\mathcal{X})$ is the $q = 0$ limit of the quantum product on $H^*(I\mathcal{X})[[q]]$.

**Exercise 1.23.** What is $H^*_\text{CR}([\mathbb{C}/\mathbb{Z}_r])$, as a vector space?
Exercise 1.24. Consider a weighted projective space $X = \mathbb{P}(w_0, w_1)$ with coprime weights $w_0$ and $w_1$. Calculate the inertia orbifold and the orbifold cohomology (as a vector space) of $X$. Although we have not explicitly defined weighted projective spaces with non-coprime weights, what do you expect are the inertia orbifold and the orbifold cohomology of $\mathbb{P}(3, 3)$? Of $\mathbb{P}(3, 6)$?

2. Fan–Jarvis–Ruan–Witten theory

Equipped with the background developed in the previous section, we are prepared to define one of our primary objects of study in these lectures: Fan–Jarvis–Ruan–Witten (FJRW) theory.

2.1. The structure of FJRW theory. Structurally, FJRW theory is analogous to the Gromov–Witten theory of a smooth projective variety $X$. In particular, it consists of:

1. a vector space $\mathcal{H}$ (the “state space”), analogous to $H^*(X)$ in Gromov–Witten theory;
2. a moduli space $\overline{M}$ equipped with a virtual fundamental cycle, analogous to $\overline{M}_{g,n}(X, \beta)$ in Gromov–Witten theory;
3. a notion of “correlators”, analogous to Gromov–Witten invariants, which are integrals against the virtual cycle of $\overline{M}$ associated to any tuple of elements in $\mathcal{H}$.

The input data for FJRW theory, though, is not a variety $X$ but a polynomial $W \in \mathbb{C}[x_1, \ldots, x_N]$. This polynomial is required to be of a particular type:

Definition 2.1. A polynomial $W \in \mathbb{C}[x_1, \ldots, x_N]$ is quasihomogeneous if there exist positive integers $w_1, \ldots, w_N$ (the weights) and $d$ (the degree) such that

$$W(\lambda^{w_1} x_1, \ldots, \lambda^{w_N} x_N) = \lambda^d W(x_1, \ldots, x_N)$$

for all $\lambda \in \mathbb{C}$.

Note, in particular, that quasihomogeneity implies that the vanishing of $W$ defines a hypersurface $X_W$ in the weighted projective space $\mathbb{P}(w_1, \ldots, w_N)$.

In addition to quasihomogeneity, in what follows, we require our polynomials $W$ to satisfy two further technical conditions:

1. **Nondegeneracy**: The hypersurface $X_W \subseteq \mathbb{P}(w_1, \ldots, w_N)$ is nonsingular as an orbifold;
2. **Invertibility**: The number of monomials of $W$ is equal to the number of variables, and the exponent matrix (whose rows correspond to monomials in $W$ and whose entries record the exponent of each variable in a monomial) is invertible.

A particularly simple example of a nondegenerate, invertible, quasihomogeneous polynomial—to which we will return repeatedly in what follows—is:

Example 2.2. Let $W(x_1, \ldots, x_5) = x_1^5 + \cdots + x_5^5$. This is a quasihomogeneous polynomial with $w_1 = \cdots = w_5 = 1$ and $d = 5$.

In the following subsections, we will define the ingredients of the FJRW theory of $W$ precisely. Before doing so, though, it is worth grounding the discussion by looking ahead to the ultimate goal: in Section 3, we present the Landau–Ginzburg/Calabi–Yau correspondence, which asserts an equivalence between the
FJRW theory of $W$ and the Gromov–Witten theory of $X_W$ in certain cases. We encourage the reader to keep this correspondence in mind as motivation for the definitions that follow.

2.2. The state space of FJRW theory. Let $W \in \mathbb{C}[x_1, \ldots, x_N]$ be a (non-degenerate, invertible) quasihomogeneous polynomial with weights $w_1, \ldots, w_N$ and degree $d$. Then we define:

**Definition 2.3.** The FJRW state space associated to $W$ is the $\mathbb{Q}$-vector space

$$
\mathcal{H}_W := H^*_{CR}(\mathbb{C}^N/J, W^{+\infty}; \mathbb{Q}),
$$

where

$$J := \left\langle \left( e^{2\pi i w_1/d}, \ldots, e^{2\pi i w_N/d} \right) \right\rangle \subseteq (\mathbb{C}^*)^N$$

acts diagonally on $\mathbb{C}^N$ and

$$W^{+\infty} := W^{-1}(\rho)$$

for any sufficiently large real number $\rho$. (The resulting vector space is independent of $\rho$ for $\rho \gg 0$, because topologically, $W^{-1}(\rho)$ is eventually independent of $\rho$.)

We note that the action of $J$ on $\mathbb{C}^N$ is cooked up so that $W(g \vec{x}) = W(\vec{x})$ for all $g \in J$, so $W$ gives a well-defined map out of the quotient $\mathbb{C}^N/J$. Furthermore, recalling that $H^*_C(X) := H^*(I_X)$ and $I_X = \{(x, g \in G_x)\}$, the FJRW state space can more explicitly be expressed as

$$
\mathcal{H}_W = \bigoplus_{g \in J} H^*(\mathbb{C}^N_g/J, W^{+\infty}; \mathbb{Q}),
$$

where $\mathbb{C}^N_g \subseteq \mathbb{C}^N$ is the fixed locus of $g$. The cohomology groups appearing in the above are ordinary (for example, de Rham) cohomology groups, for which the cohomology of a quotient $Y/G$ can be calculated as the $G$-invariant part of the cohomology of $Y$. Thus:

$$
\mathcal{H}_W = \bigoplus_{g \in J} H^*(\mathbb{C}^N_g, W^{+\infty}; \mathbb{Q}),
$$

in which the superscript denotes the $J$-invariant part.

