ORBIFOLDS AND ORBIFOLD COHOMOLOGY

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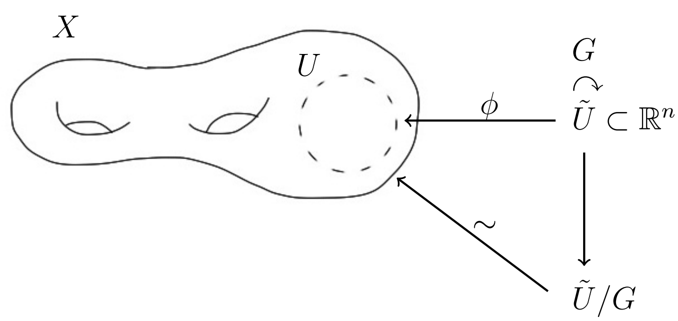
1. What is an orbifold?

Roughly speaking, an orbifold is a topological space that is locally homeomorphic to the quotient of an open subset of Euclidean space by the action of a finite group. Just as the definition of a manifold can be made precise in terms of charts, one can define an **orbifold chart** on a topological space X. Since we will not need this definition, let us simply illustrate it pictorially rather than setting it out in words:



From here, the notion of orbifold atlas can be specified, with two atlases being declared equivalent if they admit a common refinement. Then, in exact analogy to the definition of a manifold, one can define an orbifold as a topological space X (assumed to satisfy some basic niceness conditions) equipped with an equivalence class of orbifold atlases; see Definition 1.1.1 of [1].

The main observation we would like to make about this definition is that an orbifold chart contains more data than simply the topological quotient $\tilde{U}/G$. In particular, an orbifold "remembers" where the G-actions in each of its charts have isotropy.

Example 1.0.1. Let $\mathbb{Z}_n$ act on $\mathbb{C}$ by multiplication by nth roots of unity. Then there is an orbifold $\mathcal{X} = [\mathbb{C}/\mathbb{Z}_n]$ with a single, global chart $\phi = \mathrm{id} : \mathbb{R}^2 \to \mathbb{C}$. Though the quotient $\mathbb{C}/\mathbb{Z}_n$ is topologically still $\mathbb{C}$, the orbifold $\mathcal{X}$ contains the further information of the $\mathbb{Z}_n$ isotropy of the action at the origin. For this reason, $\mathcal{X}$ is typically depicted as a complex plane with an "orbifold point" at the origin— that is, a point carrying the data of the group $\mathbb{Z}_n$.

Example 1.0.2. More generally, if M is a smooth manifold and G is a finite group acting smoothly on M, then one can form an orbifold $[M/G]$; this follows from the fact that any point $x \in M$ with isotropy group $G_x \subset G$ is contained in a G_x-invariant chart. Orbifolds of this form are referred to as **global quotients**.

Example 1.0.3. Let $\mathbb{C}^*$ act on $\mathbb{C}^{n+1}$ by

$$\lambda(z_0, \ldots, z_n) = (\lambda^{c_0} z_0, \ldots, \lambda^{c_n} z_n),$$

in which the c_i are coprime positive integers. Then the quotient

$$\mathbb{P}(c_0, \ldots, c_n) := \mathbb{C}^{n+1}/\mathbb{C}^*$$

can be given the structure of an orbifold, called **weighted projective space**. The underlying manifold X is the projective space $\mathbb{P}^n$, and the coordinate points $p_i = [0 : \cdots : 1 : \cdots : 0]$ have isotropy group $\mathbb{Z}_{c_i}$, while all other points have trivial isotropy. It can be shown (Example 1.53 of [1]) that $\mathbb{P}(c_0, \ldots, c_n)$ is not presentable as a global quotient.

All of this can be cast in the language of **groupoids**— that is, categories in which every morphism is an isomorphism— and more specifically, of **Lie groupoids**, in which the objects and morphisms both form smooth manifolds and all of the structure morphisms of the category are smooth. To compare with the previous description of orbifolds, the objects of the category should be thought of as the points in the charts $\tilde{U}$, and arrows between objects as indicating elements of the local groups G sending one point to another. Certain technical conditions are required in order to ensure that this definition of orbifold agrees with the previous one; in particular, it should be the case that each object x has a finite group G_x of self-arrows and that G_x acts on a neighborhood of x in the manifold of objects. See Definition 1.38 of [1] and the discussion preceding it for details.

