Moduli of Curves and Gromov Witten Theory
Part I: Kontsevich's Formula

Emily Clader

Student Geometry and Topology Seminar, University of Michigan

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References:

- Kock and Vainsencher, *An Invitation to Quantum Cohomology.*
- Fulton and Pandharipande, *Notes on stable maps and quantum cohomology.*

Introduction and Historical Background

- One of the most basic problems of enumerative geometry is to compute the number N_d of rational curves of degree d in $\mathbb{P}^2$ passing through $3d-1$ given points in general position.
- Some easy cases (like $N_1 = 1$) have been known since antiquity, and in the 1850's Steiner and others used the somewhat questionable methods of "Schubert calculus" to compute many more.
- But it wasn't until the 1990's that the subject of enumerative geometry really came into its own. Two things happened at this time:
  - Ellingsrud and Stromme observed that equivariant cohomology could be used to tackle many enumerative quesitons. (I won't be discussing this.)
  - A synthesis with string theory led to the birth of Gromov-Witten theory.
- It was the connection with physics that led Kontsevich to prove his celebrated recursion for the N_d's in 1994. Two amazing things about this formula:
  - Any N_d can be computed recursively, starting just with $N_1 = 1$.
  - It's equivalent to the associativity of the so-called *quantum product* on $H^*(\mathbb{P}^2)$.
- The way one proves Kontsevich's formula is by studying the structure of the moduli space of degree-d maps from $\mathbb{P}^1$ to $\mathbb{P}^2$. These are parameterizations of degree-d curves, and we will ensure that the curves pass through the given points by specifying *marked points* on the domain and demanding that these map to the given points.

The Moduli Space of Curves

- An *n-pointed smooth rational curve* $(C, p_1, \ldots, p_n)$ is a projective smooth rational curve C equipped with a choice of n distinct marked points $p_1, \ldots, p_n \in C$.
- An *isomorphism* of such curves is an isomorphism $\varphi : C \xrightarrow{\sim} C'$ such that $\varphi(p_i) = p'_i$.
- A *family of n-pointed smooth rational curves* (over a base variety B) is a flat and proper map of varieties $\pi : \mathcal{X} \to B$ with n disjoint sections $\sigma_i : B \to \mathcal{X}$ such that each geometric fiber $\pi^{-1}(b)$ is a projective smooth rational curve.
  - Note that each fiber thus has the structure of and n-pointed smooth rational curve.
- An *isomorphism of families* over the same base is an isomorphism $\mathcal{X} \to \mathcal{X}'$ respecting the projections and sections.
- **Fact:** There is a variety $M_{0,n}$ whose points are in bijection with isomorphism classes of n-pointed smooth rational curves, and whose algebraic structure reflects how such curves vary in families. (Called a "fine moduli space".)
- *Problem*: We want to do intersection theory on $M_{0,n}$, but it's noncompact. (This isn't hard to believe– imagine a sequence of curves where two marked points get closer and closer together.) We need a compactification. Of course, one can just compactify the variety $M_{0,n}$, but we'd like to do so in such a way that the resulting variety still parameterizes something.
- We'll need to allow in more objects than just n-pointed smooth rational curves. (Maybe let the points coincide...?) The right thing to do is to allow reducible curves:
- A *tree of projective lines* is a connected curve such that:
  1. Each irreducible component is isomorphic to $\mathbb{P}^1$.
  2. The points of intersection are ordinary double points.
  3. There are no closed circuits (that is, if a node is removed, the curve becomes disconnected.)
- An irreducible component of a tree is called a *twig*.
- A *stable n-pointed rational curve* is a tree of projective lines with n distinct marked points that are smooth points of C such that each twig contains at least three special points. (A *special point* is a point that is either a node or a marked point.)
- The definition of stability is cooked up so that a stable n-pointed rational curve has no automorphisms, which turns out to be exactly what we need to get a fine moduli space.
- **Key Theorem:** For each $n \geq 3$, there is a smooth projective variety $\overline{M}_{0,n}$ that is a fine moduli space for stable n-pointed rational curves, and it contains $M_{0,n}$ as a dense open subvariety.