**Example 2.4.** Let $W(x_1, \ldots, x_5)x_1^5 + \cdots + x_5^5$. Then

$$J = \left\langle \left( e^{2\pi i \frac{1}{5}}, \ldots, e^{2\pi i \frac{5}{5}} \right) \right\rangle \cong \mathbb{Z}_5,$$

acting diagonally on $\mathbb{C}^5$ by multiplication by fifth roots of unity. Given that the action of 1 $\in J$ fixes all of $\mathbb{C}^5$ but the action of any nontrivial $g \in J$ fixes only the origin, we have

$$I[\mathbb{C}^5/J] = [\mathbb{C}^5/\mathbb{Z}_5] \sqcup B \mathbb{Z}_5 \sqcup B \mathbb{Z}_5 \sqcup B \mathbb{Z}_5 \sqcup B \mathbb{Z}_5.$$

It follows that

$$\mathcal{H}_W = H^*(\mathbb{C}^5, W^{+\infty}; \mathbb{Q})^\mathbb{Z}_5 \oplus \bigoplus_{g \neq 1 \in \mathbb{Z}_5} H^*(\{0\}, \emptyset; \mathbb{Q}) = H^*(\mathbb{C}^5, W^{+\infty}; \mathbb{Q})^\mathbb{Z}_5 \oplus \mathbb{Q}^4.$$

The state space decomposes into two types of elements:

**Definition 2.5.** An element $g \in J$ (or the corresponding component of $\mathcal{H}_W$) is called narrow if it fixes only $0 \in \mathbb{C}^N$, meaning that the corresponding component of $\mathcal{H}_W$ is a copy of $\mathbb{Q}$. Elements (or components) that are not narrow are called broad.
For example, in Example 2.4, the element \( g = 1 \) is broad and all other \( g \in J \) are narrow.

**Exercise 2.6.** Find an example of a quasihomogeneous polynomial for which there exists a nontrivial broad element \( g \in J \).

### 2.3. The moduli space of FJRW theory.

Let \( W \) be as in the previous subsection. Then the moduli space of FJRW theory is the parameter space \( \mathcal{M}_{g,n}^W \) of tuples \( (C; q_1, \ldots, q_n; L; \phi) \), where

- \((C; q_1, \ldots, q_n)\) is a genus-\( g \), \( n \)-pointed orbifold stable curve;
- \( L \) is an orbifold line bundle on \( C \);
- \( \phi \) is an isomorphism

\[
\phi : L^\otimes d \xrightarrow{\sim} \omega_{\log},
\]

where \( \omega_{\log} := \omega_C([q_1] + \cdots + [q_n]) \), subject to a certain stability condition on the actions of the isotropy groups on the fibers of \( L \). The term “orbifold stable curve” here has a precise meaning that we will not specify; in particular, though, it means that \( C \) has nontrivial isotropy only at special points.

**Example 2.7.** When \( W(x_1, \ldots, x_5) = x_5^5 + \cdots + x_5 \), the moduli space \( \mathcal{M}_{g,n}^W \) is otherwise known as the moduli space of 5-spin structures and is often denoted \( \mathcal{M}_{g,n}^{1/5} \).

### 2.4. The correlators of FJRW theory.

To define the correlators of FJRW theory, we must refine the definition of the moduli space by decomposing it into components depending on the “multiplicities” of the bundle \( L \).

**Definition 2.8.** Let \( C \) be an orbifold curve, let \( L \) be an orbifold line bundle on \( C \), and let \( q \in C \) be a point with isotropy group \( G_q = \mathbb{Z}_r \). Then, locally near \( q \), the curve \( C \) looks like the global quotient \([C/\mathbb{Z}_r]\), so the data of \( L \) consists of a bundle \( \tilde{L} \) on \( C \) together with an action of \( \mathbb{Z}_r \) according to which the projection map \( \tilde{L} \to C \) is equivariant. In particular, \( \mathbb{Z}_r \) acts on the fiber of \( \tilde{L} \) over \( q \). The multiplicity of \( L \) at \( q \) is defined as the number

\[
\text{mult}_q(L) := m_q \in \{0, 1, \ldots, r-1\}
\]

such that the action of \( \mathbb{Z}_r \) on \( \tilde{L} \) is

\[
\zeta \cdot (x, v) = \left(e^{2\pi i \frac{1}{r} x}, e^{2\pi i \frac{m_q}{r}} v\right)
\]
in local coordinates around \( q \).

In the case of \( \mathcal{M}_{g,n}^W \), the stability condition ensures that all of the isotropy groups have order dividing \( d \). Thus, one can view the multiplicities of \( L \) at any special point as an element of \( \{0, 1, \ldots, d-1\} \). For a tuple

\[
m_1, \ldots, m_n \in \{0, 1, \ldots, d-1\},
\]

we set

\[
\mathcal{M}_{g,(m_1, \ldots, m_n)}^W := \{(C; q_1, \ldots, q_n; L; \phi) \mid \text{mult}_{q_i}(L) = m_i \text{ for all } i\} \subseteq \mathcal{M}_{g,n}^W.
\]
The key observation needed to define FJRW correlators, at least in the narrow case, is that for any $g \in J,$
\[ g = \left( e^{2\pi i m_1}, \ldots, e^{2\pi i m_N} \right) \]
for some $m \in \{0, 1, \ldots, d - 1\}.$ In particular, the components of the FJRW state space can be indexed by the same set $\{0, 1, \ldots, d - 1\}$ that indexes the components of the FJRW moduli space.

More specifically, we define narrow correlators as follows:

**Definition 2.9.** Decompose the FJRW state space as
\[ H_W = \bigoplus_{g \text{ broad}} H^*(\mathbb{C}_g^N, W^{+\infty}; \mathbb{Q})^J \bigoplus \bigoplus_{g \text{ narrow}} \mathbb{Q}\{e_g\}, \]
where $e_g$ denotes the fundamental class on the component of $H_W$ indexed by $g.$

Choose $n$ narrow elements $\alpha_1, \ldots, \alpha_n \in H_W,$

and write $\alpha_i = c_i \cdot e_{g_i}$ for $c_i \in \mathbb{Q}$ and
\[ g_i = \left( e^{2\pi i m_1}, \ldots, e^{2\pi i m_N} \right), \]
where $m_i \in \{0, 1, \ldots, d - 1\}.$ Then we define the associated FJRW correlator as
\[ \langle \alpha_1 \cdots \alpha_N \rangle_{g,n} = c_1 \cdots c_n \int_{[\overline{\mathcal{M}}_{g,(m_1,\ldots,m_n)}]^\text{vir}} 1. \]

Two difficult fundamental facts are needed in order to make this definition complete. First, there indeed exists a virtual cycle $[\overline{\mathcal{M}}_{g,(m_1,\ldots,m_n)}]^\text{vir}$ against which the above integral can be defined; we will say some more words about how this cycle is constructed in later sections of these notes. And second, the definition of correlators can be extended to broad insertions. This second fact is harder, and we will not address it here; for the cases of interest below, narrow correlators are sufficient.