Example 1.0.4. Let $[M/G]$ be a global quotient orbifold. Then there is a category $\mathcal{X}$ in which the objects are M and the morphisms are $M \times G$, with one morphism $x \to g \cdot x$ for each $(x, g) \in M \times G$.

Though admittedly more abstract, this category-theoretic language has the advantage of generalizing immediately to the case of ineffective group actions.[1]

[1]There are other reasons why this language is preferable. One reason has to do with the notion of orbifold morphisms, which are surprisingly subtle to define but can be made precise in the groupoid context. Another is that, historically, some of

Example 1.0.5. Let G be a finite group. Then one can form an orbifold $BG := [\bullet/G]$ by allowing G to act trivially on a point. In terms of groupoids, this is the category with one object and morphisms given by G.

Example 1.0.6. In the definition of weighted projective space given in Example 1.0.3, allowing the integers $c_0, \ldots, c_n$ to have a common factor d produces an orbifold in which every chart $(\tilde{U}, G)$ has a subgroup $\mathbb{Z}_d \subset G$ acting ineffectively; in other words, every point has $\mathbb{Z}_d$ isotropy, and the coordinate points $[0 : \cdots : 1 : \cdots : 0]$ have a larger isotropy group containing $\mathbb{Z}_d$. As a groupoid, $\mathbb{P}(c_0, \ldots, c_n)$ has objects $\mathbb{C}^{n+1}$ and morphisms $\mathbb{C}^{n+1} \times \mathbb{C}^*$, just as in the groupoid construction of a global quotient.

2. Orbifold bundles and orbifold de Rham cohomology

All of the geometric objects that one might associate to a manifold can be extended to orbifolds. Most importantly for us, there is a notion of an orbifold vector bundle (and in particular, of a tangent and cotangent bundle to an orbifold) and of de Rham cohomology.

The general principle when defining the orbifold analogues of such concepts is that one should specify the appropriate manifold data on each chart, and it should be equivariant with respect to the chart's G-action. This is easiest to make precise in the case of global quotients:

Definition 2.0.7. Let $\mathcal{X} = [M/G]$ be a (not necessarily effective) global quotient orbifold; see Example 1.0.4. Then an **orbifold vector bundle** over $\mathcal{X}$ is a vector bundle $\pi : E \to M$ equipped with a G-action taking the fiber of E over $x \in M$ to the fiber over gx via a linear map.

Definition 2.0.8. A **section** of an orbifold vector bundle over $[M/G]$ is a G-equivariant section of $\pi : E \to M$.

These notions generalize to arbitrary orbifolds. An orbifold vector bundle over an orbifold presented by a groupoid $\mathcal{X}$, for example, is a vector bundle E over the objects of $\mathcal{X}$, with a linear map $E_x \to E_y$ for each arrow $g : x \to y$ of $\mathcal{X}$.

In particular, the **tangent bundle** to a groupoid can be constructed by taking the tangent bundle to the objects (in the global quotient case, this is TM) and allowing arrows to act by the derivative of their action on objects.

the first spaces whose orbifold structure was put to serious use were moduli spaces (in which the isotropy groups encode automorphisms of the objects parameterized), and these arise very naturally as categories.

In this way, one arrives at the definition of differential p-forms on an orbifold; they are sections of the orbifold bundle $\bigwedge^p T^*\mathcal{X}$. As usual, the case of global quotients is easiest to understand: a differential form on $[M/G]$ is a G-invariant differential form on M. It is straightforward to check that the exterior derivative on M (or, for a more general orbifold, on the objects of $\mathcal{X}$) preserves G-invariance. Hence, the de Rham complex and the **orbifold de Rham cohomology** can be defined.

Integration on a global quotient $\mathcal{X} = [M/G]$ is defined by

$$\int_{\mathcal{X}} \omega := \frac{1}{|G|}\int_M \omega,$$

where $\omega \in \Omega^p(M)$ is a G-invariant differential form. More generally, one can extend the definition of integration to arbitrary orbifolds by working in charts via a partition of unity.

