The Landau-Ginzburg/Calabi-Yau Correspondence for Certain Complete Intersections

by

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For my parents, who always knew I could.

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Moduli of Curves and Gromov-Witten Theory at the Institut Fourier. The work in this thesis is a generalization of their ideas.

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CHAPTER I

Introduction

In the late 1980s and early 1990s, physicists posited the existence of a Landau-Ginzburg/Calabi-Yau (LG/CY) correspondence connecting two theories associated to a collection of polynomials. The theory on the Calabi-Yau side can be understood mathematically as encoding the intersection theory— or, more specifically, the Gromov-Witten invariants— of the complete intersection cut out by the polynomials inside weighted projective space. The Landau-Ginzburg model, on the other hand, studies the polynomials not as defining equations but as singularities.

Although such models have long been well-understood in the physical context, it was not until 2007, with the series of papers [25][26][27], that a precise definition of the Landau-Ginzburg model was proposed in mathematical terms. The theory developed in those papers, known as Fan-Jarvis-Ruan-Witten (FJRW) theory, applies to hypersurfaces cut out by a single quasihomogeneous polynomial. Chiodo-Ruan proved in [11] that for the quintic polynomial $W = x_1^5 + \cdots + x_5^5$, FJRW theory indeed coincides in genus zero with the Gromov-Witten theory of the corresponding hypersurface— that is, the genus-zero LG/CY correspondence holds in that case. Later, Chiodo-Iritani-Ruan [9] generalized the genus zero correspondence to arbitrary Calabi-Yau hypersurfaces.

The primary goal of this thesis is to extend the results of [11] to certain complete intersections. In order to accomplish this, it is necessary to generalize FJRW theory, constructing a mathematical Landau-Ginzburg model associated to a collection of singularities rather than just one. The theory we construct is a "hybrid" model that combines aspects of FJRW theory and Gromov-Witten theory. Before defining the hybrid model precisely, however, we will explain how FJRW theory initially arose out of the study of a PDE known as Witten's equation. The exploration of these ideas will lead us to a perspective on the Landau-Ginzburg model based on variation of GIT quotients, and from here we will see not only why an LG/CY correspondence might be expected, but also how it should be generalized to the hybrid setting.

1.1 Witten's equation

Before Witten's equation garnered the attention of mathematicians, a different conjecture of Witten generated widespread excitement. This earlier conjecture relates to the intersection theory on the moduli space $\overline{\mathcal{M}}_{g,n}$ of stable marked curves, which can be encoded in the generating function

$$F(t_0, t_1, \ldots) = \sum_{\substack{g \geq 0, n \geq 1 \\ 2g-2+n>0}} \sum_{d_1,\ldots,d_n} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \frac{t_{d_1} \cdots t_{d_n}}{n!}$$

for integrals of ψ classes.[1] In 1991, Witten conjectured [49] that F satisfies a system of differential equations called the Korteweg-de Vries (KdV) hierarchy, which can be summarized by the equation

$$\frac{\partial^2 F}{\partial t_0 \partial t_1} = \frac{1}{2}\left(\frac{\partial^2 F}{\partial t_0^2}\right)^2 + \frac{1}{12}\frac{\partial^4 F}{\partial t_0^4}. \tag{1.1}$$

[1]The class ψ_i is defined as $c_1(\mathcal{L}_i)$, where $\mathcal{L}_i$ is the line bundle on $\overline{\mathcal{M}}_{g,n}$ whose fiber over a marked curve $(C; x_1, \ldots, x_n)$ is the cotangent line to C at x_i.

The conjecture was proved by Kontsevich [40] shortly after its announcement. When combined with the string equation and the initial value $F = t_0^3/6 + \cdots$, equation (1.1) uniquely determines all ψ integrals on $\overline{\mathcal{M}}_{g,n}$.

At around the same time, Witten proposed a generalization [50] [51] of his conjecture, in which the moduli space of curves is enhanced to the moduli space of r-spin structures, which parameterizes marked curves $(C; x_1, \ldots, x_n)$ together with a line bundle L on C satisfying $L^{\otimes r} \cong \omega_{C,\log} := \omega_C \otimes \mathcal{O}([x_1] + \cdots + [x_n])$. According to the conjecture, the intersection theory on this moduli space should be governed by the more general nKdV hierarchy.

