Math Club Talk

Fall 2010

Introduction:

The goal of this talk is to explain a way of constructing finite topological spaces that "model" more complicated spaces. To do so, I'll need to (briefly) define what I mean by a topological space (for those who haven't seen them before). Once that's done, I can define the type of space that admits a finite model (simplicial complexes) and explain how the model is constructed and the kinds of information the model captures, such as the fundamental group and higher homotopy groups. Finally, I'll discuss a generalization of these results that iterates the modeling process to obtain an even closer approximation.

References:

- McCord, *Singular Homology Groups and Homotopy Groups of Finite Topological Spaces*
- C, *Inverse Limits of Finite Topological Spaces*

What is a Topological Space?

Definition:

- **Definition.** Let X be a set. A **topology** on X is a collection $\mathcal{T}$ of subsets of X (the "open sets") such that:
  1. $\emptyset$ and X are in $\mathcal{T}$.
  2. The union of any collection of elements of $\mathcal{T}$ is in $\mathcal{T}$.
  3. The intersection of any finite collection of elements of $\mathcal{T}$ is in $\mathcal{T}$.
- **Definition.** A set together with a topology on that set is called a **topological space**.

Examples:

- **Example.** $X = \mathbb{R}^n$, $\mathcal{T} =$ sets U with the following property: for any $x \in U$, there exists $\varepsilon > 0$ such that any point of distance less than ε from x is also in U.

- **Example.** $X =$ circle, $\mathcal{T} =$ all sets obtainable by intersecting an open set in $\mathbb{R}^2$ with the circle. ("Subspace topology")
- **Example.** $X = \{a, b, c\}$, $\mathcal{T} = \{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}\}$.

- **Example.** $X = \{a, b, c, d\}$, $\mathcal{T} =$ all subsets of X. ("Discrete topology")

Motivation:

- You might wonder why it's useful for a set to come equipped with a notion of open sets. Here are two reasons:

  1. Open sets capture a notion of "nearness" without mentioning a metric. For example, one way of saying that a sequence $\{a_n\}$ of points in $\mathbb{R}$ converges to a point a is to say that any open set containing a contains (at least) one of the points a_n.
  2. Open sets let us define continuity. Specifically, a function $f : X \to Y$ between two topological spaces is **continuous** if for any open subset U of Y, $f^{-1}(U)$ is an open subset of X.

Finite Topological Spaces

- You might complain that in the finite examples above, the topology was really arbitrary. It *means* something for a subset of $\mathbb{R}^n$ to be open, whereas with the finite spaces, we just chose which sets should be open willy-nilly, our only constraint being the axioms of a topology. As a result, it looks like continuity means something totally crazy and ungrounded in the real world when we're talking about finite spaces, whereas when we were talking about $\mathbb{R}^n$ it meant something very intuitively desirable.
- So why care about finite spaces??
- It turns out that a large class of all topological spaces can be "modeled" by finite spaces. That is, for a lot of interesting infinite spaces, we can find finite spaces that capture the same topological information as the infinite space.
- This is really nice, because while infinite spaces might be the ones that are interesting to us, finite spaces are a lot easier for us to write down, get our hands dirty with, program into computers, et cetera.

Simplicial Complexes

- The class of topological spaces that can be modeled by finite spaces are called simplicial complexes. Their definition demonstrates an extremely important technique for coming up with topological spaces: take topological spaces you know and "glue" them together.
- **"Definition".** A **simplicial complex** is a geometric object built by gluing together finitely-many points, line segments, triangles, tetrahedra, and the higher-dimensional analogues of these things. (These are called **simplices**.)
- Lots of the geometric objects we care about can be built this way.
- **Example.** The circle is a simplicial complex: we can build it, for example, by gluing together four points and four line segments.
- **Example.** The sphere is also a simplicial complex: we can build it by gluing together six points, twelve line segments, and eight triangles.
- By the way, you might be wondering: if these are topological spaces, what's the topology? The easiest way to answer that question is to just imagine any of these things as sitting inside $\mathbb{R}^n$; then, we can do what we did for the circle: say a subset of a simplicial complex X is open if it's the intersection of an open set in $\mathbb{R}^n$ with S.