- Points on the boundary of $\overline{M}_{0,n}$ corresponds to reducible curves.
- One can show that if Σ_a is the subset of $\overline{M}_{0,n}$ consisting of curves with a nodes, then $\operatorname{codim}(\Sigma_a) = a$.
- In particular, there is a codimension-1 subvariety consisting of curves with one node. This subvariety typically has a number of connected components. For example, when $n = 6$, the connected components are:

- The closure of each such component is a (closed, irreducible) codimension-1 subvariety, called a *boundary divisor*.
- Boundary divisors can be specified by giving a partition $S = A \cup B$ of the marking set $\{p_1, \ldots, p_n\}$, with $\#A \geq 2$ and $\#B \geq 2$; this indicates which marked points go on which of the two twigs. Such a divisor is denoted $D(A|B)$.
- There is a *forgetful morphism*
$$\varepsilon : \overline{M}_{0,n+1} \longrightarrow \overline{M}_{0,n}$$
that forgets the last marked point (and contracts any unstable components), and it is easy to see that
$$\varepsilon^* D(A|B) = D(A \cup \{p_{n+1}\}|B) + D(A|B \cup \{p_{n+1}\}).$$
- Iterate this:
  - Consider the forgetful map
  $$\varepsilon : \overline{M}_{0,n} \longrightarrow \overline{M}_{0,4} = \mathbb{P}^1$$
  that forgets all but the last four marks.
  - There are three boundary divisors in $\overline{M}_{0,4}$: $D(ij|kl)$, $D(ik|jl)$, and $D(il|jk)$.
  - Divisors in projective spaces are determined by their degree, so these three divisors are all linearly equivalent.
  - Pull back these three divisors along ε to get three divisors in $\overline{M}_{0,n}$, which must also be linearly equivalent. It's easy to iterate the above to get formulas for these three divisors in terms of the boundary divisors. One finds:
  $$\sum_{i,j \in A, k,l \in B} D(A|B) \equiv \sum_{i,kj \in A, j,l \in B} D(A|B) \equiv \sum_{i,l \in A, j,k \in B} D(A|B).$$

The Moduli Space of Stable Maps

- We'd like a moduli space for morphisms $\mu : C \to \mathbb{P}^r$ (of a specified degree), where C is an n-pointed smooth rational curve. It's easy to come up with a fine moduli space, but as before it won't be compact. What to allow in to compactify it?
- Again, we'll need to allow our source curves to be reducible (this makes sense intuitively–draw a picture of $[s : t] \mapsto [bs^2 : t^2 : st]$ degenerating), and we'll want to make enough demands on the marked points to get rid of automorphisms.
- This isn't quite possible, but we can at least ensure that there are only finitely many automorphisms.
- A *stable n-pointed map* is a morphism $\mu : C \to \mathbb{P}^r$, where C is a tree of projective lines with n distinct marked points that are smooth points of C, and where any twig that is mapped to a point by μ is stable as a pointed curve.
- The point here is that a nonconstant map from a rational curve to $\mathbb{P}^r$ can never have infinitely many automorphisms; see this using algebra, since the function field of the source curve is a finite extension of the function field of the image curve, and automorphisms of the map correspond to automorphisms of this extension. We're adding enough marked points so that the constant maps don't have any automorphisms.
- **Key Theorem:** There exists a coarse moduli space $\overline{M}_{0,n}(\mathbb{P}^r, d)$ parameterizing isomorphism classes of stable n-pointed maps of degree d to $\mathbb{P}^r$.
- We represent a stable map by a cartoon like:
- There are forgetful maps that forget one (or more) of the marked points, as well as forgetful maps to $\overline{M}_{0,n}$ that forget the map to $\mathbb{P}^r$ (and contract any unstable twigs).
- There are also evaluation maps
$$\nu_i : \overline{M}_{0,n}(\mathbb{P}^r, d) \to \mathbb{P}^r,$$
sending a map μ to $\mu(p_i)$.
- There are boundary divisors, now indexed by partitions $A \cup B = S$ of the marked set together with partitions $d_A + d_B = d$ of the degree, satisfying the necessary constraints. Such a divisor is denoted $D(A, B; d_A, d_B)$.
- Pulling back by the forgetful morphism $\overline{M}_{0,n}(\mathbb{P}^r, d) \to \overline{M}_{0,4}$, we get another linear equivalence.