The r-spin moduli space can be understood as the space of solutions to the equation

$$\overline{\partial} u + r\overline{u}^{r-1} = 0, \tag{1.2}$$

where u is a section of an orbifold line bundle L on a curve C. In particular, after choosing a metric on C in order to induce an isomorphism $\overline{L}^{\vee} \cong L$, the condition that the two terms of equation (1.2) lie in the same space forces (C, L) to be an r-spin curve. Although the study of solutions to equation (1.2) appears initially to involve not just the underlying r-spin curve but the section u, the equation in fact decouples into

$$\overline{\partial} u = 0, \quad r\overline{u}^{r-1} = 0. \tag{1.3}$$

The second of these equations implies that u vanishes, so the moduli space of solutions to (1.2) is simply the r-spin moduli space. The key ingredient in the proof of this decoupling is the "Ramond vanishing" property of the r-spin theory, which asserts that the intersection numbers over the r-spin moduli space vanish away from components where L has nontrivial orbifold structure at every marked point; this property was conjectured by Witten and proved in [36] and [43]. We will return to it repeatedly in what follows.

Because of the decoupling of equation (1.2) into (1.3), little attention was paid by mathematicians to this equation despite their study of r-spin curves [33] [34] [35] [36]. However, Witten continued to generalize his conjecture. He replaced (1.2) with what is now called the *Witten equation*,

$$\bar{\partial} u_i + \overline{\frac{\partial W}{\partial u_i}} = 0, \tag{1.4}$$

and he dubbed the theory of solutions to this equation the "Landau-Ginzburg A-model". Here, $W = W(x_1, \ldots, x_N)$ is an arbitrary quasi-homogeneous polynomial, and u_i is again a section of an appropriate orbifold line bundle.

The mathematical study of solutions to the Witten equation was eventually taken up by Fan, Jarvis, and Ruan, who constructed a moduli space of solutions and defined Gromov-Witten-type correlators by integrating certain cohomology classes against a virtual cycle on the moduli space. The more general Witten conjecture, when placed in this framework, asserts that in certain cases, the generating functions for these correlators should satisfy specific integrable hierarchies. More precisely, the conjecture applies to ADE-type singularities:

$$A_n: \; W = x^{n+1}, \;\; n \geq 1;$$

$$D_n: \; W = x^{n-1} + xy^2, \;\; n \geq 4;$$

$$E_6: \; W = x^3 + y^4;$$

$$E_7: \; W = x^3 + xy^3;$$

$$E_8: \; W = x^3 + y^5.$$

There is also an ADE classification of integrable hierarchies, constructed in two equivalent versions by Drinfeld-Sokolov [22] and Kac-Wakimoto [38]. These are the hierarchies that the generating function of solutions to the Witten equation were conjectured to satisfy, when W is of ADE type.

1.2 FJRW theory

In this section, we will give a brief account of FJRW theory. For more details, see the papers [25] [26] [27], or for the special case of the quintic, see [11].

Let W be a quasi-homogeneous polynomial. That is, $W \in \mathbb{C}[x_1, \ldots, x_N]$, and there exist weights $c_1, \ldots, c_N \in \mathbb{Z}^{>0}$ and a degree d such that

$$W(\lambda^{c_1} x_1, \ldots, \lambda^{c_N} x_N) = \lambda^d W(x_1, \ldots, x_N)$$

for any $\lambda \in \mathbb{C}$. We assume furthermore that W is nondegenerate, which means that the weights are uniquely determined by W and that the hypersurface in weighted projective space

$$\{W = 0\} \subset \mathbb{P}(c_1, \ldots, c_N)$$

is nonsingular.

