Toric Varieties and the Secondary Fan

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1 Motivation

• The Batyrev mirror symmetry construction for Calabi-Yau hypersurfaces goes roughly as follows:
  
  – Start with an $n$-dimensional reflexive polytope $\Delta$ in $M_{\mathbb{R}}$.
  
  – This gives a toric variety $\mathbb{P}_{\Delta}$. (One way to describe it is it’s the toric variety associated to the normal fan $\Sigma$ in $N_{\mathbb{R}}$, which, since $\Delta$ is reflexive, simply means the fan obtained by taking cones over the proper faces of $\Delta^\circ$.)
  
  – Weil divisors on $\mathbb{P}_{\Delta}$ correspond to polytopes in $M_{\mathbb{R}}$. (One way to state the correspondence is as follows: Given a cone $\rho$ of $\Sigma$, we get an affine variety $X_{\rho} = \text{Spec}(\mathbb{C}[S_{\rho}]) \subset X_{\Sigma} = \mathbb{P}_{\Delta}$, where $S_{\rho} = M \cap \rho$. Points of this affine variety are in bijective correspondence with semigroup homomorphisms $\gamma : S_{\rho} \to \mathbb{C}$, and there is a distinguished point $\gamma_{\rho}$ given by the homomorphism
    
    $$m \mapsto \begin{cases} 
    1 & m \in S_{\rho} \cap \rho^\perp \\
    0 & \text{otherwise}.
    \end{cases}$$
  
    If $\rho$ is a one-dimensional cone of $\sigma$, then let $D_{\rho}$ be the closure of the torus-orbit of $\gamma_{\rho}$, an irreducible subvariety in $\mathbb{P}_{\Delta}$. We have seen that any Weil divisor on $\mathbb{P}_{\Delta}$ can be expressed as $D = \sum_{\rho \in \Sigma(1)} a_{\rho} D_{\rho}$ for integers $a_{\rho}$. We then let
    
    $$\Delta_D = \{m \in M_{\mathbb{R}} \mid \langle m, \rho \rangle \geq -a_{\rho} \ \forall \rho \in \Sigma(1)\},$$
  
  and this is the polytope corresponding to $D$.)
  
  – The polytope corresponding to the anticanonical divisor is precisely $\Delta$.
  
  – Any hypersurface $Y \subset \mathbb{P}_{\Delta}$ whose divisor class is the anticanonical class is Calabi-Yau. (This follows from the adjunction formula, which says that for $D$ a divisor on a complex manifold $M$, one has $K_D \cong (K_M(D))|_D$. The result in this case is that $K_{Y} \cong K_{\mathbb{P}_{\Delta}}(-K_{\mathbb{P}_{\Delta}})|_Y = \mathcal{O}_Y$.)
  
  – Thus, the reflexive polytope $\Delta$ determines a family of Calabi-Yau hypersurfaces in the toric variety $\mathbb{P}_{\Delta}$.
This isn’t quite the family for which we will construct a mirror; first, we want to desingularize it as much as possible while still remaining Calabi-Yau. A desingularization of $\mathbb{P}_\Delta$ is given by adding new cone generators to $\Sigma$ until it is smooth— that is, every cone has generators as part of a $\mathbb{Z}$ basis of $N$. If $Z$ is the toric variety obtained by adding a new cone generator $\rho$ to $\Sigma$, then we get a birational map of toric varieties $f : Z \to \mathbb{P}_\Delta$, and

$$K_Z = f^*(K_{\mathbb{P}_\Delta}) - (\langle \rho, m_F \rangle + 1)D$$

where $\rho$ lies in the facet $\langle u, m_F \rangle = -1$ of $\Delta^\circ$. In particular, the resulting toric variety has the same canonical class as long as the new cone generators we add are chosen from $N \cap \Delta^\circ$. So the most we can desingularize while retaining the property that an anticanonical hypersurface is Calabi-Yau is by taking the toric variety associated to a projective simplicial fan $\Sigma'$ with $\Sigma'(1) = (N \cap \Delta^\circ) \setminus \{0\}$.

