INTRODUCTION TO THE LANDAU-GINZBURG MODEL

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In this lecture, we will discuss the basic ingredients of the Landau-Ginzburg model associated to certain polynomials $W \in \mathbb{C}[x_1, \ldots, x_N]$. The mathematical framework for the Landau-Ginzburg model is referred to as **FJRW theory**, since it was developed by Fan-Jarvis-Ruan [3] [4] [5] and was motivated by a proposal of Witten. Like the Gromov-Witten theory of a space X, the FJRW theory of a polynomial consists, schematically, of the following pieces of information:

(1) A "state space" $\mathcal{H}$, which is a vector space equipped with a nondegenerate pairing. This is analogous to the role played by $H^*(X)$ with its Poincaré pairing in Gromov-Witten theory.
(2) A moduli space $\mathcal{W}^{W,G}_{g,n}$ equipped with a virtual cycle. This is analogous to the moduli space of stable maps; as in that case, it parameterizes genus-g n-marked curves with some additional structure.
(3) A notion of correlators of any genus g, which are maps associating to any collection of n elements of $\mathcal{H}$ an integral over $\mathcal{W}^{W,G}_{g,n}$ against its virtual cycle.

The definitions of these objects for the Landau-Ginzburg model may appear strange at a first glance. Two facts are useful to keep in mind in order to keep the discussion slightly more grounded. First, this theory generalizes the better-known r-spin theory, which relates to curves equipped with an rth root of the canonical bundle. Second, according to the Landau-Ginzburg/Calabi-Yau correspondence, the Landau-Ginzburg model associated to certain W should "match", in a precise sense, the Gromov-Witten theory of the hypersurface cut out by W. We will describe this correspondence in detail in the next lecture, and show how it can be used to recover all of the definitions on the Landau-Ginzburg side in a way that much more closely parallels the Gromov-Witten setting.

1. Quasihomogeneous singularities and their symmetries

We begin by specifying the types of polynomials for which FJRW theory is defined.

Definition 1.0.1. A polynomial $W \in \mathbb{C}[x_1, \ldots, x_N]$ is **quasihomogeneous** if there exist positive integers $c_1, \ldots, c_N$ (known as **weights**) and d (the **degree**) such that

$$W(\lambda^{c_1} x_1, \ldots, \lambda^{c_N} x_N) = \lambda^d W(x_1, \ldots, x_N)$$

for all $\lambda \in \mathbb{C}$. The rational numbers $q_i := c_i/d$ are referred to as **charges** of W.

Definition 1.0.2. A quasihomogeneous polynomial W is **nondegenerate** if

(1) W defines a nonsingular hypersuface X_W in the weighted projective space $\mathbb{P}(c_1, \ldots, c_N)$;
(2) the charges are uniquely determined by W.

Definition 1.0.3. Given a nondegenerate quasihomogeneous polynomial W, the **maximal diagonal symmetry group** is

$$G_W := \{(\alpha_1, \ldots, \alpha_N) \in (\mathbb{C}^*)^N \mid W(\alpha_1 x_1, \ldots, \alpha_N x_N) = W(x_1, \ldots, x_N)\}.$$

Nondegeneracy can be used to show that G_W is finite. Furthermore, it always contains the element

$$J := \left(e^{2\pi i \frac{1}{q_1}}, \ldots, e^{2\pi i \frac{1}{q_N}}\right),$$

which plays a special role in the theory.

2. State space

Let W be a nondegenerate quasihomogeneous polynomial, and let $G \subset G_W$ be a subgroup containing J.

2.1. The state space as a vector space.

Definition 2.1.1. The (A-model) **Landau-Ginzburg state space** associated to (W, G) is the vector space

$$\mathcal{H}_{W,G} = H^*_{CR}([\mathbb{C}^N/W], W^{+\infty}; \mathbb{Q}),$$

in which $W^{+\infty} = \mathrm{Re}(W)^{-1}(\rho, \infty)$ for $\rho \gg 0$.