By searching for line bundles $L_1, \ldots, L_N$ such that the two terms of (1.4) lie in the same space when $u_i \in \Gamma(L_i)$, Fan-Jarvis-Ruan arrived at the notion of a W-*structure*. By definition, a W-structure on an orbifold stable curve[2] C is a choice of orbifold line bundles $L_1, \ldots, L_N$ and isomorphisms

$$\varphi_j : L_j^{\otimes d} \xrightarrow{\sim} \omega_{C,log}^{\otimes c_j} \text{ for } j \in \{1, \ldots, N\}$$

that combine to give an isomorphism

$$W_i(L_1, \ldots, L_N) \xrightarrow{\sim} \omega_{C,\log}$$

[2]See [2] or Section 3.1 below for a precise definition of orbifold stable curve.

for each monomial W_i of W. Here, inserting a line bundle into a monomial is defined by ignoring the coefficient and treating powers of the variables as tensor products on the line bundles. One further stability condition on the bundles is needed to ensure that there is a good moduli space of W-structures: at each point $y \in C$, the representation

$$\rho_y : G_y \to (\mathbb{C}^*)^N$$

of the isotropy group at y on the fiber of $\bigoplus_{i=1}^N L_i$ is required to be faithful.[3] A curve together with a W-structure is referred to as a W-*curve*.

After specifying an appropriate notion of morphism between W-curves, Theorem 2.2.6 of [27] shows that there is a smooth, compact Deligne-Mumford stack $\mathscr{W}_{g,n}$ parameterizing W-structures on genus-g, n-pointed orbifold curves up to isomorphism.

The isotropy group G_{x_i} at a marked point of a stable orbifold curve is always cyclic, so one can associate to each marked point in a W-curve an element

$$\gamma_i = \rho_{x_i}(1) \in (\mathbb{C}^*)^N$$

describing the action of the generator $1 \in G_{x_i}$ on the fiber of $\bigoplus_{i=1}^N L_i$ over x_i. This element is referred to as the *multiplicity* at x_i. Let

$$\mathscr{W}_{g,n}(\gamma_1, \ldots, \gamma_n) \subset \mathscr{W}_{g,n}$$

be the (open and closed) substack of W-curves in which the multiplicity at x_i is γ_i.

It is straightforward to check (Lemma 2.1.17 of [27]) that these substacks must have $\gamma_i \in G_W$, where

$$G_W := \{(\alpha_1, \ldots, \alpha_N) \in (\mathbb{C}^*)^N \mid W(\alpha_1 x_1, \ldots, \alpha_N x_N) = W(x_1, \ldots, x_N)\}$$

[3] In particular, this prevents one from giving points in C arbitrarily large isotropy groups, which would lead to a noncompact moduli space.

is the group of diagonal symmetries of W. This is the first indication that the group G_W is intimately connected to solutions of the Witten equation.

Having defined the moduli space $\mathscr{W}_{g,n}$, which is the background data for solving the Witten equation, the next step is to construct a virtual cycle against which to integrate. Fan-Jarvis-Ruan's original construction is analytic in nature. It relies on the observation that, because W has only a single, highly-degenerate critical point at $\mathbf{x} = 0$, solving the Witten equation is very difficult. It is much easier to solve a perturbed equation associated to a polynomial $W + W_0$ whose restriction to the fixed point set $\text{Fix}(\gamma) \subset \mathbb{C}^N$ for each $\gamma \in G$ is a holomorphic Morse function whose critical values have distinct imaginary parts. Such a W_0 is called a *strongly regular perturbation*.

Given a strongly regular perturbation, Fan-Jarvis-Ruan [26] construct a virtual cycle on a different moduli space $\mathscr{W}^s_{g,n}(\gamma_1, \ldots, \gamma_n)$ that admits a proper, quasi-finite map to a component $\mathscr{W}_{g,n}(\gamma_1, \ldots, \gamma_n)$. However, the cycle depends on the perturbation W_0. It changes in a controlled way whenever W_0 crosses a wall where the imaginary parts of its critical values collide, described by a wall-crossing formula in terms of *Lefschetz thimbles*— that is, elements in the relative homology groups

$$H_{N_{\gamma_i}}(\mathbb{C}^{N_\gamma}, W^{+\infty}_{\gamma_i}; \mathbb{C}),$$

where

$$W^{+\infty} = (\text{Re}W)^{-1}(M, \infty)$$

for $M \gg 0$, and N_γ is the complex dimension of $\text{Fix}(\gamma)$. Thus, even after pushing forward from $\mathscr{W}^s_{g,n}(\gamma_1, \ldots, \gamma_n)$ to $\mathscr{W}_{g,n}(\gamma_1, \ldots, \gamma_n)$, a virtual cycle that is independent of the perturbation will necessarily belong to

$$H_*(\mathscr{W}_{g,n}(\gamma_1,\ldots,\gamma_n);\mathbb{Q})\otimes\prod_{i=1}^{n}H_{N_{\gamma_i}}(\mathbb{C}^{N_{\gamma_i}},W_{\gamma_i}^{+\infty};\mathbb{C})^{G_W}.$$