We conclude this section with several exercises to practice with the notion of multiplicity:

**Exercise 2.10.** Let $(C; q_1, \ldots, q_n)$ be a smooth orbifold curve at which each marked point $q_i$ has isotropy group $\mathbb{Z}_d,$ and let $L$ be an orbifold line bundle on $C.$

A fundamental fact about the multiplicities of $L$ is that the bundle
\[ |L| := L \otimes \mathcal{O} \left( - \sum_{i=1}^{n} \frac{m_i}{d} [q_i] \right) \]
is pulled back from the coarse underlying curve $C.$ Use this fact to explain why the multiplicities must satisfy
\[ \sum_{i=1}^{n} m_i \equiv 0 \mod d. \tag{2.1} \]

**Exercise 2.11.** Generalizing to nodal curves of compact type, convince yourself that the multiplicity of $L$ at each branch of each node is determined by (2.1) together with the "kissing condition" that $m' + m'' \equiv 0 \mod d$ for the multiplicities $m', m''$ at opposite branches of the same node. What happens if the curve is not of compact type—in other words, if it has a non-separating node?
Exercise 2.12. Work out the “fundamental fact” from Exercise 2.10 explicitly in local coordinates, by considering the diagram

\[
L = [(\mathbb{C} \times \mathbb{C})/\mathbb{Z}_d] \twoheadrightarrow [(\mathbb{C} \times \mathbb{C})/\mathbb{Z}_d] = |L|
\]

where \( L \) is defined by the action \( \zeta(x, v) = (\zeta x, \zeta^m v) \) and \( |L| \) is defined by the action \( \zeta(x, v) = (\zeta x, v) \) of \( \mathbb{Z}_d \) on \( \mathbb{C} \times \mathbb{C} \). (See [19, Section 2.1.4] for related discussion.)

3. The Landau–Ginzburg/Calabi–Yau correspondence

Two questions would be reasonable for the reader to ask at this point: where do all of these FJRW theory definitions come from, and what does all of this have to do with the Gromov–Witten theory of the hypersurface defined by \( W \)?

The answers to both of these questions come from putting FJRW theory in the context of the Landau–Ginzburg/Calabi–Yau (LG/CY) correspondence, which we describe in this section. In order to tell that story in a more readable form, we restrict our attention to the case where

\[
W = W(x_1, \ldots, x_5) = x_1^5 + \cdots + x_5^5
\]

whose vanishing locus is the quintic threefold \( Q \subseteq \mathbb{P}^r \). We describe, in this case, an isomorphism between the state spaces \( H^*(Q) \) and \( H_W \) of Gromov–Witten and FJRW theory, respectively, and a “matching” between the generating functions of genus-zero Gromov–Witten and FJRW correlators under a particular identification of the state spaces.

3.1. The state space correspondence. The first step in the LG/CY correspondence is to explain why the state spaces \( H^*(Q) \) and \( H_W \) are isomorphic. To do so, we introduce a larger space out of which both of these can be built.

Consider the space \( \mathbb{C}^5 \times \mathbb{C} \) with coordinates \((x_1, \ldots, x_5, p)\), and let \( \mathbb{C}^* \) act on this space by

\[
\lambda(x_1, \ldots, x_5, p) = (\lambda x_1, \ldots, \lambda x_5, \lambda^{-5} p).
\]

Then the polynomial

\[
\overline{W}(x_1, \ldots, x_5, p) = p(x_1^5 + \cdots + x_5^5)
\]

is \( \mathbb{C}^* \)-invariant, so it gives a well-defined map out of the quotient \((\mathbb{C}^5 \times \mathbb{C})/\mathbb{C}^*\).

This quotient is not separated, but it admits two maximal separated subquotients; in the language of geometric invariant theory, these are the GIT quotients \((\mathbb{C}^5 \times \mathbb{C})// \theta \mathbb{C}^* \) associated to the positive and negative characters \( \theta \in \chi(\mathbb{C}^*) \cong \mathbb{Z} \). Namely, they are:

\[
X_+ := \frac{(\mathbb{C}^5 \setminus \{0\}) \times \mathbb{C}}{\mathbb{C}^*} = \mathcal{O}_{\mathbb{C}^*}(-5)
\]

and

\[
X_- := \frac{\mathbb{C}^5 \times (\mathbb{C} \setminus \{0\})}{\mathbb{C}^*} = [\mathbb{C}^5/\mathbb{Z}_5].
\]
If one calculates the orbifold cohomology of $X_-$ relative to a fiber $W^+\infty$ of $W : X_- \to C$, the result, by definition, is the FJRW state space $H_W$. On the other hand, the (orbifold) cohomology of $X_+$ relative to a fiber $W^+\infty$ of $W : X_+ \to C$ is

$$H^*(O_{P^4}(-5), W^+\infty) \cong H^*(P^4, P^4 \setminus Q) \cong H^*(Q),$$

where the first isomorphism follows from the deformation retraction of $O_{P^4}(-5)$ onto $P^4$, which carries $W^+\infty$ bijectively onto $P^4 \setminus Q$ and the second isomorphism is the Thom isomorphism.

Via this argument, the state spaces of Gromov–Witten and FJRW theory can be viewed as arising in completely analogous ways from the larger quotient $(C^5 \times C)/C^*$. In fact, this framework can be leveraged—with the help of a number of exact sequences—to prove that the two state spaces are isomorphic:

$$H^*(Q) \cong H^W.$$

This is the first piece of the LG/CY correspondence.

Exercise 3.1. Show, furthermore, that there is an isomorphism between the ambient part of the cohomology of $H^*(Q)$—that is, the cohomology classes pulled back from $P^4$—and the narrow part of $H_W$.

3.2. The moduli spaces of Gromov–Witten and FJRW theory. The second, more substantial, piece of the correspondence is a comparison of correlators, and in particular, of the moduli spaces and their virtual cycles.

The basic structure of this comparison is modeled on the argument for the state space correspondence explained above; namely, just as the state space correspondence arose by placing both $H^*(Q)$ and $H_W$ in the context of a larger but badly-behaved quotient $(C^5 \times C)/C^*$, the moduli comparison begins by finding a larger but badly-behaved moduli space out of which $M_{g,n}(Q, \beta)$ and $M_W^{g,n}$ can both be built.

That larger moduli space, denoted $X_{g,n,\beta}$, parameterizes tuples

$$(C; q_1, \ldots, q_n; L, x_1, \ldots, x_5, p),$$

in which $(C; q_1, \ldots, q_n)$ is again a genus-$g$, $n$-pointed orbifold stable curve, $L$ an orbifold line bundle of degree $\beta$ on $C$, and

$$(x_1, \ldots, x_5, p) \in \Gamma \left( L^{\oplus 5} \oplus (L^{\oplus -5} \otimes \omega_{\log}) \right),$$

subject to certain stability and compatibility conditions that we will not make explicit here.