3. The need for a new cohomology theory

For the remainder of these notes, $\mathcal{X}$ will be assumed to be a complex orbifold; to put it concisely, this means that the defining data of the groupoid are not just smooth but holomorphic. An "orbifold curve" will refer to an orbifold of complex dimension one.

The first indication that orbifold de Rham cohomology is insufficient for a true study of orbifolds comes from the following theorem:

Theorem 3.0.9 (Satake). *There is an isomorphism*

$$H^*_{dR}(\mathcal{X}) \cong H^*(|\mathcal{X}|;\mathbb{R}),$$

where $|\mathcal{X}|$ is the **orbit space** *of $\mathcal{X}$— that is, the quotient of the objects of $\mathcal{X}$ by the identification $x \sim y$ if there exists an arrow $x \to y$— and the right-hand side denotes singular cohomology.*

This implies that orbifold de Rham cohomology sees nothing of the isotropy groups, but only the topological quotients (or "coarse underlying spaces") $\tilde{U}/G$ in each chart.

The appropriate definition of cohomology for orbifolds is, in fact, inspired by Gromov-Witten theory: one should begin by defining not ordinary cohomology but *quantum* cohomology, and then restrict to the degree-zero part to recover a definition of cohomology for orbifolds.

How, then, should orbifold quantum cohomology be defined? In pointing toward the correct definition, the first key observation is that the structure of evaluation morphisms should be somewhat richer in this setting. The reason for this lies in the definition of a morphism between orbifolds. While we will not make this definition precise (indeed, to do so is somewhat subtle, as the atlas on the source may need to be refined; see Section 2.4 of [1]), we remark that as a consequence, an orbifold morphism $\mathcal{X} \to \mathcal{Y}$ includes the data of homomorphisms on isotropy groups

$$\lambda_x : G_x \to G_z$$

for each object x of $\mathcal{X}$, in which z is the image (or, to be more precise, *an* image) of x. See Section 2.5 of [1] for a more precise version of this statement.

Suppose, then, that we have defined a moduli space $\overline{\mathcal{M}}_{0,n}(\mathcal{X}, \beta)$ of maps f from a genus-zero n-pointed complex *orbifold* curve $\mathcal{C}$ to a fixed complex orbifold $\mathcal{X}$. Then each marked point $x_i \in \mathcal{C}$ carries two pieces of local data: the image of x_i under f, and the homomorphism λ_{x_i} from the isotropy group of $\mathcal{C}$ at x_i to the isotropy group of $\mathcal{X}$ at $f(x_i)$. In fact, it is a consequence of the definition of orbifold curves that their isotropy groups are necessarily cyclic and are equipped with a preferred generator, so the information of the homomorphism λ_{x_i} is encoded by an element of the isotropy group at $f(x_i)$.

The upshot of this discussion is that the target of the evaluation maps should not be $\mathcal{X}$ but the following:

Definition 3.0.10. Given an orbifold $\mathcal{X}$, the **inertia stack** $I\mathcal{X}$ of $\mathcal{X}$ is an orbifold groupoid whose objects consist of pairs (x, g), where x is an object of $\mathcal{X}$ and $g \in G_x$ is an element of the isotropy group of $\mathcal{X}$ at x. The orbifold structure on $I\mathcal{X}$ is given by putting an arrow

$$(x, g) \to (hx, hgh^{-1})$$

for each arrow h of $\mathcal{X}$ whose source is x.

The evaluation morphisms map

$$ev_i : \overline{\mathcal{M}}_{0,n}(\mathcal{X}, \beta) \to I\mathcal{X}$$

via

$$(f : \mathcal{C} \to \mathcal{X}; x_1, \ldots, x_n) \mapsto (f(x_i), \lambda_{x_i}(1_{x_i})),$$

in which $1_{x_i} \in G_{x_i}$ is the canonical generator.

In fact, since we will only be concerned with degree-zero, three-pointed maps, all of this can be made much more explicit. A degree-zero morphism $\mathcal{C} \to \mathcal{X}$ factors through $\mathcal{C} \to BG_x$, where $x \in \mathcal{X}$ is the image point. There is a simple classification of orbifold morphisms into an orbifold of the form BG:

Fact 3.0.11. A morphism $\mathcal{Y} \to BG$ is equivalent to a principal G-bundle $E \to \mathcal{Y}$.