We have not defined relative Chen-Ruan cohomology, but the definition is a generalization of the contents of the previous lecture. In particular, it follows that

$$\mathcal{H}_{W,G} = \left(\bigoplus_{g \in G} H^*(\mathbb{C}^N_g, W_g^{+\infty}; \mathbb{Q}) \right)^G,$$

where $\mathbb{C}^N_g$ is the fixed locus of g and W_g is the restriction of W to this fixed locus. The action of G should take the summand indexed by g to the summand indexed by $h^{-1}gh$ via pullback under multiplication by h; since any group of diagonal symmetries is abelian, though, this is actually a G-action on each sector separately.

The relative cohomology groups appearing in each of the above summands vanish away from degree $N_g = \dim_{\mathbb{C}}(\mathbb{C}^N_g)$, so we can write

$$\mathcal{H}_{W,G} = \left(\bigoplus_{g \in G} H^{N_g}(\mathbb{C}^N_g, W_g^{+\infty}; \mathbb{Q}) \right)^G.$$

Example 2.1.2. Let $W = x_1^d + \cdots + x_N^d$, the **Fermat polynomial** of degree d. Then

$$G_W = \{(\alpha, \ldots, \alpha) \mid \alpha^d = 1\} \cong \mathbb{Z}_d,$$

so it is generated by $J = (e^{2\pi i \frac{1}{d}}, \ldots, e^{2\pi i \frac{1}{d}})$. The identity element $1 \in G_W$ obviously fixes all of $\mathbb{C}^N$, while any nontrivial g fixes only the origin. Thus:

$$\begin{aligned} \mathcal{H}_{W,G_W} &= \left(H^N(\mathbb{C}^N, W^{+\infty}) \oplus \bigoplus_{g \neq 1 \in \mathbb{Z}_d} H^0(\{0\}, \emptyset) \right)^{\mathbb{Z}_d} \\ &= H^N(\mathbb{C}^N, W^{+\infty})^{\mathbb{Z}_d} \oplus \mathbb{Q}^{d-1}. \end{aligned}$$

Example 2.1.3. Let $W(x,y,z) = x^3y + yz^2 + z^4$. It is straightforward to check that this is quasihomogeneous with weights

$$c_x = 2,\ c_y = 6,\ c_z = 3$$

and degree $d = 12$. The maximal diagonal symmetry group is

$$G_W = \{(\lambda_x, \lambda_x^{-3}, \lambda_z) \mid \lambda_z^4 = 1,\ \lambda_z^2 = \lambda_x^3\},$$

while

$$J = (e^{2\pi i \frac{1}{6}}, e^{2\pi i \frac{1}{2}}, e^{2\pi i \frac{1}{4}}),$$

which generates a proper subgroup. Even when $G = J$, there are sectors of $\mathcal{H}_{W,G}$ for which $N_g = 3$ (the nontwisted sector), $N_g = 2$, $N_g = 1$, or $N_g = 0$.

Just as Chen-Ruan cohomology has a decomposition induced by the twisted sectors of the inertia stack, the Landau-Ginzburg state space has a decomposition into sectors indexed by $g \in G$. These sectors come in two basic types:

Definition 2.1.4. If the action of $g \in G$ on $\mathbb{C}^N$ fixes only $\{0\}$, then the sector $H^{N_g}(\mathbb{C}^N_g, W^{+\infty}_g) \cong \mathbb{Q}$ of $\mathcal{H}_{W,G}$ is called **narrow**. Otherwise, the sector is called **broad**.[1]

2.2. **Pairing.** Again in analogy to the Poincaré pairing on Chen-Ruan cohomology, the pairing on $\mathcal{H}_{W,G}$ is given by matching the sector indexed by g with the sector indexed by g^{-1}.