Just as in the case of the quotient $(C^5 \times C)/C^*$, this moduli space is non-separated; furthermore, it is non-compact (as the sections can be scaled by any complex number) and not Deligne–Mumford (as scaling the line bundle leads to infinitely many automorphisms). Nevertheless, it contains two separated Deligne–Mumford substacks, $X^+_{g,n,\beta}$ and $X^-_{g,n,\beta}$, by imposing that either $\vec{x}$ or $p$ is nowhere-vanishing.

When $\vec{x}$ is nowhere-vanishing, it is equivalent to the data of a map $f : C \to P^4$, and the stability condition on $X_{g,n,\beta}$ implies that this map is stable. Furthermore, from this perspective, we have $L = f^* O(1)$, so

$$X^+_{g,n,\beta} = \{(C; q_1, \ldots, q_n; f; p) \mid f : C \to P^4 \text{ stable}, p \in \Gamma(f^* O(-5) \otimes \omega_{\log})\}.$$
On the other hand, when \( p \) is nowhere-vanishing, it trivializes \( L^{\otimes -5} \otimes \omega_{\text{log}} \), so

\[
X_{g,n,\beta}^- = \{(C; q_1, \ldots, q_n; L; x_1, \ldots, x_5; \varphi) \mid x_1 \in \Gamma(L), \varphi : L^{\otimes -5} \cong \omega_{\text{log}} \}.
\]

These two moduli spaces, while separated and Deligne–Mumford, are still not compact, since each still includes an unconstrained section of a bundle. Analogously to the way in which we passed, in the state space correspondence, to cohomology relative to the fiber \( W^+ \), we restrict on the moduli level to the loci \( Z_{g,n,\beta}^\pm \subseteq X_{g,n,\beta}^\pm \) on which the section \( (x_1, \ldots, x_5, p) \) lands in the critical locus

\[
\text{Crit}(W) \subseteq \mathbb{C}^5 \times \mathbb{C}.
\]

More specifically:

**Exercise 3.2.** Viewing \( W \) as a function on \( X_+ = \mathcal{O}_{\mathbb{P}^4}(-5) \), verify that the critical locus of \( W \) is the quintic threefold \( Q \) inside the zero section \( \mathbb{P}^4 \subseteq X_+ \). Similarly, viewing \( W \) as a function on \( X_- = \mathbb{C}^5/\mathbb{Z}_5 \), verify that the critical locus of \( W \) is \( \{0\}/\mathbb{Z}_5 \).

In particular, then, we have

\[
Z_{g,n,\beta}^+ = \{(C; q_1, \ldots, q_n; f; p) \mid f : C \to Q \text{ stable, } p = 0\} = \overline{M}_{g,n}(Q, \beta),
\]

which is the moduli space of Gromov–Witten theory, whereas

\[
Z_{g,n,\beta}^- = \{(C; q_1, \ldots, q_n; L; x_1, \ldots, x_5; \varphi) \mid x_1 = \ldots = x_5 = 0, \varphi : L^{\otimes -5} \cong \omega_{\text{log}} \} = \overline{M}_W^{\text{FJRW}},
\]

which is the moduli space of FJRW theory.

We have thus succeeded in constructing the moduli spaces of the two theories in entirely analogous ways. In fact, this framework can be upgraded to give a uniform construction of the virtual fundamental cycles on the two moduli spaces, via the cosection construction of Kiem–Li [20]. The basic idea, which was first applied to this setting by Chang–Li–Li [2, 3], is that \( X_{g,n,\beta}^\pm \) has a natural deformation theory—and hence, a natural virtual fundamental cycle—coming from the cohomology of \( L^{\otimes 5} \oplus (L^{\otimes -5} \otimes \omega_{\text{log}}) \), and the cosection construction is designed to produce a virtual fundamental cycle supported on a critical locus inside a larger space.

One caution is in order here: unlike in the state space correspondence, no equivalence between virtual cycles or correlators falls out immediately from the fact that the two moduli spaces arise in analogous ways. Rather, the above argument should be viewed, at this point, as merely a heuristic framework for motivating a relationship between the two theories.

Nevertheless, a correspondence of correlators does hold in genus zero, as Chiodo and Ruan proved:

**Theorem 3.3 (Chiodo–Ruan).** The genus-zero FJRW theory of \( W \) can be encoded in a generating function \( J^{\text{FJRW}}(t) \) taking values in \( \mathcal{H}_W[[z^{-1}, z]] \), and the genus-zero Gromov–Witten theory of \( Q \) can be encoded in a generating function \( J^{GW}(q) \) taking values in \( H^*(Q)[[z^{-1}, z]] \). After choosing a specific isomorphism \( \mathcal{H}_W \cong H^*(Q) \), these two generating functions are related by changes of variables in \( q \) and \( t \), an identification \( q = t^{-5} \), and analytic continuation between the \( t = 0 \) and \( q = 0 \) expansions.

We return in the last section to discuss some ideas in the proof of the genus-zero LG/CY correspondence. Before doing so, however, we address another natural question: if FJRW corresponds to the Gromov–Witten theory of the hypersurface
Q, what theory corresponds in this way to the Gromov–Witten theory of a complete intersection?

4. The hybrid model and the GLSM

Let

\[ W_1, \ldots, W_r \in \mathbb{C}[x_1, \ldots, x_N] \]

be a collection of quasihomogeneous polynomials of the same weights \( w_1, \ldots, w_N \) and the same degree \( d \), defining a nonsingular complete intersection

\[ Z = \{ W_1 = \cdots = W_r = 0 \} \subseteq \mathbb{P}(w_1, \ldots, w_N). \]

The answer to the question posed at the end of the previous section is provided by a theory referred to as the “hybrid model,” whose definition occupies this section. Just as in the FJRW case, the theory consists of a state space, a moduli space (equipped with a virtual fundamental cycle), and a notion of correlators.

4.1. The state space of the hybrid model. To understand what the appropriate FJRW-type state space associated to \( Z \) might be, we return to the variation of GIT perspective from the previous section. Namely, introduce \( r \) auxiliary variables \( p_1, \ldots, p_r \), and let \( \mathbb{C}^* \) act on \( \mathbb{C}^N \times \mathbb{C}^r \) by

\[ \lambda(x_1, \ldots, x_N, p_1, \ldots, p_r) = (\lambda w_1 x_1, \ldots, \lambda w_N x_N, \lambda^{-d} p_1, \ldots, \lambda^{-d} p_r). \]

Then the polynomial

\[ \overline{W}(x_1, \ldots, x_N, p_1, \ldots, p_r) = p_1 W_1(\vec{x}) + \cdots + p_r W_r(\vec{x}) \]

gives a well-defined map out of the quotient \( (\mathbb{C}^N \times \mathbb{C}^r)/\mathbb{C}^* \).