Of course, we have not defined principal bundles over orbifolds, so this fact is still rather imprecise. Nevertheless, Definition 2.0.7 should give a flavor of the correct definition of principal bundle, and in particular, should point to the fact that E will restrict to an ordinary principal G-bundle on the locus of points in $\mathcal{Y}$ with trivial isotropy.

Thus, a degree-zero morphism $f : \mathcal{C} \to BG_x \subset \mathcal{X}$ will yield a principal G_x-bundle on the three-punctured sphere $\mathcal{C} \setminus \{x_1, x_2, x_3\}$. A careful study of the definitions (and of Fact 3.0.11) shows that the element $\lambda_{x_i}(1_{x_i}) \in G_x$ is nothing but the monodromy of this bundle around the puncture at x_i.

These monodromies are sufficient to capture the data of the principal bundle. More precisely, a principal G_x-bundle on $\mathcal{C} \setminus \{x_1, x_2, x_3\}$ is specified by a homomorphism

$$\pi_1(\mathcal{C} \setminus \{x_1, x_2, x_3\}) \to G_x.$$

Hence, it is given by the three monodromies $\lambda_{x_i}(1_{x_i})$ around the independent loops of $\mathcal{C} \setminus \{x_1, x_2, x_3\}$, subject to the condition that

$$\prod_{i=1}^{3} \lambda_{x_i}(1_{x_i}) = 1.$$

Two such homomorphisms correspond to the same principal G_x-bundle if they are conjugate under the action of G_x.

This indicates that objects of $\overline{\mathcal{M}}_{0,3}(\mathcal{X}, 0)$ should be given by tuples $(x, (g_1, g_2, g_3))$ with $g_i \in G_x$ satisfying $g_1 g_2 g_3 = 1$, and that each such object should have automorphism group G_x from the conjugation action. We can put this more carefully in the language of groupoids: the objects are

$$\mathrm{Obj}\big(\overline{\mathcal{M}}_{0,3}(\mathcal{X}, 0)\big) = \{(x, (g_1, g_2, g_3)) \mid g_i \in G_x,\ g_1 g_2 g_3 = 1\},$$

and there are arrows

$$(x, (g_1, g_2, g_3)) \to (hx, (hg_1h^{-1}, hg_2h^{-1}, hg_3h^{-1}))$$

for each $h \in G_x$.

In these terms, the evaluation maps are simply

$$\mathrm{ev}_i : \overline{\mathcal{M}}_{0,3}(\mathcal{X}, 0) \to I\mathcal{X}$$

$$(x, (g_1, g_2, g_3)) \mapsto (x, g_i).$$

4. Chen-Ruan cohomology

Equipped with a definition of $\overline{\mathcal{M}}_{0,3}(\mathcal{X}, 0)$ and its evaluation maps, the path to degree-zero quantum cohomology should be clear by analogy to the non-orbifold case. We will require a virtual class on $\overline{\mathcal{M}}_{0,3}(\mathcal{X}, 0)$, which will yield a definition of three-point invariants:

$$\langle \alpha\ \beta\ \gamma \rangle^{\mathcal{X}}_{0,3,0} = \int_{[\overline{\mathcal{M}}_{0,3}(\mathcal{X},0)]^{\mathrm{vir}}} \mathrm{ev}_1^*(\alpha)\mathrm{ev}_2^*(\beta)\mathrm{ev}_3^*(\gamma).$$

We will further require a Poincaré pairing $\langle\ ,\ \rangle$, so a product $*$ can be defined via

$$\langle \alpha * \beta, \gamma \rangle = \langle \alpha\ \beta\ \gamma \rangle^{\mathcal{X}}_{0,3,0}.$$

Note, though, that since the evaluation maps land in $I\mathcal{X}$, the three-point invariants take $\alpha, \beta, \gamma \in H^*_{dR}(I\mathcal{X})$ as insertions. Thus, the Poincaré pairing should be defined on $I\mathcal{X}$, and the resulting $*$ will be a product on $H^*_{dR}(I\mathcal{X})$.