Kontsevich's Formula:

- Let's compute the number N_2 of plane conics through five given points.
- Fix two lines L_1 and L_2 and four points Q_1, Q_2, Q_3, Q_4 in $\mathbb{P}^2$.
- Consider the variety $\overline{M}_{0,6}(\mathbb{P}^2, 2)$. Let

  $$Y = \nu_{m_1}^{-1}(L_1) \cap \nu_{m_2}^{-1}(L_2) \cap \nu_{p_1}^{-1}(Q_1) \cap \cdots \cap \nu_{p_4}^{-1}(Q_r)$$

  be the subvariety of maps sending m_i to a point in L_i and p_i to Q_i.
- An easy dimension count shows that Y is 1-dimensional. Therefore (as long as the lines and points are sufficiently general), its intersection with each boundary divisor in $\overline{M}_{0,6}$ is a finite collection of points. We will count these points.
- Denote the marked points in $\overline{M}_{0,6}$ by $m_1, m_2, p_1, \ldots, p_4$.
- Consider the forgetful morphism

  $$\varepsilon : \overline{M}_{0,6}(\mathbb{P}^2, 2) \longrightarrow \overline{M}_{0,\{m_1,m_2,p_1,p_2\}}.$$

  By the fundamental linear equivalence:

  $$Y \cap \varepsilon^* D(m_1, m_2 | p_1, p_2) \equiv Y \cap \varepsilon^* D(m_1, p_1 | m_2, p_2).$$

  Each of these is a finite collection of points, so the equivalence simply says that they are the same number of points.
- Now,

  $$\varepsilon^* D(m_1, m_2 | p_1, p_2) = \sum D(A, B; d_A, d_B),$$

  where the sum is over 2-weighted partitions of the marking set with $m_1, m_2 \in A$ and $p_1, p_2 \in B$. The terms are:

- What is the intersection of Y with each of these irreducible divisors?
- First column:
  - C_A maps to a point $z \in \mathbb{P}^2$. Since m_1 and m_2 both map to z, it must be the case that $\{z\} = L_1 \cap L_2$.
  - If there were any p_i's on C_A, these would also map to z, meaning we'd have $Q_i \in L_1 \cap L_2$, contradicting genericity.
  - So the intersection of Y with the bottom three divisors is empty.
  - The intersection of Y with the top-left divisor consists of maps from C_B to a conic, which must pass through the five points $Q_1, \ldots, Q_4, z$. This intersection thus consists of N_2 points.
- Second column:
  - The first three irreducible divisors have three p_i's mapping to a line, contradicting genericity of the points Q_i. Hence, the intersection of Y with these is empty.
  - The bottom irreducible divisor corresponds to maps of C_A and C_B onto the two lines $\overline{Q_3Q_4}$ and $\overline{Q_1Q_2}$. There is only one such map, so Y intersects this irreducible divisor in 1 point.
- Third column:
  - In each case, $d_B = 0$, so C_B maps to a point. This means that $Q_1 = Q_2$, contradicting genericity. Thus, the intersection of Y with any of these is empty.
- In total, we find that $Y \cap \varepsilon^* D(m_1, m_2 | p_1, p_2) = N_2 + 1$.
- Repeat this exact reasoning to get $Y \cap \varepsilon^* D(m_1, p_1 | m_2, p_2) = 2$.
- The conclusion is that $N_2 = 1$.
- The exact idea of this computation generalizes to give Kontsevich's formula:

$$N_d = \sum_{d_A + d_B = d} N_{d_A} N_{d_B} d_A^2 d_B \left(d_B \binom{3d-4}{3d_A - 2} - d_A \binom{3d-4}{3d_A - 1} \right).$$