For any such $\Sigma'$, denote the family of anticanonical hypersurfaces on $X_{\Sigma'}$ by $\hat{Y}$.

Then we obtain the mirror family of Calabi-Yau toric hypersurfaces by replacing $\Delta$ by $\Delta^\circ$.

- The sense in which these families are expected to be mirror to one another is that the moduli of complex structures on $\hat{Y}$ corresponds to the moduli of Kähler structures on $\hat{Y}^\circ$ and vise versa.

- But this is a local correspondence in the sense that not all of the possible complex/Kähler structures on the underlying Calabi-Yau are represented as it varies through the family of anticanonical hypersurfaces on $X_{\Sigma'}$.

- Indeed, it could not possibly be a global correspondence, because the full moduli space of complex structures forms a quasi-projective variety, while the moduli space of Kähler structures is the quotient of a bounded domain by a finite group, which is a smaller object.

- We would like a global version of the construction, which gives an isomorphism between the moduli space of complex structures on $\hat{Y}$ and an enlarged version of the moduli space of Kähler structures on $\hat{Y}^\circ$.

- The way to obtain this is to allow $\hat{Y}$ to vary not just over the anticanonical hypersurfaces in $X_{\Sigma'}$ for a fixed fan $\Sigma'$, but over all anticanonical hypersurfaces in $X_{\Sigma'}$ for all possible fans $\Sigma'$ satisfying $\Sigma'(1) = (N \cap \Delta^\circ) \setminus \{0\}$.

- The resulting $\hat{Y}$'s are thought to be related by flops, which implies that they have the same moduli space of complex structures.

- Their Kähler cones, however, may be different, and the secondary fan describes how they fit together. This is a larger object than the individual Kähler cones, but it isn’t quite big enough yet. The so-called “enlarged secondary fan” is actually the object that is conjecturally mirror to the moduli space of complex structures.
2 The Secondary Fan

• Once we have fixed a reflexive polytope $\Delta$, we want to examine all fans for which the generators of the one-dimensional cones are the fixed set $(N \cap \Delta^o) \setminus \{0\}$, and ask how the Kähler cones of these are related.

• So fix a finite set of strongly convex rational 1-dimensional cones $\Xi$ in $N_\mathbb{R}$ and consider fans $\Sigma$ with $\Sigma(1) = \Xi$.

**Note:** $A_{n-1}(X) \otimes \mathbb{R}$ depends only on $\Xi$, because it is determined by the exact sequence

$$0 \rightarrow M \rightarrow \mathbb{Z}^{\Sigma(1)} \rightarrow A_{n-1}(X) \rightarrow 0,$$

so we will denote it $A(\Xi)$. The same is true of the effective divisor classes $A_{n-1}^+ \otimes \mathbb{R}$, so we will denote this by $A^+(\Xi)$. It is a cone in $A(\Xi)$.

• Recall, we defined what it means for an effective divisor class $a = \sum a_\rho[D_\rho]$ to be convex. Namely, for each $\sigma \in \Sigma$, we know that we can find $m_\sigma \in M_\mathbb{R}$ such that $\langle m_\sigma, v_\rho \rangle = -a_\rho$ for all $\rho \in \sigma(1)$ ($v_\rho$ is the unique integral generator of $\rho$), because these generators are assumed to be linearly independent. We say that $a$ is convex if for each $\sigma \in \Sigma$, we have $\langle m_\sigma, v_\rho \rangle \geq -a_\rho$ for all $\rho \in \Sigma(1)$.

• Let $cpl(\Sigma) = \{ a \in A^+(\Xi) \mid a \text{ is convex} \}$. This depends on $\Sigma$, but all lie in the common space $A^+(\Xi)$.