In the end, then, the **Chen-Ruan cohomology** of $\mathcal{X}$ will be defined as

$$H^*_{CR}(\mathcal{X}) := H^*_{dR}(I\mathcal{X})$$

with ring structure given by the above product.

4.1. **Poincaré pairing.** We begin by defining the Poincaré pairing. To do so, we will require the decomposition of $I\mathcal{X}$ into **twisted sectors**. In the case where $\mathcal{X} = [M/G]$ is a global quotient, this relies on a fairly simple observation: we have

$$I[M/G] = \left[\left(\coprod_{g \in G} M^g \right) / G \right],$$

in which an element $h \in G$ acts on the disjoint union by sending

$$M^g \to M^{hgh^{-1}}$$

via multiplication by h. This is "equivalent" to

$$\coprod_{g \in \mathrm{Conj}(G)} [M^g / C(g)],$$

where $\mathrm{Conj}(G)$ denotes the set of conjugacy classes and $C(g)$ is the centralizer of g. The notion of equivalence here means, in particular, that this new version of $I[M/G]$ has the same orbit space (and hence the same de Rham cohomology) as well as the same isotropy groups as our original definition— thus, replacing $I[M/G]$ by the above does not affect integrals. In what follows, we will write

$$I[M/G] = \coprod_{(g) \in \mathrm{Conj}(G)} [M^g / C(g)] \tag{1}$$

and refer to the components of (1) as **twisted sectors** of $[M/G]$. Notice that the sector corresponding to the conjugacy class of $1 \in G$ is isomorphic to $[M/G]$ itself; this is called the **nontwisted sector**.

It is a slightly nontrivial fact that there is an analogous decomposition of $I\mathcal{X}$ (or, to be precise, an orbifold equivalent to $I\mathcal{X}$ in the above sense) for an arbitrary $\mathcal{X}$:

$$I\mathcal{X} = \bigsqcup_{(g)\in T} \mathcal{X}_{(g)} \tag{2}$$

Here, T denotes the set of equivalence classes of pairs $(x, g) \in I\mathcal{X}$ under a certain notion of equivalence. This equivalence should be thought of as conjugacy, but some work is required to make sense of what it means for (x, g) and (y, h) to be conjugate when x and y lie in different charts.

Example 4.1.1. Let $\mathcal{X} = \mathbb{P}(2, 3)$, a one-dimensional weighted projective space that looks like $\mathbb{P}^1$ with isotropy group $\mathbb{Z}_2$ at ∞ and $\mathbb{Z}_3$ at 0. Then the twisted sector decomposition of the inertia stack is

$$I\mathcal{X} = \mathbb{P}(2,3) \sqcup [\{(\infty, \zeta_2)\}/\mathbb{Z}_2] \sqcup [\{(0, \zeta_3)\}/\mathbb{Z}_3] \sqcup [\{(0, \zeta_3^2)\}/\mathbb{Z}_3],$$

where $\zeta_2 = e^{2\pi i \frac{1}{2}}$ and $\zeta_3 = e^{2\pi i \frac{1}{3}}$.

One important feature of this decomposition is that there is an isomorphism

$$I : \mathcal{X}_{(g)} \to \mathcal{X}_{(g^{-1})}$$

for any $(g) \in T$; in the global quotient case, this is simply the statement that $X^g = X^{g^{-1}}$.

Using this, the **Poincaré pairing** on $I\mathcal{X}$ is defined as the direct sum of the pairings

$$\langle\ ,\ \rangle_{(g)} : H^*(\mathcal{X}_{(g)}) \otimes H^*(\mathcal{X}_{(g^{-1})}) \to \mathbb{R}$$

$$\langle \alpha, \beta \rangle_{(g)} = \int_{\mathcal{X}_{(g)}} \alpha \wedge I^*\beta.$$

4.2. **Virtual class.** We will not describe the construction of the virtual class in any detail. Instead, let us simply make two remarks.

