Exercises on the Gauged Linear Sigma Model

Crossing the Walls in Enumerative Geometry

Lecture 1: Preliminaries on Orbifolds and GW Theory

1. Recall that, for a topological space X, an *orbifold chart* is an open set $U \subseteq X$ together with an open set $\widetilde{U} \subseteq \mathbb{R}^n$ equipped with a smooth action of a finite group G and a G-invariant map $\phi : \widetilde{U} \to U$ that descends to a homeomorphism $\overline{\phi} : \widetilde{U}/G \to U$. Make a guess about the definitions of a *subchart* (for $U' \subseteq U \subseteq X$) and *compatibility* of charts (for $\emptyset \neq U \cap U' \subseteq X$). See, for example, Definitions 1.13 and 1.14 of [2] for the solutions.

2. Let $X = \mathbb{P}(w_0, \ldots, w_n)$ be a weighted projective space where $w_0, \ldots, w_n$ are pairwise coprime. Carefully write down the orbifold structure on X; in particular, verify that each coordinate point p_i is contained in an orbifold chart $U_i \cong \mathbb{C}/\mathbb{Z}_{w_i}$ and every other point is contained in an orbifold chart with trivial action.

3. Consider the action of $\mathbb{Z}_2$ on the torus $\mathbb{T} = S^1 \times S^1 \subseteq \mathbb{C}^* \times \mathbb{C}^*$, where the nontrivial element of $\mathbb{Z}_2$ acts by
$$(e^{it_1}, e^{it_2}) \mapsto (e^{-it_1}, e^{-it_2}).$$
Find an explicit atlas of orbifold charts on $X = \mathbb{T}/\mathbb{Z}_2$. What, topologically, is X?

4. Consider the action of the symmetric group S_3 on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ given by permuting the three coordinates; the resulting quotient is the *symmetric product* $X = S^3(\mathbb{P}^1)$. At which points does the action of S_3 have nontrivial isotropy, and what is the isotropy group? Find an explicit atlas of orbifold charts on X, and convince yourself that, topologically, $X \cong \mathbb{P}^3$.

5. Let X be a smooth projective variety, and define the quantum product $*$ on $H^*(X)[[q]]$ by
$$(\alpha * \beta, \gamma) := \sum_{\beta} \langle \alpha\ \beta\ \gamma \rangle^X_{0,3,\beta}.$$
Verify that setting $q = 0$ recovers the usual cup product on $H^*(X)$.

Lecture 2: FJRW Theory

1. A basic example of an orbifold morphism is $f : [\mathbb{C}/\mathbb{Z}_r] \to [\mathbb{C}/\mathbb{Z}_n]$, described by a map
$$f_0 : \mathbb{C} \to \mathbb{C}$$
together with a homomorphism
$$f_1 : \mathbb{Z}_r \to \mathbb{Z}_n$$
such that
$$f_0(\zeta \cdot z) = f_1(\zeta) \cdot f_0(z)$$
for all $z \in \mathbb{C}$; note that the map
$$\overline{f} : \mathbb{C}/\mathbb{Z}_r \to \mathbb{C}/\mathbb{Z}_n$$
$$\overline{f}([z]) = [f_0(z)]$$
is well-defined. Such a morphism is said to be *representable* if f_1 is injective. Suppose that f is representable and, under the isomorphisms $\mathbb{C}/\mathbb{Z}_r \cong \mathbb{C}$ and $\mathbb{C}/\mathbb{Z}_n \cong \mathbb{C}$, the coarse underlying map $\overline{f}$ is $z \mapsto z^d$. Show that
$$r = \frac{n}{\gcd(d,n)}$$
and f_1 is multiplication by d. (This shows, in particular, that f is fully determined by its behavior on coarse underlying spaces. See Lemma II.12 of [1] for a solution.)

2. Calculate the inertia stack of the orbifold $\mathcal{X} = [\mathbb{C}/\mathbb{Z}_r]$, and using this, calculate $H^*_{\mathrm{CR}}([\mathbb{C}/\mathbb{Z}_r])$ as a vector space.

3. Consider a weighted projective space $\mathcal{X} = \mathbb{P}(w_0, w_1)$ with coprime weights w_0 and w_1. Calculate the inertia stack and the Chen–Ruan cohomology (as a vector space) of $\mathcal{X}$. Although we have not explicitly defined weighted projective spaces with non-coprime weights, what do you expect are the inertia stack and Chen–Ruan cohomology of $\mathbb{P}(3,3)$? Of $\mathbb{P}(3,6)$?

4. Find an example of a quasihomogeneous polynomial for which there exists a nontrivial broad element $g \in J$.

Lecture 3: The LG/CY Correspondence

1. Let $(\mathcal{C}; q_1, \ldots, q_n)$ be a smooth orbifold curve at which each marked point q_i has isotropy group $\mathbb{Z}_d$, and let L be an orbifold line bundle on $\mathcal{C}$. A fundamental fact about the multiplicities of L is that the bundle
$$|L| := L \otimes \mathcal{O}\left(-\sum_{i=1}^{n} \frac{m_i}{d}[q_i]\right)$$
is pulled back from the coarse underlying curve C. Use this fact to explain why
$$\sum_{i=1}^{n} m_i \equiv 0 \mod d. \tag{1}$$

Generalizing to nodal curves of compact type, convince yourself that the multiplicity of L at each branch of each node is determined by (1) together with the "kissing condition" that $m' + m'' \equiv 0 \mod d$ for the multiplicities m', m'' at opposite branches of the same node. What happens if the curve is not of compact type—in other words, if it has a non-separating node?