Let us begin with the nontwisted sector $g = 1$, which pairs with itself. First of all, we observe that there is a pairing

$$\langle\ ,\ \rangle^{-} : H^N(\mathbb{C}^N, W^{-\infty}; \mathbb{Q}) \otimes H^N(\mathbb{C}^N, W^{+\infty}; \mathbb{Q}) \to \mathbb{Q}, \tag{1}$$

where

$$W^{-\infty} = (\mathrm{Re}W)^{-1}(-\infty, -\rho)$$

for $\rho \gg 0$. To see this, note that the vector spaces $H^N(\mathbb{C}^N, W^{\pm\infty}; \mathbb{Q})$ are dual to the homology $H_N(\mathbb{C}^N, W^{\pm\infty}; \mathbb{Q})$, and the latter has a known basis in terms of **Lefschetz thimbles**. Namely, one can choose a perturbation W_a of W to a holomorphic Morse function, and then choose a collection of nonintersecting paths in $\mathbb{C}$ that begin at critical values of W_a and move in the direction of $\mathrm{Re}(z) = \infty$, eventually becoming horizontal. The preimages of these paths in $\mathbb{C}^N$ form a basis for $H_N(\mathbb{C}^N, W^{+\infty}; \mathbb{Q})$, and similarly, the "opposite" thimbles form a basis for $H_N(\mathbb{C}^N, W^{-\infty}; \mathbb{Q})$. (See Figure 1.)

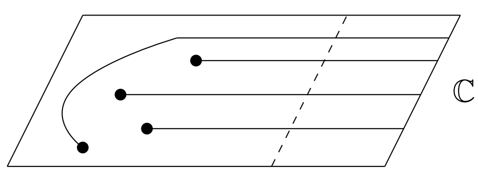

FIGURE 1. The preimages in $\mathbb{C}^N$ of these paths are closed at one end and open at the other, giving them the appearance of infinite "thimbles".

The pairing (1) is obtained by dualizing the intersection pairing

$$H_N(\mathbb{C}^N, W^{-\infty}; \mathbb{Q}) \otimes H_N(\mathbb{C}^N, W^{+\infty}; \mathbb{Q}) \to \mathbb{Q}.$$

[1] These situations were referred to as **Neveu-Schwarz** and **Ramond** in some previous literature.

From here, choose a dth root of unity ξ and define

$$I : \mathbb{C}^N \to \mathbb{C}^N$$

$$(x_1, \ldots, x_N) \mapsto (\xi^{c_1} x_1, \ldots, \xi^{c_N} x_N),$$

which interchanges $W^{+\infty}$ and $W^{-\infty}$. The desired pairing on the nontwisted sector of $\mathcal{H}_{W,G}$ is

$$\langle \, , \, \rangle : H^N(\mathbb{C}^N, W^{+\infty}; \mathbb{Q})^G \otimes H^N(\mathbb{C}^N, W^{+\infty}; \mathbb{Q})^G \to \mathbb{Q}$$

$$\langle \alpha, \beta \rangle = \langle \alpha, I^* \beta \rangle^-.$$

The restriction to J-invariants (and, in particular, to G-invariants for any G containing J) is sufficient to ensure that this pairing does not depend on the choice of ξ.

Having defined the pairing on the nontwisted sector, the pairing between twisted sectors indexed by g and g^{-1} is defined by applying the above construction to the restricted polynomial W_g.

2.3. **Grading.** Just as in the case of Chen-Ruan cohomology, a grading on $\mathcal{H}_{W,G}$ should be chosen so that elements pair nontrivially only when their degrees add up to the "dimension". The correct notion of dimension in this case is the following:

Definition 2.3.1. The **central charge** of a quasihomogeneous polynomial W is

$$c_W := \sum_{i=1}^{N} (1 - 2q_i).$$

For example, when $\sum q_i = 1$, the polynomial is said to be **Calabi-Yau**, since it defines a Calabi-Yau hypersurface $X_W \subset \mathbb{P}(c_1, \ldots, c_N)$. In this case, $c_W = N - 2$, which is the complex dimension of X_W.