First, $\overline{\mathcal{M}}_{0,3}(\mathcal{X}, 0)$ is smooth, and the virtual class can be expressed as

$$[\overline{\mathcal{M}}_{0,3}(\mathcal{X}, 0)]^{\text{vir}} = [\overline{\mathcal{M}}_{0,3}(\mathcal{X}, 0)] \cap e(\text{Ob})$$

for an obstruction bundle Ob.

Second, there is a decomposition of $\overline{\mathcal{M}}_{0,3}(\mathcal{X}, 0)$ into components, and a formula for the virtual dimension can be given on each of these. Namely,

$$\overline{\mathcal{M}}_{0,3}(\mathcal{X}, 0) = \bigsqcup_{(g_1,g_2,g_3)\in T_3} \overline{\mathcal{M}}_{0,(g_1,g_2,g_3)}(\mathcal{X}, 0),$$

where

$$\overline{\mathcal{M}}_{0,(g_1,g_2,g_3)}(\mathcal{X}, 0) = \text{ev}_1^{-1}(\mathcal{X}_{(g_1)}) \cap \text{ev}_2^{-1}(\mathcal{X}_{(g_2)}) \cap \text{ev}_3^{-1}(\mathcal{X}_{(g_3)}).$$

Here, T_3 is a set of equivalence classes of elements $(x, (g_1, g_2, g_3)_{G_x}) \in \overline{\mathcal{M}}_{0,3}(\mathcal{X}, 0)$ analogous to the set T above. For example, when $\mathcal{X} = [X/G]$ is a global quotient, we have

$$T_3 = \{(g_1, g_2, g_3) \mid g_i \in G \; g_1g_2g_3 = 1\}/\sim,$$

where $(g_1, g_2, g_3) \sim (hg_1h^{-1}, hg_2h^{-1}, hg_3h^{-1})$.

The virtual dimension formula is

$$\text{vdim}(\overline{\mathcal{M}}_{0,(g_1,g_2,g_3)}(\mathcal{X}, 0)) = 2\text{dim}_{\mathbb{C}}(X) - 2\iota_{(g_1)} - 2\iota_{(g_2)} - 2\iota_{(g_3)}.$$

Here, the definition of $\iota_(g)$ is as follows:

Definition 4.2.1. Let (x, g) be an object of the inertia stack $I\mathcal{X}$, where $g \in G_x$ and x is an object of $\mathcal{X}$. Viewing $\mathcal{X}$ as a groupoid, let X_0 denote the set of objects. Then G_x acts on the tangent space T_xX_0 by the derivative of its action on a neighborhood of x, and this action induces a homomorphism

$$\rho_x : G_x \to GL_n(\mathbb{C}).$$

Since $g \in G_x$ has finite order, the matrix $\rho_x(g)$ is diagonalizable; write the diagonalized matrix as

$$\begin{pmatrix} e^{2\pi i \frac{m_{1,g}}{m_g}} & & \\ & \ddots & \\ & & e^{2\pi i \frac{m_{n,g}}{m_g}} \end{pmatrix},$$

where $n = \dim_{\mathbb{C}} X_0$, m_g is the order of $\rho_x(g)$, and $0 \leq m_{i,g} < m_g$.

The **degree-shifting number** (or **age shift**) of (x, g) is

$$\iota_{(g)} := \sum_{i=1}^{n} \frac{m_{i,g}}{m_g}.$$

One can check that ι defines a locally constant function $I\mathcal{X} \to \mathbb{Q}$, and hence it depends only on the twisted sector in which (x, g) lies.

The reason for the name "degree-shifting number" will be made clear in the next subsection.

4.3. **Grading.** We have now completed (modulo an explicit construction of the virtual cycle) the definition of the vector space and ring structure on the Chen-Ruan cohomology. However, there is one ingredient that we have not yet addressed: the grading.

The easiest way in which to understand the grading is via the Poincaré pairing: if $\mathcal{X}$ has complex dimension n, then elements of $H^d_{CR}(\mathcal{X})$ should pair nontrivially only with elements of $H^{2n-d}_{CR}(\mathcal{X})$. It is easy to check that, under the definition of the Poincaré pairing given above, this will *not* be the case if we allow $H^*_{CR}(\mathcal{X}) = H^*_{dR}(I\mathcal{X})$ to be graded in the same way as the cohomology of the inertia stack.

