Toric Varieties and the Secondary Fan

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1 Motivation

- The Batyrev mirror symmetry construction for Calabi-Yau hypersurfaces goes roughly as follows:
  - Start with an n-dimensional reflexive polytope Δ in $M_{\mathbb{R}}$.
  - This gives a toric variety $\mathbb{P}_\Delta$. (One way to describe it is it's the toric variety associated to the normal fan Σ in $\mathbb{N}_{\mathbb{R}}$, which, since Δ is reflexive, simply means the fan obtained by taking cones over the proper faces of Δ°.)
  - Weil divisors on $\mathbb{P}_\Delta$ correspond to polytopes in $M_{\mathbb{R}}$. (One way to state the correspondence is as follows: Given a cone ρ of Σ, we get an affine variety $X_\rho = \mathrm{Spec}(\mathbb{C}[S_\rho]) \subset X_\Sigma = \mathbb{P}_\Delta$, where $S_\rho = M \cap \check{\rho}$. Points of this affine variety are in bijective correspondence with semigroup homomorphisms $\gamma : S_\rho \to \mathbb{C}$, and there is a distinguished point γ_ρ given by the homomorphism

    $$m \mapsto \begin{cases} 1 & m \in S_\rho \cap \rho^\perp \\ 0 & \text{otherwise.} \end{cases}$$

    If ρ is a one-dimensional cone of σ, then let D_ρ be the closure of the torus-orbit of γ_ρ, an irreducible subvariety in $\mathbb{P}_\Delta$. We have seen that any Weil divisor on $\mathbb{P}_\Delta$ can be expressed as $D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho$ for integers a_ρ. We then let

    $$\Delta_D = \{m \in M_{\mathbb{R}} \mid \langle m, \rho \rangle \geq -a_\rho \ \ \forall \rho \in \Sigma(1)\},$$

    and this is the polytope corresponding to D.)
  - The polytope corresponding to the anticanonical divisor is precisely Δ.
  - Any hypersurface $Y \subset \mathbb{P}_\Delta$ whose divisor class is the anticanonical class is Calabi-Yau. (This follows from the adjunction formula, which says that for D a divisor on a complex manifold M, one has $K_D \cong (K_M(D))|_D$. The result in this case is that $K_Y \cong K_{\mathbb{P}_\Delta}(-K_{\mathbb{P}_\Delta})|_Y = \mathcal{O}_Y$.)
  - Thus, the reflexive polytope Δ determines a family of Calabi-Yau hypersurfaces in the toric variety $\mathbb{P}_\Delta$.

  - This isn't quite the family for which we will construct a mirror; first, we want to desingularize it as much as possible while still remaining Calabi-Yau. A desingularization of $\mathbb{P}_\Delta$ is given by adding new cone generators to Σ until it is smooth– that is, every cone has generators as part of a $\mathbb{Z}$ basis of N. If Z is the toric variety obtained by adding a new cone generator ρ to Σ, then we get a birational map of toric varieties $f : Z \to \mathbb{P}_\Delta$, and

    $$K_Z = f^*(K_{\mathbb{P}_\Delta}) - (\langle \rho, m_F \rangle + 1)D$$

    where ρ lies in the facet $\langle u, m_F \rangle = -1$ of Δ°. In particular, the resulting toric variety has the same canonical class as long as the new cone generators we add are chosen from $N \cap \Delta^\circ$. So the most we can desingularize while retaining the property that an anticanonical hypersurface is Calabi-Yau is by taking the toric variety associated to a projective simplicial fan Σ' with $\Sigma'(1) = (N \cap \Delta^\circ) \setminus \{0\}$.
  - For any such Σ', denote the family of anticanonical hypersurfaces on $X_{\Sigma'}$ by $\hat{Y}$.
  - Then we obtain the mirror family of Calabi-Yau toric hypersurfaces by replacing Δ by Δ°.

- The sense in which these families are expected to be mirror to one another is that the moduli of complex structures on $\hat{Y}$ corresponds to the moduli of Kähler structures on $\hat{Y}^\circ$ and vise versa.

- But this is a *local* correspondence in the sense that not all of the possible complex/Kähler structures on the underlying Calabi-Yau are represented as it varies through the family of anticanonical hypersurfaces on $X_{\Sigma'}$.

- Indeed, it could not possibly be a global correspondence, because the full moduli space of complex structures forms a quasi-projective variety, while the moduli space of Kähler structures is the quotient of a bounded domain by a finite group, which is a smaller object.

- We would like a global version of the construction, which gives an isomorphism between the moduli space of complex structures on $\hat{Y}$ and an enlarged version of the moduli space of Kähler structures on $\hat{Y}^\circ$.

- The way to obtain this is to allow $\hat{Y}$ to vary not just over the anticanonical hypersurfaces in $X_{\Sigma'}$ for a fixed fan Σ', but over all anticanonical hypersurfaces in $X_{\Sigma'}$ for *all* possible fans Σ' satisfying $\Sigma'(1) = (N \cap \Delta^\circ) \setminus \{0\}$.