Associated to each $\gamma \in G_W$, and hence to each sector of the state space, there is a degree-shifting number

$$\iota_\gamma = \sum_{i=1}^{N} (\Theta_i^\gamma - q_i),$$

where $\Theta_i^\gamma \in [0, 1)$ is defined by writing

$$\gamma = (e^{2\pi i \Theta_1^\gamma}, \ldots, e^{2\pi i \Theta_N^\gamma}) \in (\mathbb{C}^*)^N.$$

It should be noted that this is *not* the same degree-shifting number as in the definition of $H^*_{CR}([\mathbb{C}^N/G])$, since it includes extra terms coming from the charges of W.

The grading on $\mathcal{H}_{W,G}$ is given by

$$\mathcal{H}^i_{W,G} = \bigoplus_{\gamma \in G} \mathcal{H}_\gamma^{i-2\iota_\gamma},$$

in which

$$\mathcal{H}_\gamma = H^{N_\gamma}(\mathbb{C}^{N_\gamma}, W_\gamma^{+\infty}; \mathbb{Q})^G$$

is the sector indexed by γ. It is straightforward to check that, indeed, the pairing on $\mathcal{H}_{W,G}$ is the direct sum of pairings

$$\mathcal{H}^i_{W,G} \otimes \mathcal{H}^{2c_W - i}_{W,G} \to \mathbb{Q}.$$

2.4. **B-model.** We have referred to $\mathcal{H}_{W,G}$ as the "A-model" state space, so we should mention briefly its B-model analogue.

Definition 2.4.1. The **Milnor ring** of a polynomial $W \in \mathbb{C}[x_1, \ldots, x_N]$ is

$$\mathcal{Q}_W := \frac{\mathbb{C}[x_1, \ldots, x_N]}{\left(\frac{\partial W}{\partial x_1}, \ldots, \frac{\partial W}{\partial x_N}\right)}.$$

This much more concrete object actually coincides with the state space defined above:

Fact 2.4.2. [7][8][9] There is an isomorphism of vector spaces

$$\mathcal{Q}_W \cong H^N(\mathbb{C}^N, W^{+\infty}; \mathbb{C}), \tag{2}$$

which restricts to an isomorphism on G_W-invariant subspaces under an appropriate definition of the G_W-action on the Milnor ring. Furthermore, the Milnor ring can be given a natural pairing that matches the pairing on the A-model state space.

There is a precisely analogous "orbifolding" construction

$$\mathcal{Q}_{W,G} := \left(\bigoplus_{g \in G} \mathcal{Q}_{W_g} \right)^G,$$

where W_g is again the restriction of W to the fixed locus of g. Combining the isomorphisms obtained from (2) on each sector, one obtains an isomorphism

$$\mathcal{Q}_{W,G} \cong \mathcal{H}_{W,G}$$

that respects the pairing on each side. However, we should note that the left-hand side also admits a natural grading, and this "B-model grading" does not match the grading defined above.

We will not use the B-model description of the state space in our discussion of the Landau-Ginzburg model, but it is a very useful computational tool, since it is rather more concrete than the A-model perspective.

3. Moduli space

Write a quasihomogeneous polynomial $W = \sum_{i=1}^{s} c_i W_i$ as a sum of distinct monomials $W_i = \prod_{j=1}^{N} x_\ell^{b_{ij}}$ with coefficient one, where $c_i \neq 0$. In order to avoid certain technicalities, we will assume that $s = N$, a condition referred to as "invertibility".