Instead, the grading is shifted on each twisted sector by the corresponding degree-shifting number:

$$H^d_{CR}(\mathcal{X}) = \bigoplus_{(g)\in T} H^{d-2\iota_{(g)}}(\mathcal{X}_{(g)}).$$

From here, it is a fairly straightforward exercise to show that, for $\alpha, \beta \in H^*_{CR}(\mathcal{X})$, the pairing $\langle \alpha, \beta \rangle$ is nonzero only when

$$\deg(\alpha) + \deg(\beta) = 2n.$$

5. Examples

5.1. Let $\mathcal{X} = BG$. Then the decomposition of $\mathcal{X}$ into twisted sectors is

$$I(BG) = \coprod_{(g)\in \mathrm{Conj}(G)} B(C(g)),$$

so

$$\mathcal{X}_{(g)} = B(C(g)),$$

where $C(g)$ denotes the centralizer of g and $B(C(g))$ is the orbifold $[\bullet/C(g)]$. As a vector space, $H^*_{CR}(BG) = H^*_{dR}(I(BG))$ is generated by the elements $1_{(g)}$ for $(g) \in \mathrm{Conj}(G)$, where $1_{(g)}$ is the constant function 1 on the sector $\mathcal{X}_{(g)}$.

The Poincaré pairing is

$$\langle 1_{(g)}, 1_{(g^{-1})} \rangle = \int_{\mathcal{X}_{(g)}} 1_{(g)} \cup I^* 1_{(g^{-1})} = \int_{B(C(g))} 1 = \frac{1}{|C(g)|}.$$

The moduli space is

$$\overline{\mathcal{M}}_{0,3}(BG, 0) = \{(g_1, g_2, g_3) \mid g_i \in G, g_1 g_2 g_3 = 1\},$$

with an arrow $(g_1, g_2, g_3) \to (hg_1h^{-1}, hg_2h^{-1}, hg_3h^{-1})$ for $h \in G$. In particular,

$$\overline{\mathcal{M}}_{0,(g_1,g_2,g_3)}(BG, 0) = \begin{cases} B\big(C(g_1) \cap C(g_2)\big) & \text{if } g_1g_2g_3 = 1 \\ \emptyset & \text{otherwise.} \end{cases}$$

Since the tangent space to the objects of $\mathcal{X}$ is 0-dimensional, all of the degree-shifting numbers, and hence the virtual dimensions of all the nonempty components of the moduli space, are equal to zero.

Thus,

$$\langle 1_{(g_1)}\ 1_{(g_2)}\ 1_{(g_3)} \rangle = \sum_{\substack{(h_1,h_2,h_3)\in T_3 \\ (g_i)=(h_i)}} \frac{1}{|C(h_1) \cap C(h_2)|}.$$

Recall, here, that $T_3 = \{(h_1, h_2, h_3) \mid h_1 h_2 h_3 = 1\}/G$, where G acts by simultaneous conjugation on all three factors.

Combining this with the Poincaré pairing computed above, we find that

$$1_{(g_1)} \cup 1_{(g_2)} = \sum_{\substack{(h_1,h_2,h_3)\in T_3 \\ (g_i)=(h_i)}} \frac{|C(h_1 h_2)|}{|C(h_1) \cap C(h_2)|}.$$

In particular, this shows that $H^*_{CR}(BG)$ is isomorphic as a ring to the center of the group algebra $\mathbb{C}G$.

5.2. Let $\mathcal{X} = \mathbb{P}(w_1, w_2)$ for coprime integers w_1 and w_2. Recall that the coarse underlying space of $\mathcal{X}$ is $\mathbb{P}^1$, and there are orbifold points $p_1 = [1 : 0]$ with isotropy $\mathbb{Z}_{w_1}$ and $p_2 = [0 : 1]$ with isotropy $\mathbb{Z}_{w_2}$. Thus, in addition to the nontwisted sector $\mathcal{X} \subset I\mathcal{X}$, the inertia stack contains the points (p_1, g) for each nontrivial $g \in \mathbb{Z}_{w_1}$ and (p_2, g) for each nontrivial $g \in \mathbb{Z}_{w_2}$.