• We have stated previously that the Kähler cone of $X_\Sigma$ is the interior of $cpl(\Sigma)$, so our question about how the Kähler cones are related can be rephrased as asking how the $cpl$’s are related.

• It would be nice if the answer were that they form a fan that fills up all of $A^+(\Xi)$, but this isn’t quite true yet.

• To make it true, we need to vary over all projective simplicial fans $\Sigma$ satisfying the weaker condition that $\Sigma(1) \subset \Xi$.

• We need to modify the definition of convexity slightly in order for $cpl(\Sigma)$ to still be a subset of $A^+(\Xi)$ for these more general fans. Namely, define $a = \sum a_\rho[D_\rho] \in A^+(\Xi)$ to be $\Sigma$-convex if $\langle m_\sigma, v_\rho \rangle \geq -a_\rho$ for all $\sigma \in \Sigma$ and $\rho \in \Xi$, where $m_\sigma$ is now defined by the condition that $\langle m_\sigma, v_\rho \rangle = -a_\rho$ only for those $v_\rho$’s that generate elements of $\Sigma(1)$. Let

$$cpl(\Sigma) = \{ a \in A^+(\Xi) \mid a \text{ is } \Sigma\text{-convex} \}.$$

**Theorem:** Let $\Xi$ be a finite set of strongly convex rational 1-dimensional cones in $N_\mathbb{R}$. As $\Sigma$ ranges over all projective simplicial fans with $\Sigma(1) \subset \Xi$, the cones $cpl(\Sigma)$ and their faces form a fan in $A(\Xi)$ whose support is $A^+(\Xi)$. (This is called the secondary fan or GKZ decomposition.)
3 The Enlarged Secondary Fan

- **Remark:** Cox and Katz just call this the secondary fan, and they call the above the GKZ decomposition.

- The enlarged secondary fan is a fan that contains the above as a subfan, but which fills up all of \( A(\Xi) \). It is defined as follows:
  - Identify \( \Xi \) with a subset of \( N_\mathbb{R} \) by associating each 1-dimensional cone to its primitive integral generator.
  - Define \( \Xi^+ = (\Xi \cup \{0\}) \times \{1\} \subset N_\mathbb{R} \times \mathbb{R} \).
  - A triangulation of \( \Xi^+ \) means a triangulation of the convex hull \( \text{Conv}(\Xi^+) \) all of whose vertices are elements of \( \Xi^+ \). Such a triangulation is called regular if there is a function on \( \text{Conv}(\Xi^+) \) that’s linear on simplices and strictly convex.
  - The exact sequence
    \[
    0 \to M_\mathbb{R} \to \mathbb{R}^\Xi \to A(\Xi) \to 0
    \]
    can be extended in an obvious way to give
    \[
    0 \to M_\mathbb{R} \oplus \mathbb{R} \to \mathbb{R}^{\Xi^+} \to A(\Xi) \to 0.
    \]
  - To each \( v \in \Xi^+ \) we associate the element \( e_v^* \in A(\Xi) \) which is the image of the standard basis element of \( \mathbb{R}^{\Xi^+} \) corresponding to \( v \). In fact, for \( v = (\rho, 1) \neq (0, 1) \), \( e_v^* \) is nothing but the divisor class \([D_\rho]\) in \( X_\Sigma \) corresponding to \( \rho \), where \( \Sigma \) is any fan such that \( \Sigma(1) = \Xi \).
  - Given any regular triangulation \( T \) of \( \Xi^+ \), we define a cone \( C(T) \subset A(\Xi) \) as follows: For each simplex \( \sigma \) of the triangulation, let \( C(\sigma) \) be the cone generated by those \( e_v^* \) for which \( v \) is not a vertex of \( \sigma \). Then, set
    \[
    C(T) = \bigcap_{\sigma \in T} C(\sigma).
    \]
  - These cones and their faces form the enlarged secondary fan.