To define the moduli space associated to (W, G), we first treat the case where $G = G_W$. Then $\mathcal{W}_{g,n}^{W,G_W}$ parameterizes tuples

$$(C; x_1, \ldots, x_n; L_1, \ldots, L_N; \phi_1, \ldots, \phi_s),$$

in which

(1) $(C; x_1, \ldots, x_n)$ is a genus-g, n-pointed stable orbifold curve—this means, in particular, that C is a curve with at worst nodal singularities and with nontrivial orbifold structure only at special points[2];
(2) $L_1, \ldots, L_N$ are orbifold line bundles on C;
(3) $\phi_1, \ldots, \phi_s$ are isomorphisms $W_i(L_1, \ldots, L_N) \cong \omega_{C,log}$, in which inserting line bundles into a monomial is defined by interpreting multiplication as tensor product.

Such a tuple is referred to as a W-**structure**[3] on the pointed curve C.

Example 3.0.3. When $W = x^r$, we have $G_W = \mathbb{Z}_r$, and $\mathcal{W}_{g,n}^{W,G_W}$ is the moduli space of r-spin structures.

From this example, we can already see the necessity for the presence of orbifold structure in the definition of the moduli space. Indeed, the line bundle $\omega_{C,\log}$ might not have an rth tensor root on an ordinary curve C, if its degree $2g-2+n$ is not a multiple of r; worse yet, the number of such roots can change when a curve degenerates. If C is allowed orbifold structure at the marked points and nodes, however, then these problems do not arise.

An important feature of orbifold line bundles is the notion of multiplicity.

[2] A careful definition of orbifold curve can be found in [1]. One other important technical point in the definition is that nodes should be "balanced", meaning that they are given locally by $\{xy = 0\}/\mathbb{Z}_k$ in which $\mathbb{Z}_k$ acts by $(x, y) \mapsto (\zeta_k x, \zeta_k^{-1} y)$ for a kth root of unity ζ_k.

[3] To be more precise, a condition is required on the isotropy groups of C at special points in order to ensure that the moduli space of W-structures is proper. There are different ways to achieve this. In [5], the order of the isotropy group at a point p is fixed by requiring that the representation $G_p \to (\mathbb{C}^*)^N$ defined by the multiplicities of $L_1, \ldots, L_N$ is faithful; see below. In [2], every special point is declared to have isotropy group $\mathbb{Z}_d$.

Definition 3.0.4. Let L be an orbifold line bundle on an orbifold curve C. Then, for any point $x \in C$, the isotropy group G_x acts linearly on the fiber of L over x. Recall that $G_x \cong \mathbb{Z}_r$ for some r and is equipped with a choice of generator 1_r. The **multiplicity** of L at x is defined as the number $m \in \{0, 1, \ldots, r-1\}$ for which the action of 1_r on the fiber of L is given by multiplication by $e^{2\pi i m/r}$.

Proposition 3.0.5. *Let $(C; x_1, \ldots, x_n; L_1, \ldots, L_N; \phi_1, \ldots, \phi_s)$ be a W-structure, and let x_i be a marked point with isotropy group $\mathbb{Z}_{r_i}$. Denote by $m_{ij} \in \{0, 1, \ldots, r_i - 1\}$ the multiplicity of L_j at x_i. Then*

$$(e^{2\pi i \frac{m_{i1}}{r_i}}, \ldots, e^{2\pi i \frac{m_{iN}}{r_i}}) \in (\mathbb{C}^*)^N$$

lies in G_W.

The proof of this proposition is a straightforward computation using the fact that $\omega_{C,\log}$ is pulled back from the coarse underlying curve $|C|$, and hence has multiplicity zero; see Lemma 2.1.18 of [5].

There is a decomposition of $\mathcal{W}_{g,n}^{W,G}$ into open and closed substacks dictated by these multiplicities:

$$\mathcal{W}_{g,n}^{W,G} = \bigsqcup_{(\gamma_1, \ldots, \gamma_n) \in G_W^n} \mathcal{W}_{g,n}(\gamma_1, \ldots, \gamma_n), \tag{3}$$

where $\mathcal{W}_{g,n}(\gamma_1, \ldots, \gamma_n)$ denotes the substack in which the multiplicities of $(L_1, \ldots, L_N)$ at ith marked point are given by the element $\gamma_i \in G_W \subset (\mathbb{C}^*)^N$.