- The resulting $\hat{Y}$'s are thought to be related by flops, which implies that they have the same moduli space of complex structures.

- Their Kähler cones, however, may be different, and the secondary fan describes how they fit together. This is a larger object than the individual Kähler cones, but it isn't quite big enough yet. The so-called "enlarged secondary fan" is actually the object that is conjecturally mirror to the moduli space of complex structures.

2 The Secondary Fan

- Once we have fixed a reflexive polytope Δ, we want to examine all fans for which the generators of the one-dimensional cones are the fixed set $(N \cap \Delta^\circ) \setminus \{0\}$, and ask how the Kähler cones of these are related.

- So fix a finite set of strongly convex rational 1-dimensional cones Ξ in $N_\mathbb{R}$ and consider fans Σ with $\Sigma(1) = \Xi$.

- **Note**: $A_{n-1}(X) \otimes \mathbb{R}$ depends only on Ξ, because it is determined by the exact sequence

$$0 \to M \to \mathbb{Z}^{\Sigma(1)} \to A_{n-1}(X) \to 0,$$

  so we will denote it $A(\Xi)$. The same is true of the effective divisor classes $A^+_{n-1} \otimes \mathbb{R}$, so we will denote this by $A^+(\Xi)$. It is a cone in $A(\Xi)$.

- Recall, we defined what it means for an effective divisor class $a = \sum a_\rho [D_\rho]$ to be convex. Namely, for each $\sigma \in \Sigma$, we know that we can find $m_\sigma \in M_\mathbb{R}$ such that $\langle m_\sigma, v_\rho \rangle = -a_\rho$ for all $\rho \in \sigma(1)$ (v_ρ is the unique integral generator of ρ), because these generators are assumed to be linearly independent. We say that a is convex if for each $\sigma \in \Sigma$, we have $\langle m_\sigma, v_\rho \rangle \geq -a_\rho$ for *all* $\rho \in \Sigma(1)$.

- Let

$$\text{cpl}(\Sigma) = \{a \in A^+(\Xi) \mid a \text{ is convex}\}.$$

  This depends on Σ, but all lie in the common space $A^+(\Xi)$.

- We have stated previously that the Kähler cone of X_Σ is the interior of $\text{cpl}(\Sigma)$, so our question about how the Kähler cones are related can be rephrased as asking how the cpl's are related.

- It would be nice if the answer were that they form a fan that fills up all of $A^+(\Xi)$, but this isn't quite true yet.

- To make it true, we need to vary over all projective simplicial fans Σ satisfying the weaker condition that $\Sigma(1) \subset \Xi$.

- We need to modify the definition of convexity slightly in order for $\text{cpl}(\Sigma)$ to still be a subset of $A^+(\Xi)$ for these more general fans. Namely, define $a = \sum a_\rho [D_\rho] \in A^+(\Xi)$ to be Σ-convex if $\langle m_\sigma, v_\rho \rangle \geq -a_\rho$ for all $\sigma \in \Sigma$ and $\rho \in \Xi$, where m_σ is now defined by the condition that $\langle m_\sigma, v_\rho \rangle = -a_\rho$ only for those v_ρ's that generate elements of $\Sigma(1)$. Let

$$\text{cpl}(\Sigma) = \{a \in A^+(\Xi) \mid a \text{ is } \Sigma\text{-convex}\}.$$

- **Theorem**: Let Ξ be a finite set of strongly convex rational 1-dimensional cones in $N_\mathbb{R}$. As Σ ranges over all projective simplicial fans with $\Sigma(1) \subset \Xi$, the cones $\text{cpl}(\Sigma)$ and their faces form a fan in $A(\Xi)$ whose support is $A^+(\Xi)$. (This is called the secondary fan or GKZ decomposition.)