Again, while this original quotient is not separated, it admits two maximal separated subquotients,

\[ X_+ = \{ \vec{x} \neq 0 \} = \bigoplus_{j=1}^r \mathcal{O}_{\mathbb{P}(w_1, \ldots, w_N)}(-d) \]

and

\[ X_- = \{ \vec{p} \neq 0 \} = \bigoplus_{i=1}^N \mathcal{O}_{\mathbb{P}(d, \ldots, d)}(-w_i). \]

Taking cohomology relative to a fiber \( \overline{W}^{+\infty} \) of \( \overline{W} \) yields

\[ H^*_{CR} \left( \bigoplus_{j=1}^r \mathcal{O}_{\mathbb{P}(w_1, \ldots, w_N)}(-d), \overline{W}^{+\infty} \right) \cong H^*_{CR}(\mathbb{P}(w_1, \ldots, w_N), \mathbb{P}(w_1, \ldots, w_N) \setminus Z) \cong H^*_{CR}(Z). \]

Here, the second isomorphism is the Thom isomorphism. The first isomorphism can be deduced from the fact that \( \overline{W}^{+\infty} \) intersects the fibers of \( \mathcal{O}_{\mathbb{P}(w_1, \ldots, w_N)}(-d)^{\oplus r} \) in an affine space (with trivial cohomology) over points in \( \mathbb{P}(w_1, \ldots, w_N) \setminus Z \), while it does not intersect the fibers of \( \mathcal{O}_{\mathbb{P}(w_1, \ldots, w_N)}(-d)^{\oplus r} \) over points in \( Z \) at all.

With this in mind, we define the hybrid-model state space as

\[ \mathcal{H}_{\overline{W}} := H^*_{CR} \left( \bigoplus_{i=1}^N \mathcal{O}_{\mathbb{P}(d, \ldots, d)}(-w_i), \overline{W}^{+\infty} \right). \]

Note that, by the definition of orbifold cohomology, this space can be decomposed as a direct sum over \( g \in \mathbb{Z}_d \). Analogously to the FJRW case, we refer to \( g \) as narrow
if its fixed locus is $\mathbb{P}(d, \ldots, d) \subseteq X_-$ or broad otherwise. The narrow elements of $\mathbb{Z}_d$ each contribute a copy of $H^*(\mathbb{P}(d, \ldots, d)) = H^*(\mathbb{P}^{N-1})$ to the hybrid-model state space.

### 4.2. The moduli space of the hybrid model.

As for the moduli space, we repeat the analogous variation-of-GIT argument once again. Namely, let $X_{g,n,\beta}$ be the (non-compact, non-separated, Artin) stack parameterizing tuples

$$(C; q_1, \ldots, q_n; L; x_1, \ldots, x_N; p_1, \ldots, p_r),$$

in which $(C; q_1, \ldots, q_n)$ is a genus-$g$, $n$-pointed orbifold stable curve and

$$(x_1, \ldots, x_N, p_1, \ldots, p_r) \in \Gamma \left( \bigoplus_{i=1}^N L^{\otimes w_i} \oplus \bigoplus_{j=1}^r \omega_{L^{\otimes -d} \otimes \omega_{\log}} \right).$$

Within $X_{g,n,\beta}$, there are two separated Deligne–Mumford substacks, which we again denote by $X^+_{g,n,\beta}$ and $X^-_{g,n,\beta}$, given by imposing that either $\overline{\mathcal{X}}$ or $\overline{\mathcal{P}}$ is nowhere-vanishing.

When $\overline{\mathcal{X}}$ is nowhere-vanishing, the situation is exactly as in the previous section. Namely,

$$X^+_{g,n,\beta} = \{(C; q_1, \ldots, q_n; f; p_1, \ldots, p_r) \mid f : C \to \mathbb{P}(\overline{w}), \ p_j \in \Gamma(f^*\mathcal{O}(-d) \otimes \omega_{\log}) \}.$$

When $\overline{\mathcal{P}}$ is nowhere-vanishing, on the other hand, whereas it led to a trivialization in the setting of FJRW theory, it now corresponds to a map $f : C \to \mathbb{P}^{r-1}$. Thus, we have

$$X^-_{g,n,\beta} = \{(C; q_1, \ldots, q_n; L; x_1, \ldots, x_N; f) \mid x_i \in \Gamma(L^{\otimes w_i}), \ f : C \to \mathbb{P}^{r-1}, \ f^*\mathcal{O}(1) \cong L^{\otimes -d} \otimes \omega_{\log} \}.$$

**Remark 4.1.** It is perhaps worth pausing to note at this point that $X^\pm_{g,n,\beta}$ and $X^-_{g,n,\beta}$ are nearly equal to the moduli spaces of stable maps to $\mathbb{P}^N$, respectively; the only difference is the appearance of the bundle $\omega_{\log}$ in various factors. For a much more detailed study of how the theories with and without $\omega_{\log}$ relate to one another, we refer the reader to [21] or [11].

Continuing on in the template laid out in the FJRW case, we observe that $X^\pm_{g,n,\beta}$ are not compact, and we restrict to the loci $Z^\pm_{g,n,\beta} \subseteq X^\pm_{g,n,\beta}$ on which the section $(x_1, \ldots, x_N, p_1, \ldots, p_r)$ lands in the critical locus

$$\text{Crit}(\overline{W}) \subseteq \mathbb{C}^N \times \mathbb{C}^r.$$  

Specifically:

**Exercise 4.2.** Viewing $\overline{W}$ as a function on $X_+ = \bigoplus_{j=1}^r \mathcal{O}_{\mathbb{P}(\overline{w})}(-d)$, verify that the critical locus of $\overline{W}$ is the complete intersection $Z$ in the zero section $\mathbb{P}(\overline{w}) \subseteq X_+$.

Similarly, viewing $\overline{W}$ as a function on $X_- = \bigoplus_{i=1}^N \mathcal{O}_{\mathbb{P}(w_i)}(-w_i)$, verify that the critical locus of $\overline{W}$ is the zero section.