Although computationally complicated, one important consequence of Fan-Jarvis-Ruan's construction of the virtual cycle is that it indicates what the state space for FJRW theory must be. To obtain a number by integrating against the virtual cycle, one must first pair it against elements $\alpha_1,\ldots,\alpha_n$ with

$$\alpha_i\in\mathcal{H}_{\gamma_i}=H^{N_{\gamma_i}}(\mathbb{C}^{N_{\gamma_i}},W_{\gamma_i}^{+\infty};\mathbb{C})^{G_W}$$

to cancel the wall-crossing contributions. Thus, the *state space* of FJRW theory is defined as

$$\mathcal{H}_W=\bigoplus_{\gamma\in G_W}\mathcal{H}_\gamma.$$

In other words, $\mathcal{H}_W$ is the Chen-Ruan cohomology $H^*_{CR}([\mathbb{C}^N/G_W],W^{+\infty};\mathbb{C})$.

Given $\alpha_1,\ldots,\alpha_n\in\mathcal{H}_W$ and integers $l_1,\ldots,l_n\geq 0$, an FJRW correlator is defined as

$$\langle\tau_{l_1}(\alpha_1),\ldots,\tau_{l_n}(\alpha_n)\rangle_{g,n}^{FJRW}:=c_{g,\gamma}\int_{[\mathscr{W}_{g,n}(\gamma_1,\ldots,\gamma_n)]^{\mathrm{vir}}}\alpha_1\cdots\alpha_n\cdot\overline{\psi}_1^{l_1}\cdots\overline{\psi}_n^{l_n},$$

where $\gamma_1,\ldots,\gamma_n$ are determined by the fact that $\alpha_i\in\mathcal{H}_{\gamma_i}\subset\mathcal{H}_W$, and the $\overline{\psi}$ classes are defined by pullback via the forgetful map $\mathscr{W}_{g,n}\to\overline{\mathcal{M}}_{g,n}$. Here, $c_{g,\gamma}$ is a constant depending on the genus and the orbifold decorations $\gamma_1,\ldots,\gamma_n$; it is needed to ensure that the correlators define a Cohomological Field Theory, but we will not bother specifying it here.

When all γ_i are "narrow"— that is, $\mathrm{Fix}(\gamma_i)=\{0\}$— these definitions simplify substantially. Then $\mathcal{H}_{\gamma_i}=\mathbb{C}$, and the choice of $\alpha_1,\ldots,\alpha_n$ amounts to a choice of component of the moduli space in which the line bundles have prescribed non-trivial multiplicity. Furthermore, on such components of the moduli space, the analytic construction of the virtual cycle can be replaced by an algebraic construction. In the generalization described in this thesis, we will restrict to the narrow situation, which will allow us to circumvent some of the complications in Fan-Jarvis-Ruan's setup.

The genus-g FJRW invariants are encoded in a generating function

$$\mathcal{F}^g_{FJRW}(\mathbf{t}) = \sum_n \frac{1}{n!}\langle \mathbf{t}(\overline{\psi_1}), \ldots, \mathbf{t}(\overline{\psi_n})\rangle^{FJRW}_{g,n},$$

where $\mathbf{t}(z) = t_0 + t_1 z + t_2 z^2 + \cdots$ is a general element of $\mathcal{H}_W[z]$. The Lagrangian cone of FJRW theory is defined as

$$\mathcal{L}_{FJRW} = \{(\mathbf{p}, \mathbf{q}) \mid \mathbf{p} = d_{\mathbf{q}}\mathcal{F}^0_{FJRW}\} \subset T^*(\mathcal{H}_W[z]) \cong \mathcal{H}_W((z^{-1})),$$

in which the variable $\mathbf{q}$ is related to $\mathbf{t}$ via the dilaton shift. See Section 3.1 of [11] or Section 4.1.3 below for a more complete definition of $\mathcal{L}_{FJRW}$ and its role in Givental's formalism.