- To see that this contains the secondary fan, we observe that a regular triangulation \( T \) in which \( (0, 1) \) is one of the vertices yields a fan \( \Sigma \) in \( N_\mathbb{R} \) whose cones are formed by taking the cones from \( (0, 1) \) over the faces of simplices of \( T \). This fan satisfies
    \[
    \Sigma(1) \subset \Xi
    \]
    and
    \[
    C(T) = \text{cpl}(\Sigma).
    \]
    Furthermore, \( T \) can be recovered from any \( \Sigma \) such that \( \Sigma(1) \subset \Xi \), so all of the cones in the secondary fan appear among the cones \( C(T) \) of the enlarged secondary fan. (Proof by picture.)
4 Relation to the Moment Map

- Recall that for a toric variety $X = X_\Sigma$ determined by a fan $\Sigma$ in $N_\mathbb{R} \cong \mathbb{R}^n$, there is a moment map $\mu_\Sigma : \mathbb{C}^{[\Sigma(1)]} \to \mathbb{R}^{r-n}$.

- Since the group $G$ for which this is the moment map depends only on $\Sigma(1)$, we can denote it $G(\Xi)$. For the same reason, the moment map depends only on $\Sigma(1)$. Moreover, its image is $A^+(\Xi)$, so we can denote it $\mu_\Xi : \mathbb{C}^\Xi \to A^+(\Xi)$.

- **Theorem**: If $\Sigma$ is a projective simplicial fan with $\Sigma(1) \subset \Xi$ and $a \in A^+(\Xi)$ is in the interior of $cpl(\Sigma)$, then there is a natural orbifold diffeomorphism $\mu_\Xi^{-1}(a)/G(\Xi)_\mathbb{R} \cong X_\Sigma$.

- Thus, once we have the secondary fan, we can use the moment map $\mu_\Xi$ to reconstruct all of the toric varieties whose Kähler cones are represented.

5 More General Secondary Fans

- The secondary fan is actually a more general construction associated to a closed group $G \subset (\mathbb{C}^*)^r$. Its purpose is to classify all of the toric varieties as GIT quotients of $\mathbb{C}^r$ by $G$.

- A GIT quotient is denoted $\mathbb{C}^r//_\chi G$, as it depends on the choice of a character $\chi \in \hat{G}$.

- Let $C \subset \hat{G}_\mathbb{R}$ be the cone spanned by the vectors $\chi_i \otimes 1$, where $\chi_i \in \hat{G}$ is the character $g \mapsto g_i$.

- The secondary fan is a fan whose support is $C$, and its key property is that whenever $\chi \otimes 1$ lies in the relative interior of the same cone in the secondary fan, the resulting GIT quotient $\mathbb{C}^r//_\chi G$ is the same up to isomorphism. (And it can be described explicitly.)

- Any toric variety $X_\Sigma$ is a GIT quotient, where

  $$G = \text{Hom}_\mathbb{Z}(A_{n-1}(X_\Sigma), \mathbb{C}^*)$$

  and we view $G$ as a subgroup of $(\mathbb{C}^*)^{[\Sigma(1)]}$ by dualizing

  $$0 \to M \to \mathbb{Z}^{\Sigma(1)} \to A_{n-1}(X_\Sigma) \to 0$$

  to get

  $$0 \to G \to (\mathbb{C}^*)^{\Sigma(1)} \to \text{Hom}(M, \mathbb{C}^*) \to 0.$$  

  In this case, $\hat{G} = A_{n-1}(X_\Sigma)$, so a character $\chi \in \hat{G}$ corresponds to a divisor class in $X_\Sigma$. As long as this divisor class is chosen to be ample, the resulting GIT quotient $\mathbb{C}^{[\Sigma(1)]}//_\chi G$ is $X_\Sigma$.

- Note that all of this depends only on $\Sigma(1)$, which we again denote by $\Xi$.

- The secondary fan associated to $G$ is exactly the secondary fan associated to $\Xi$ described above.