2. Work out the "fundamental fact" from the previous problem explicitly in local coordinates, by considering the diagram

$$\begin{tikzcd} L = [(\mathbb{C} \times \mathbb{C})/\mathbb{Z}_d] \arrow[rr, "{[x,v] \mapsto [x, x^{-m}v]}"] \arrow[dr] & & {[(\mathbb{C} \times \mathbb{C})/\mathbb{Z}_d] = |L|} \arrow[dl] \\ & \mathcal{C} = [\mathbb{C}/\mathbb{Z}_d] & \end{tikzcd},$$

where L is defined by the action $\zeta(x, v) = (\zeta x, \zeta^m v)$ and $|L|$ is defined by the action $\zeta(x, v) = (\zeta x, v)$ of $\mathbb{Z}_d$ on $\mathbb{C} \times \mathbb{C}$. (See [1, Section 2.1.4] for related discussion.)

3. Let $W = x_1^5 + \cdots + x_5^5$ and let $Q = \{W = 0\} \subseteq \mathbb{P}^4$. Show that there is an isomorphism between the ambient part of the cohomology $H^*(Q)$—that is, the cohomology classes pulled back from $\mathbb{P}^4$—and the narrow part of the FJRW state space $\mathcal{H}_W$.

4. As in the lectures, let
$$\overline{W}(x_1, \ldots, x_5, p) = p(x_1^5 + \cdots + x_5^5),$$
which gives a well-defined polynomial function out of both $X_+ := \mathcal{O}_{\mathbb{P}^4}(-5)$ and $X_- := [\mathbb{C}^5/\mathbb{Z}_5]$. Verify that the critical locus of $\overline{W}$ in X_+ is $Q \subseteq \mathbb{P}^4$ and the critical locus of $\overline{W}$ in X_- is $[\{0\}/\mathbb{Z}_5]$.

Lecture 4: The Hybrid Model

1. Let $Z = \{W_1 = \cdots = W_r = 0\} \subseteq \mathbb{P}^{N-1}$ be a nonsingular complete intersection in projective space. Show that there is an isomorphism between the ambient part of the cohomology $H^*(Z)$ and the narrow part of the hybrid state space $\mathcal{H}_{\overline{W}}$.

2. Now let $Z = \{W_1 = \cdots = W_r = 0\} \subseteq \mathbb{P}(w_1, \ldots, w_N)$ be a nonsingular complete intersection of hypersurfaces of degree d in weighted projective space. As in the lectures, let

$$\overline{W}(x_1, \ldots, x_N, p_1, \ldots, p_r) = p_1 W_1(\vec{x}) + \cdots + p_r W_r(\vec{x}),$$

which gives a well-defined polynomial function out of both

$$X_+ := \bigoplus_{j=1}^{r} \mathcal{O}_{\mathbb{P}(\vec{w})}(-d) \qquad \text{and} \qquad X_- := \bigoplus_{i=1}^{N} \mathcal{O}_{\mathbb{P}(\vec{d})}(-w_i).$$

Verify that the critical locus of $\overline{W}$ in X_+ is $Z \subseteq \mathbb{P}(\vec{w})$ and the critical locus of $\overline{W}$ in X_- is the zero section $\mathbb{P}(\vec{d})$.

3. Let
$$(C; L; x_1, \ldots, x_N, p_1, \ldots, p_r) \in X^{\text{hyb}}_{g,0,\beta}.$$
Show, using Serre duality, that there is a map
$$\sigma : \bigoplus_{i=1}^{N} H^1(L^{\otimes w_i}) \oplus \bigoplus_{j=1}^{r} H^1(L^{\otimes -d} \otimes \omega) \to \mathbb{C}$$
$$\sigma(u_1, \ldots, u_N, v_1, \ldots, v_r) = \sum_{i=1}^{N} \frac{\partial \overline{W}}{\partial x_i}(\vec{x}, \vec{p}) \cdot u_i + \sum_{j=1}^{r} \frac{\partial \overline{W}}{\partial p_j}(\vec{x}, \vec{p}) \cdot v_j,$$
and that this map fails to be surjective if and only if
$$(C; L; x_1, \ldots, x_N, p_1, \ldots, p_r) \in Z^{\text{hyb}}_{g,0,\beta}.$$
(These fiberwise maps are used to define the "cosection" on $X^{\text{hyb}}_{g,0,\beta}$ from which the virtual cycle on $Z^{\text{hyb}}_{g,0,\beta}$ is constructed, and they are the reason why the ω's are needed in the sections p_j.)

References

[1] P. Johnson. Equivariant Gromov-Witten theory of one dimensional stacks. Thesis (Ph.D.)–University of Michigan, 2009.

[2] D. Zvonkine An introduction to moduli spaces of curves and their intersection theory. *Handbook of Teichmüller theory. Volume III*, 667–716. IRMA Lect. Math. Theor. Phys., 17, *Eur. Math. Soc., Zürich*, 2012.