As a result, then, we have

$$Z^+_{g,n,\beta} = \mathcal{M}_{g,n}(Z, \beta),$$

whereas we define the hybrid model state space as

$$Z^-_{g,n,\beta} = \{(C; q_1, \ldots, q_n; L; f) \mid f : C \to \mathbb{P}^{r-1}, \ f^*\mathcal{O}(1) \cong L^{\otimes -d} \otimes \omega_{\log} \}.$$
Perhaps, at this point, the name “hybrid” makes some sense; the above moduli space combines the Gromov–Witten theory of $\mathbb{P}^{r-1}$ with the FJRW theory of a degree-$d$ polynomial.

4.3. The correlators of the hybrid model. Just as in the FJRW case, because $Z_{g,n,\beta}^{\pm}$ arise as critical loci inside a larger moduli space with a natural deformation theory, they have induced virtual fundamental cycles via the cosection construction \([2, 3]\). Somewhat more explicitly:

**Exercise 4.3.** Let
\[(C; L; x_1, \ldots, x_N, p_1, \ldots, p_r) \in X_{g,0,\beta}^{\text{hyb}}.\]
Show, using Serre duality, that there is a map
\[\sigma: \bigoplus_{i=1}^N H^1(L^\otimes w_i) \oplus \bigoplus_{j=1}^r H^1(L^\otimes -d \otimes \omega) \to \mathbb{C}\]
\[\sigma(u_1, \ldots, u_N, v_1, \ldots, v_r) = \sum_{i=1}^N \frac{\partial W}{\partial x_i}(\vec{x}, \vec{p}) \cdot u_i + \sum_{j=1}^r \frac{\partial W}{\partial p_j}(\vec{x}, \vec{p}) \cdot v_j,\]
and that this map fails to be surjective if and only if
\[(C; L; x_1, \ldots, x_N, p_1, \ldots, p_r) \in Z_{g,0,\beta}^{\text{hyb}}.\]
(These fiberwise maps are used to define the cosection on $X_{g,0,\beta}^{\text{hyb}}$ from which the virtual cycle on $Z_{g,0,\beta}^{\text{hyb}}$ is constructed, and they are the reason why the $\omega_{\log}$’s are needed in the sections $p_j$.)

Furthermore, both of these moduli spaces have evaluation maps
\[\text{ev}_i: Z_{g,n,\beta}^{\pm} \to X_{g}^{\pm}\]
for each $i \in \{1, \ldots, n\}$. To see this, we need the following fundamental fact (see, for example, [25] for further explanation):

**Fact 4.4.** Let $\mathcal{C}$ be the universal curve over $\mathcal{M}_{g,n}$, and let $\pi: \mathcal{C} \to \mathcal{M}_{g,n}$ be the projection map. If $\Delta_i \subseteq \mathcal{C}$ denotes the divisor corresponding to the $i$th marked point, then $\omega_{\log}|_{\Delta_i}$ is trivial. Furthermore, the analogous statement is true for $X_{g,n,\beta}$, $X_{g,n,\beta}^{\pm}$, or $Z_{g,n,\beta}^{\pm}$.

With this fact in mind, consider the moduli space $X_{g,n,\beta}$, with its universal curve $\pi: \mathcal{C} \to X_{g,n,\beta}$, its universal line bundle $\mathcal{L}$ on $\mathcal{C}$, and its universal section
\[\sigma \in \Gamma \left( \bigoplus_{i=1}^N \mathcal{L}^\otimes w_i \oplus \bigoplus_{j=1}^r (\mathcal{L}^\otimes -d \otimes \omega_{\log}) \right).\]
Then, for any $i \in \{1, \ldots, n\}$, we have
\[\sigma|_{\Delta_i} \in \Gamma \left( \bigoplus_{i=1}^N \mathcal{L}^\otimes w_i \oplus \bigoplus_{j=1}^r \mathcal{L}^\otimes -d \right).\]
M back under the inclusion. These correlators are defined only for ambient insertions—those pulled back under the inclusion. Here, the state space is drawn from the wrong vector space.

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be encoded in a generating function of complete intersections, at least under restrictive assumptions:

indeed do agree, up to a sign $[2]$. The case where choosing a specific isomorphism taking values in $H^*(\mathbb{P}(\bar{w}))$; thus, in the same way that hybrid correlators are defined only for narrow insertions, these correlators are defined only for ambient insertions—those pulled back under the inclusion $Z \rightarrow \mathbb{P}(\bar{w})$. If we define the virtual cycle of $Z_{g,n,\beta}^+$ via the cosection construction, however, it is not at all obvious that the resulting correlators recover the usual ambient Gromov–Witten invariants of $Z$. Chang and Li proved, in the case where $Z$ is the quintic threefold, that these two versions of the correlators indeed do agree, up to a sign $[2]$. Having developed the definition of the hybrid model correlators, we are now prepared to state the generalization of the LG/CY correspondence to the setting of complete intersections, at least under restrictive assumptions:

Theorem 4.9. Suppose that $Z$ is a Calabi–Yau threefold and $w_1 = \cdots = w_N = 1$. The genus-zero hybrid theory can be encoded in a generating function $J^{hyb}(t)$ taking values in $\mathcal{H}_W[[z^{-1}, z]]$, and the genus-zero Gromov–Witten theory of $Z$ can be encoded in a generating function $J^{GW}(q)$ taking values in $H^*(Q)[[z^{-1}, z]]$. After choosing a specific isomorphism $\mathcal{H}_W \cong H^*(Q)$, these two generating functions are related by changes of variables in $q$ and $t$, an identification between $q$ and $t$, and analytic continuation between the $t = 0$ and $q = 0$ expansions.
4.4. The general gauged linear sigma model. The story we have told for FJRW theory and the hybrid model can be vastly generalized; it is an instance of the gauged linear sigma model (GLSM), which was developed mathematically by Fan, Jarvis, and Ruan [15].

In full generality, the GLSM is a theory that depends on three pieces of input data:

1. a GIT quotient $X = [V \sslash G]$, where $V$ is a complex vector space and $G \subseteq \text{GL}(V)$;
2. a polynomial function $W : X \rightarrow \mathbb{C}$ known as the "superpotential";
3. an action of $\mathbb{C}^*$ on $V$ known as the "R-charge."

The rough idea, from here, is to form a moduli space parameterizing "Landau–Ginzburg maps" from curves into the critical locus of $W$; these look nearly the same as ordinary maps to the critical locus of $W$, but there are additional occurrences of $\omega_{\text{log}}$ dictated by the $R$-charge.