3 The Enlarged Secondary Fan

- (**Remark**: Cox and Katz just call this the secondary fan, and they call the above the GKZ decomposition.)
- The enlarged secondary fan is a fan that contains the above as a subfan, but which fills up all of $A(\Xi)$. It is defined as follows:
  - Identify Ξ with a subset of $N_{\mathbb{R}}$ by associating each 1-dimensional cone to its primitive integral generator.
  - Define $\Xi^+ = (\Xi \cup \{0\}) \times \{1\} \subset N_{\mathbb{R}} \times \mathbb{R}$.
  - A triangulation of Ξ^+ means a triangulation of the convex hull $\mathrm{Conv}(\Xi^+)$ all of whose vertices are elements of Ξ^+. Such a triangulation is called regular if there is a function on $\mathrm{Conv}(\Xi^+)$ that's linear on simplices and strictly convex
  - The exact sequence
    $$0 \to M_{\mathbb{R}} \to \mathbb{R}^{\Xi} \to A(\Xi) \to 0$$
    can be extended in an obvious way to give
    $$0 \to M_{\mathbb{R}} \oplus \mathbb{R} \to \mathbb{R}^{\Xi^+} \to A(\Xi) \to 0.$$
  - To each $v \in \Xi^+$ we associate the element $e_v^* \in A(\Xi)$ which is the image of the standard basis element of $\mathbb{R}^{\Xi^+}$ corresponding to v. In fact, for $v = (\rho, 1) \neq (0, 1)$, e_v^* is nothing but the divisor class $[D_\rho]$ in X_Σ corresponding to ρ, where Σ is any fan such that $\Sigma(1) = \Xi$.
  - Given any regular triangulation $\mathcal{T}$ of Ξ^+, we define a cone $\mathcal{C}(\mathcal{T}) \subset A(\Xi)$ as follows: For each simplex σ of the triangulation, let $C(\sigma)$ be the cone generated by those e_v^* for which v is not a vertex of σ. Then, set
    $$C(\mathcal{T}) = \bigcap_{\sigma \in \mathcal{T}} C(\sigma).$$
  - These cones and their faces form the enlarged secondary fan.
- To see that this contains the secondary fan, we observe that a regular triangulation $\mathcal{T}$ in which $(0, 1)$ is one of the vertices yields a fan Σ in $N_{\mathbb{R}}$ whose cones are formed by taking the cones from $(0, 1)$ over the faces of simplices of $\mathcal{T}$. This fan satisfies
  $$\Sigma(1) \subset \Xi$$
  and
  $$\mathcal{C}(\mathcal{T}) = \mathrm{cpl}(\Sigma).$$
  Furthermore, $\mathcal{T}$ can be recovered from any Σ such that $\Sigma(1) \subset \Xi$, so all of the cones in the secondary fan appear among the cones $\mathcal{C}(\mathcal{T})$ of the enlarged secondary fan. (Proof by picture.)

4 Relation to the Moment Map

- Recall that for a toric variety $X = X_\Sigma$ determined by a fan Σ in $N_\mathbb{R} \cong \mathbb{R}^n$, there is a moment map $\mu_\Sigma : \mathbb{C}^{|\Sigma(1)|} \to \mathbb{R}^{r-n}$.
- Since the group G for which this is the moment map depends only on $\Sigma(1)$, we can denote it $G(\Xi)$. For the same reason, the moment map depends only on $\Sigma(1)$. Moreover, its image is $A^+(\Xi)$, so we can denote it
$$\mu_\Xi : \mathbb{C}^\Xi \to A^+(\Xi).$$
- **Theorem**: If Σ is a projective simplicial fan with $\Sigma(1) \subset \Xi$ and $a \in A^+(\Xi)$ is in the interior of $\mathrm{cpl}(\Sigma)$, then there is a natural orbifold diffeomorphism $\mu_\Xi^{-1}(a)/G(\Xi)_\mathbb{R} \cong X_\Sigma$.
- Thus, once we have the secondary fan, we can use the moment map μ_Ξ to reconstruct all of the toric varieties whose Kähler cones are represented.

5 More General Secondary Fans

- The secondary fan is actually a more general construction associated to a closed group $G \subset (\mathbb{C}^*)^r$. Its purpose is to classify all of the toric varieties as GIT quotients of $\mathbb{C}^r$ by G.
- A GIT quotient is denoted $\mathbb{C}^r//_\chi G$, as it depends on the choice of a character $\chi \in \hat{G}$.
- Let $C \subset \hat{G}_\mathbb{R}$ be the cone spanned by the vectors $\chi_i \otimes 1$, where $\chi_i \in \hat{G}$ is the character $g \mapsto g_i$.
- The secondary fan is a fan whose support is C, and its key property is that whenever $\chi \otimes 1$ lies in the relative interior of the same cone in the secondary fan, the resulting GIT quotient $\mathbb{C}^r//_\chi G$ is the same up to isomorphism. (And it can be described explicitly.)
- Any toric variety X_Σ is a GIT quotient, where
$$G = \mathrm{Hom}_\mathbb{Z}(A_{n-1}(X_\Sigma), \mathbb{C}^*)$$
and we view G as a subgroup of $(\mathbb{C}^*)^{|\Sigma(1)|}$ by dualizing
$$0 \to M \to \mathbb{Z}^{\Sigma(1)} \to A_{n-1}(X_\Sigma) \to 0$$
to get
$$0 \to G \to (\mathbb{C}^*)^{\Sigma(1)} \to \mathrm{Hom}(M, \mathbb{C}^*) \to 0.$$
In this case, $\hat{G} = A_{n-1}(X_\Sigma)$, so a character $\chi \in \hat{G}$ corresponds to a divisor class in X_Σ. As long as this divisor class is chosen to be ample, the resulting GIT quotient $\mathbb{C}^{\Sigma(1)}//_\chi G$ is X_Σ.
- Note that all of this depends only on $\Sigma(1)$, which we again denote by Ξ.
- The secondary fan associated to G is exactly the secondary fan associated to Ξ described above.