Furthermore, there is a compatibility condition on $\gamma_1, \ldots, \gamma_n$ that is necessary (and sufficient) in order for them to index a nonempty substack. Let us work this out explicitly in the r-spin case, in which a W-structure is an orbifold line bundle L satisfying

$$L^{\otimes r} \cong \omega_{C,\log}.$$

Let $\epsilon : C \to |C|$ be the morphism that forgets the orbifold structure on C. This is a flat morphism, so $|L| := \epsilon_* L$ is a line bundle on $|C|$. However, it is no longer necessarily a root of the log-canonical bundle, but instead satisfies the equation

$$|L|^{\otimes r} \cong \omega_{|C|,\log} \otimes \mathcal{O}(-m_1[x_1] - \cdots - m_n[x_n]),$$

where m_i is the multiplicity of L at the ith marked point. (This fact can be proved by finding a local trivialization for $|L|$, as explained, for example, in Section 2.1.4 of [6].) In particular, since $|L|$ is an honest, non-orbifold line bundle, it has integral degree, and hence:

$$\sum_{i=1}^{n} m_i \equiv 0 \mod r. \tag{4}$$

We find, then, that if $\zeta_r = e^{2\pi i/r}$, the substack $\mathcal{W}_{g,n}(\zeta_r^{m_1}, \ldots, \zeta_r^{m_n})$ of the moduli space of r-spin structures is nonempty only when (4) is satisfied.

From the decomposition (3), we are finally ready to define the moduli space for (W, G) when G is a proper subgroup of G_W. Namely, we restrict to those components indexed by elements of G:

$$\mathcal{W}_{g,n}^{W,G} := \bigsqcup_{(\gamma_1,\ldots,\gamma_n)\in G^n} \mathcal{W}_{g,n}(\gamma_1, \ldots, \gamma_n).$$

4. Correlators

Let us assume, for the time being, that the moduli space $\mathcal{W}_{g,n}^{W,G}$ admits a virtual cycle. We will give a construction of the virtual cycle, at least in certain cases, in our discussion of the LG/CY correspondence next lecture.

Correlators should be defined by integrating cohomology classes against the virtual cycle, and these cohomology classes should be dictated by elements of $\mathcal{H}_{W,G}$. It is far from obvious, though, how one should associate classes on $\mathcal{W}_{g,n}^{W,G}$ to state space vectors.

Let us first consider the case of narrow insertions. Recall that a sector $\mathcal{H}_\gamma \subset \mathcal{H}_{W,G}$ indexed by $\gamma \in G$ is called narrow if γ fixes only the origin in $\mathbb{C}^N$. In this case, $\mathcal{H}_\gamma = \mathbb{Q}$, and we denote the generator by e_γ.

Suppose that $\phi_1, \ldots, \phi_n \in \mathcal{H}_{W,G}$ are all drawn from narrow sectors, so we can write $\phi_i = c_i e_{\gamma_i}$. Then define **correlators** by

$$\langle \tau_{a_1}(\phi_1), \ldots, \tau_{a_n}(\phi_n)\rangle_{g,n} := \left(\prod_{i=1}^{n} c_i^{\circ}\right.$$

$2cmight)\cdot\int_{[\mathcal{W}_{g,n}(\gamma_1,\ldots,\gamma_n)]^{\mathrm{vir}}} \psi_1^{a_1}\cdots\psi_n^{a_n}$for any non-negative integers $a_1, \ldots, a_n$. The ψ classes appearing in this formula are defined on the moduli space $\mathcal{W}_{g,n}^{W,G}$ in exactly the same way that they are defined in Gromov-Witten theory, as the first Chern class of the cotangent line bundles at the marked points.