The decomposition into twisted sectors is:

$$I\mathcal{X} = \mathcal{X} \sqcup \bigsqcup_{g\neq 1\in\mathbb{Z}_{w_1}} B\mathbb{Z}_{w_1} \sqcup \bigsqcup_{g\neq 1\in\mathbb{Z}_{w_2}} B\mathbb{Z}_{w_2}.$$

The sector indexed by $g \in \mathbb{Z}_{w_1}$ or $g \in \mathbb{Z}_{w_2}$ will be denoted $\mathcal{X}_g$.

To compute the degree-shifting numbers, notice that if $g = e^{2\pi ik/w_1} \in \mathbb{Z}_{w_1}$ for some $1 \leq k < w_1$, then g acts on the standard chart $U_{p_1} \cong \mathbb{C}$ around p_1 by multiplication, so the derivative of this action is equal to itself. It follows that $\rho_{p_1}(g) = e^{2\pi ik/w_1} \in GL_1(\mathbb{C})$, so

$$\iota_{(g)} = \frac{k}{w_1}.$$

A similar computation holds for the sectors indexed by $g \in \mathbb{Z}_{w_2}$.

For each $1 \leq k < w_1$ and $1 \leq \ell < w_2$, let

$$\alpha^k \in H^{2k/w_1}_{CR}(\mathcal{X}) = H^0(\mathcal{X}_{e^{2\pi ik/w_1}}) = H^0(B\mathbb{Z}_{w_1}) = \mathbb{C}$$

and

$$\beta^\ell \in H^{2\ell/w_2}_{CR}(\mathcal{X}) = H^0(\mathcal{X}_{e^{2\pi i\ell/w_2}}) = H^0(B\mathbb{Z}_{w_2}) = \mathbb{C}$$

denote the constant functions 1 on the various twisted sectors.

It is easy to see that $\alpha * \beta = 0$. Indeed, this product is defined by the three-point invariants $\langle\alpha\ \beta\ \gamma\rangle$. The insertion α forces the first marked point to map to the twisted sector $\mathcal{X}_{e^{2\pi i/w_1}}$, so on coarse underlying spaces, it goes to $p_1 \in \mathbb{P}^1$. The insertion β, similarly, forces the second marked point to map to $p_2 \in \mathbb{P}^1$. Since we are considering degree-zero morphisms, this is impossible.

One can check, furthermore, that $\alpha^{k_1} * \alpha^{k_2} = \alpha^{k_1+k_2}$ whenever $k_1 + k_2 < w_1$, and a similarly property holds for powers of β. For degree reasons, this must be true up to a constant, and the determination of the constant is a straightforward application of the definitions; the obstruction bundle has rank zero, so it does not play a role.

Finally, we have $\alpha^{w_1-1} * \alpha = \beta^{w_2-1} * \beta = H$, the hyperplane class in the nontwisted sector $H^2_{CR}(\mathcal{X}) = H^2(\mathbb{P}^1)$. Once again, degree constraints force this to be true up to a constant, and the constant can be computed by showing

$$\langle \alpha^{w_1-1}\ \alpha\ 1\rangle = \int_{[\overline{\mathcal{M}}_{0,(\zeta^{w_1-1},\zeta^{w_1},1)}(\mathcal{X},0)]^{\mathrm{vir}}} \mathrm{ev}_1^*(1) \cup \mathrm{ev}_2^*(1) \cup \mathrm{ev}_3^*(1) = 1,$$

where $\zeta = e^{2\pi i/w_1}$, and similarly for the case of β.

In summary, we have shown that the Chen-Ruan cohomology of $\mathbb{P}(w_1, w_2)$ for coprime weights w_i is generated as a ring by the two twisted classes $\alpha \in H^{2/w_1}_{CR}(\mathcal{X})$ and $\beta \in H^{2/w_2}(\mathcal{X})$, subject to the relations

$$\alpha \cup \beta = 0,\ \alpha^{w_1} = \beta^{w_2},\ \alpha^{w_1+1} = \beta^{w_2+1} = 0.$$

References

[1] A. Adem, J. Leida, and Y. Ruan. *Orbifolds and Stringy Topology*, volume 171. Cambridge University Press, 2007.