In order for the resulting moduli space to be compact, it is sometimes necessary to weaken the notion of maps to “quasimaps”, an idea first introduced by Ciocan-Fontanine and Kim [7, 8]. Quasimaps also play a key role in the proof of the genus-zero LG/CY correspondence, so we discuss them further in the final section of these notes.

5. Wall-crossing and the proof of the LG/CY correspondence

We turn, in this final section, to the proof of the LG/CY correspondence in genus zero. A number of proof strategies have been developed at this point, including Chiodo and Ruan’s Lagrangian cone approach [5] (based on ideas of [13]) and Lee–Priddd–Shoemaker’s approach via the crepant transformation conjecture [21, 11] (which leverages [14]). In what follows, we describe a proof via quasimaps and wall-crossing, which was first applied to the LG/CY correspondence by Ross and Ruan [23]; one advantage of this approach is that it points toward a strategy for higher genus.

5.1. Quasimaps. As in the previous section, let $W_1, \ldots, W_r \in \mathbb{C}[x_1, \ldots, x_N]$ be quasihomogeneous polynomials of the same weights $w_1, \ldots, w_N$ and the same degree $d$, defining a nonsingular complete intersection $Z \subseteq \mathbb{P}(w_1, \ldots, w_N)$. Recall that the moduli spaces of Gromov–Witten theory and the hybrid model were both constructed as substacks of the larger (Artin) stack $X_{g, n, \beta}$ parameterizing tuples

$$(C; q_1, \ldots, q_n; L; x_1, \ldots, x_N; p_1, \ldots, p_r)$$

with

$$(x_1, \ldots, x_n, p_1, \ldots, p_r) \in \Gamma \left( \bigoplus_{i=1}^{N} L^{\otimes w_i} \oplus \bigoplus_{j=1}^{r} (L^{\otimes -d} \otimes \omega_{\text{log}}) \right),$$

by first imposing that either $\vec{x}$ or $\vec{p}$ was nowhere-vanishing.
Suppose, now, that we weaken this requirement, allowing \( \vec{x} \) (on the Gromov–Witten side) or \( \vec{p} \) (on the hybrid side) to have zeroes of bounded order. Specifically, on the Gromov–Witten side, we consider the following definition (due to Ciocan-Fontanine and Kim \([7, 8]\)):

**Definition 5.1.** Fix a positive rational number \( \epsilon \), and define \( X^{+,\epsilon}_{g,n,\beta} \subseteq X_{g,n,\beta} \) to be the locus where:

1. \( \vec{x}(q) = 0 \) for at most finitely many points \( q \in C \), all of which are nonspecial and satisfy
   \[
   \text{ord}_q(\vec{x}) \leq 1/\epsilon,
   \]
   where \( \text{ord}_q(\vec{x}) \) denotes the order of vanishing of \( \vec{x} \);
2. the bundle \( L^{\otimes \epsilon} \otimes \omega_{\log} \) is ample.

Elements of \( X^{+,\epsilon}_{g,n,\beta} \) are referred to as \( \epsilon \)-stable quasi maps to \( X^+ \).

The second condition in Definition 5.1 is equivalent to requiring that \( \deg(L_{|C'}) > 1/\epsilon \) on any genus-zero components \( C' \subseteq C \) with only one special point (such components are referred to as “rational tails”) and \( \deg(L_{|C'}) > 0 \) on any genus-zero components \( C' \subseteq C \) with only two special points (referred to as “rational bridges”).

The intuition behind this condition is the following. Having imposed a bound on the order of vanishing of \( \vec{x} \), properness of the moduli space forces one to specify what happens in the limit as a nowhere-vanishing \( \vec{x} \) approaches a zero of order \( \beta > 1/\epsilon \). Just as in the stable maps setting, the limit in this case is a curve with a new rational component on which \( L \) has degree \( \beta \). The second condition in Definition 5.1 is concocted to allow precisely such rational components; allowing any more would lead to non-uniqueness of limits, preventing the moduli space from being separated.

Having constructed the moduli space \( X^{+,\epsilon}_{g,n,\beta} \), we mimic the previous section and restrict further to the locus

\[
Z^{+,\epsilon}_{g,n,\beta} \subseteq X^{+,\epsilon}_{g,n,\beta}
\]

where \( (\vec{x}, \vec{p}) \) lands in \( \text{Crit}(\overline{W}) = Z \subseteq X \)—that is, where

\[
W_j(\vec{x}) \equiv 0 \in \Gamma(L^{\otimes d}) \quad \text{for all } j \in \{1, \ldots, r\}
\]

and

\[
\vec{p} \equiv 0.
\]

This recovers Ciocan-Fontanine and Kim’s moduli space of \( \epsilon \)-stable quasi maps to \( Z \).

When \( \epsilon > 2 \), condition (1) of Definition 5.1 implies that \( \vec{x} \) is nowhere-vanishing and hence gives rise to an honest map \( f : C \to \mathbb{P}(w_1, \ldots, w_N) \), and condition (2) implies that \( L \) has positive degree on all rational components with fewer than three special points, so \( f \) is stable. Thus, for such \( \epsilon \), the moduli space of \( \epsilon \)-stable quasi maps to \( Z \), which we denote by \( Z^{+,\infty}_{g,n,\beta} \), is nothing but the usual moduli space of stable maps.

Taking \( \epsilon \to 0 \), on the other hand (that is, requiring conditions (1) and (2) of Definition 5.1 for all \( \epsilon > 0 \)) corresponds to allowing arbitrary isolated zeroes of \( \vec{x} \) and disallowing all rational tails. The resulting moduli space, which we denote by \( Z^{+0}_{g,n,\beta} \), is the moduli space of stable quotients studied by Marian, Oprea, and Pandharipande \([22]\).
All of this can be carried out analogously on the hybrid side. Namely, inside $X_{g,n,\beta}$, we replace the requirement that $\vec{p}$ is nowhere zero by a bound on the order of its zeroes:

**Definition 5.2.** Fix a positive rational number $\epsilon$, and define $X^{-,\epsilon}_{g,n,\beta} \subseteq X_{g,n,\beta}$ to be the locus where:

1. $\vec{p}(q) = 0$ for at most finitely many points $q \in C$, all of which are nonspecial and satisfy

   $\text{ord}_q(\vec{p}) \leq 1/\epsilon$,

   where $\text{ord}_q(\vec{p})$ denotes the order of vanishing of $\vec{p}$;

2. the bundle $(L^\otimes d \otimes \omega \log)^{\otimes \epsilon} \otimes \omega \log$ is ample.