For some polynomials W, the above is sufficient to define all correlators. When $W = x_1^d + \cdots + x_N^d$, for example, one can simply set a correlator to zero if any of its insertions come from the broad sector. In general, however, such a definition would not yield a Cohomological Field Theory, so a nontrivial definition of broad correlators must be given. This is difficult for two reasons: first, broad sectors are more than 1-dimensional, so we must find a more subtle way to arrive at a cohomology class on the moduli space against which to integrate. Furthermore, as we will see in the next lecture, the method we will use to define the virtual cycle breaks down on components $\mathcal{W}_{g,n}(\gamma_1, \ldots, \gamma_n)$ for which at least one γ_i is broad.

There is an alternative (analytic, rather than algebraic) construction of the virtual cycle that is valid on all components. This construction is quite complicated to define, but it has the interesting feature that it naturally yields an element

$$[\mathcal{W}_{g,n}(\gamma_1, \ldots, \gamma_n)]^{\text{vir}} \in H^*(\mathcal{W}_{g,n}(\gamma_1, \ldots, \gamma_n)) \otimes \mathcal{H}_{\gamma_1}^{\vee} \otimes \cdots \otimes \mathcal{H}_{\gamma_n}^{\vee} \tag{5}$$

rather than simply a class on the moduli space. Thus, given such a virtual cycle, one immediately obtains correlators

$$\langle \tau_{a_1}(\phi_1), \ldots, \tau_{a_n}(\phi_n) \rangle_{g,n}$$

by pairing $\phi_i \in \mathcal{H}_{\gamma_i}$ against the $\mathcal{H}_{\gamma_i}^{\vee}$ part of the virtual cycle, and then integrating $\psi_1^{a_1} \cdots \psi_n^{a_n}$ against the cohomology part.

Let us give a very vague indication of why the virtual class lives in this tensor product. Associated to a nondegenerate quasihomogeneous polynomial is an equation known as the **Witten equation**:

$$\overline{\partial} u_i + \overline{\frac{\partial W}{\partial u_i}} = 0, \tag{6}$$

where $u_1, \ldots, u_n$ are sections of an appropriate line bundle. Witten conjectured that the theory of solutions to this equation should be governed by a certain integral hierarchy, generalizing his conjecture that the intersection theory of the moduli space of curves is governed by the KdV hierarchy. It is not clear, however, that the two terms of (6) are even sections of the same bundle; the moduli space $\mathcal{W}_{g,n}^{W,G}$ is constructed precisely so that, if u_i is a section of L_i, then equation (6) makes sense.

Since W has only a single, highly-degenerate critical point at $x = 0$, it is very difficult to solve the Witten equation even after making sense of it. It is much easier to solve a perturbed equation $W + W_0$ whose restriction to the fixed point set $\text{Fix}(\gamma) \subset \mathbb{C}^N$ for each $\gamma \in G$ is a holomorphic Morse function whose critical values have distinct imaginary parts. Such a W_0 is called a **strongly regular perturbation**.

Given a strongly regular perturbation, Fan-Jarvis-Ruan [4] constructed a virtual cycle on a different moduli space $\mathcal{W}_{g,n}^{s}(\gamma_1, \ldots, \gamma_n)$ that admits a proper, quasi-finite map to $\mathcal{W}_{g,n}(\gamma_1, \ldots, \gamma_n)$. It can be pushed forward to the moduli space we care about, but the result will depend on the perturbation W_0. More precisely, there is a chamber structure on the space of perturbations, in which the walls are places where the imaginary parts of the critical values of $W + W_0$ collide. There is a wall-crossing formula describing the way Fan-Jarvis-Ruan's virtual cycle changes when W_0 crosses such a wall, and this formula is given in terms of Lefschetz thimbles. Thus, a virtual cycle that is independent of the perturbation will necessarily involve the duals of these Lefschetz thimbles, yielding (5).

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