How to Build a Finite Model of a Simplicial Complex

- Here's a recipe for building a finite space from a simplicial complex (McCord):
  - Give your finite space an element corresponding to each simplex.
  - For each simplex v in your simplicial complex, form the set of all simplices containing v. That set of simplices corresponds to a set of elements of your space. Declare that set to be open.
  - Also declare to be open any set you can form by taking unions of the sets you've already got.
  - That's your topological space.
- **Example:** The circle.

Sidebar: Topologies = Relations

- **Fact:** Suppose X is a finite set. Then imposing a topology on X is equivalent to imposing a reflexive, transitive relation $\leq$ on the elements of X.
- **Topology $\to$ Relation:** For any element $x \in X$, let U_x be the intersection of all open sets containing x. Then, say $x \leq y$ if $U_x \subset U_y$.
- **Relation $\to$ Topology:** For any element $x \in X$, let $U_x = \{y \in X \mid y \leq x\}$. Declare all of the sets U_x, and all the sets you can get by taking unions of these, to be open.
- The topology we imposed on the finite space constructed above corresponds to the relation coming from inclusion on simplices: $x \leq y$ iff $\sigma_x \subset \sigma_y$, where σ_x and σ_y are the corresponding simplices.

In What Sense Is This a Model?

- There are many, many properties we can ascribe to topological spaces, but one of the most important and easiest to describe is the number of "holes".
- **"Definition".** Let X be a topological space. The **fundamental group** is the set of all different continuous maps from a circle into X, where we consider two different maps from the circle into X to be the same if we can "deform" one into the other.
- Think of a continuous map from the circle into X as being a way of drawing a loop in X.
- **Example.** $X = \mathbb{R}^2$. There is only one distinct way to draw a circle in X, because any other way can be deformed into this one.
- **Example.** $X = \mathbb{R}^2 \setminus \{0\}$. There is more than one way to map a circle into X. For example, a circle that goes around the hole cannot be deformed into one that doesn't go around the hole. Also, a circle that goes around the hole twice cannot be deformed into one that goes around only once.
- If you've seen some abstract algebra, the name of the fundamental group might clue you into the fact that it has more structure than I'm letting on. In fact, the group structure comes from concatenating two loops to get a new loop.
- There are higher-dimensional versions of the fundamental group, called **homotopy groups**. For example, the second homotopy group measures how many different continuous maps there are from a sphere into X.
- Knowing all the homotopy groups of a topological space tells you a lot of information about that space.
- **Punchline:** The sense in which the finite space formed previously "models" the simplicial complex is that it has all of the same homotopy groups as the simplicial complex.

Conclusion: My Work

- In my thesis, I iterated McCords construction to obtain progressively "better" finite models of a simplicial complex. Taking a type of limit (called an inverse limit) over all of these finite models, I proved that you get something that's homotopy equivalent (an even stronger form of topological similarity) to your original simplicial complex.
- Start with a simplicial complex X.
- Let X_0 be the finite model as above.
- Take a "barycentric subdivision" of X. This means that we add one new vertex in the center of each existing simplex, and add to our simplicial complex all simplices whose vertices were either already in the complex or are now:
- Let X_1 be the finite model of this barycentric subdivision.
- Barycentrically subdivide again, and let X_2 be the resulting finite model.
- Continue indefinitely.
- Notice that we have maps $X_0 \xleftarrow{p_1} X_1 \xleftarrow{p_2} X_2 \xleftarrow{p_3} \cdots$.
- Let $\tilde{X} = \varprojlim X_i$. That is:

$$\tilde{X} = \{(x_0, x_1, x_2, \ldots) \mid x_i \in X_i,\ p_i(x_i) = x_{i-1}\}.$$

  The result of my thesis was that $\tilde{X} \simeq X$. This is (almost) the strongest kind of topological similarity.