Then, within $X^{-,\epsilon}_{g,n,\beta}$, we restrict further to the locus $Z^{-,\epsilon}_{g,n,\beta}$ on which $\vec{p}$ lands in $\text{Crit}(\mathcal{W}) = \mathbb{P}(d, \ldots, d) \subseteq X,$, or in other words, where $\vec{p} \equiv 0$. As above, taking $\epsilon \gg 0$ in the definition of $Z^{-,\epsilon}_{g,n,\beta}$ recovers the hybrid moduli space $Z^{-}$.

### 5.2. Proof of genus-zero LG/CY

One of the most important reasons for our interest in quasimaps is their intimate connection to mirror symmetry, which leads ultimately to the LG/CY correspondence.

The idea of genus-zero mirror symmetry—at least, one of its manifestations—is to calculate the genus-zero Gromov–Witten invariants of the complete intersection $Z$ by relating their generating function to an explicit power series. In particular, Givental showed in [16] that the generating function $J_{GW}(q)$ of genus-zero Gromov–Witten invariants of $Z$ (mentioned in Theorem 4.9) is related by change of variables to the "I-function"

$$I_{GW}(q) = \sum_{\beta \geq 0} q^\beta \frac{\prod_{b=1}^{d} (dH + bz)^\epsilon}{\prod_{b=1}^{d} (H + bz)^{N+1}},$$

a power series in $q$ with coefficients in $H^*(Z)[[z^{-1}, z]]$, where $H \in H^*(Z)$ denotes the hyperplane class. This is a powerful result that allows the genus-zero Gromov–Witten invariants of $Z$ to be explicitly computed.

The original appearance of the I-function in the literature was in terms of period integrals and Picard–Fuchs equations, but Ciocan-Fontanine and Kim gave an alternative explanation for its role. Namely, they showed that one can define a generating function $J^{GW, \epsilon}(q)$ of genus-zero $\epsilon$-stable quasimap invariants for any $\epsilon$, and $J^{GW, \epsilon}(q)$ coincides with $J^{GW}(q)$ when $\epsilon \gg 0$ whereas $J^{GW,0}(q)$ is precisely the I-function. Furthermore, it is apparent from the definition of quasimaps that the invariants change only when $1/\epsilon$ crosses an integer value, so $J^{GW, \epsilon}(q)$ changes only when $\epsilon$ crosses one of a discrete set of "walls." From this perspective, genus-zero mirror symmetry becomes the result of a "wall-crossing" formula exhibiting how $J^{GW, \epsilon}(q)$ varies when $\epsilon$ crosses a wall; this re-proof of mirror symmetry was carried out by Ciocan-Fontanine and Kim in [7].

Precisely the same story can be told on the hybrid side, using wall-crossing to relate the hybrid $J$-function $J^{hyb}(t) = J^{hyb, \infty}(t)$ to the function $J^{hyb,0}(t)$, which is the "hybrid I-function" and can be calculated explicitly as a hypergeometric series; this was done by Ross–Ruan for FJRW theory and by Ross and the author for the hybrid model [23, 12].
From here, one proves the LG/CY correspondence by relating the two $I$-functions to one another. More specifically, if one expands $I^{GW}(q)$ in the form

$$I^{GW}(q) = I_0(q) \cdot 1 + I_1(q) \cdot H + I_2(q) \cdot H^2 + \cdots + I_{N-1-r}(q)H^{N-1-r},$$

then the coefficients $I_i(q)$ form a basis of solutions to a differential equation in $q$ known as the Picard–Fuchs equation. In the same way, the hybrid $I$-function assembles a basis of solutions to a differential equation in $t$, and these two differential equations coincide after a particular identification between $q$ and $t$. (For example, in the case of the quintic threefold, the identification is $q = t^{-5}$.) Thus, by analytically continuing the solutions in the $q$-coordinate patch to solutions in the $t$-coordinate patch, and then applying a linear isomorphism of the state space to change from one basis of solutions to another, one shows that $I^{GW}(q)$ and $I^{hyb}(t)$ are related by analytic continuation and a state space isomorphism. This, combined with the wall-crossing changes of variables on both sides, proves the genus-zero LG/CY correspondence.

5.3. Towards higher-genus LG/CY. The proof of the genus-zero LG/CY correspondence outlined in the previous subsection suggests a path toward a proof in all genus: first, prove all-genus wall-crossing theorems exhibiting the dependence of the generating functions of Gromov–Witten or hybrid invariants on $\epsilon$, and second, relate the $\epsilon \to 0$ Gromov–Witten generating function to the $\epsilon \to 0$ hybrid generating function.

The first step of this program has been successfully carried out, by Ciocan-Fontanine and Kim on the Gromov–Witten side [6] and by Janda, Ruan, and the author on the hybrid side [9]:

**Theorem 5.3.** For any complete intersection $Z \subseteq \mathbb{P}(w_1, \ldots, w_N)$ of hypersurfaces of the same degree, we have

$$[Zgw/hyb_{g,n,\beta}, I^{GW/hyb, \epsilon}_{g,n,\beta}]_{vir} = \sum_{\tilde{\beta}_0 + \tilde{\beta}_1 + \cdots + \tilde{\beta}_k = \beta} b_{\tilde{\beta}} \left( \prod_{i=1}^k \epsilon_{n+i}^{\cdot \sum_{\beta_i}^{GW/hyb, \epsilon} (\mu_{\beta_i}^{GW/hyb, \epsilon} (-\psi_{n+i}))} \cap [Z_{g,n+k,\beta_0}]_{vir} \right).$$

Here, $b_{\tilde{\beta}} : Z^{GW/hyb}_{g,n+k,\beta_0} \to Z^{GW/hyb}_{g,n,\beta}$ is a certain comparison map that converts marked points to zeroes of $\tilde{x}$ or $\tilde{p}$, and $\mu_{\beta_i}^{GW/hyb, \epsilon}$ is a certain coefficient of the Gromov–Witten or hybrid $I$-function.

We should also note that the proof of [9] yields an alternative proof of the wall-crossing on the Gromov–Witten side [10], and on the hybrid side, a different proof of Theorem 5.3 was independently given by Y. Zhou [24] in the case where $r = 1$.

Theorem 5.3 represents significant progress toward the higher-genus LG/CY correspondence, but the second step of the program—relating the $\epsilon \to 0$ theories on the two sides—remains almost entirely mysterious. It has been established in genus zero (as described in the previous subsection) and in genus one [17, 18]; the latter is the subject of another of the mini-courses at this workshop.

**References**