A particularly important slice of the Lagrangian cone is given by the J-function,

$$J_{FJRW}(\mathbf{t}, z) = 1 \cdot z + \mathbf{t} + \sum_n \frac{1}{n!}\left\langle \mathbf{t}(\overline{\psi}), \ldots, \mathbf{t}(\overline{\psi}), \frac{\phi_\alpha}{z - \overline{\psi}}\right\rangle^{FJRW}_{0,n+1} \phi^\alpha,$$

in which ϕ_α runs over a basis for $\mathcal{H}_W$ with dual basis $\{\phi^\alpha\}$. The definitions of these objects all precisely mimic the corresponding definitions in Gromov-Witten theory; an exposition in that setting can be found, for example, in [20].

All of the ideas described in this section (the moduli space of W-structures, the state space, the correlators, and their generating functions) can be developed more generally with respect to a choice of subgroup $G \subset G_W$, under a certain admissibility condition on G. In the theory for G, one considers only components of $\mathscr{W}_{g,n}$ on which all of the multiplicities lie in G, and the state space is

$$\mathcal{H}_{W,G} = \bigoplus_{\gamma \in G} \mathcal{H}_\gamma.$$

In particular, in this thesis we will restrict to the choice where G is equal to

$$J = \left\langle \left(e^{2\pi i \frac{c_1}{d}}, \ldots, e^{2\pi i \frac{c_N}{d}} \right) \right\rangle \subset G_W. \tag{1.5}$$

One consequence of this restriction (Proposition 2.3.13 of [11]) is that the N line bundles parameterized by $\mathcal{H}_{W,J}$ are all tensor powers of a single bundle L.

1.3 The LG/CY correspondence for the quintic

As described in Section 1.1, the original motivation for FJRW theory was to make sense of Witten's ADE conjecture. This was accomplished (with a necessary modification to the conjecture) in Corollary 6.1.4 of [27]. However, the fact that Witten described the solutions to equation (1.4) as the "Landau-Ginzburg A-model" suggests, motivated by predictions from physics, that it should also have another mathematical function: it should fit into a Landau-Ginzburg/Calabi-Yau correspondence. Framed in mathematical terms, this means that there should be an "equivalence" of some form between the FJRW theory associated to a polynomial W and the Gromov-Witten theory of the hypersurface $X_W := \{W = 0\} \subset \mathbb{P}(c_1, \ldots, c_N)$, assuming W satisfies a Calabi-Yau condition.

For the case where $W = x_1^5 + \cdots + x_5^5$ is the quintic and FJRW theory is considered with respect to the group J defined in (1.5), Chiodo and Ruan [11] made this equivalence precise and proved it at the level of genus-zero invariants. In their formulation, the LG/CY correspondence involves proving two statements:

1. A state space isomorphism (sometimes called the *cohomological LG/CY correspondence*), which is the statement that there is a degree-preserving isomorphism $\mathcal{H}_{W,J} \cong H^*(X_W)$ under an appropriate grading on each.

2. The existence of a degree-preserving symplectic transformation $\mathbb{U}$ on the

space $\mathcal{H}_{W,J}((z^{-1})) \cong H^*(X_W)((z^{-1}))$ that maps the Lagrangian cone $\mathcal{L}_{FJRW} \subset \mathcal{H}_{W,J}((z^{-1}))$ encoding genus-zero FJRW invariants to an analytic continuation of the Lagrangian cone $\mathcal{L}_{GW} \subset H^*(X_W)((z^{-1}))$ encoding genus-zero Gromov-Witten invariants.

Their proof is quite complicated, and in particular involves placing both Gromov-Witten theory and FJRW theory in the context of mirror symmetry. It is only in the B-model, the other side of the mirror, that a relationship between the two theories is provided via an extra complex parameter on that side. Specifically, Chiodo-Ruan exhibit I-functions (B-model generating functions) for both theories, and by understanding the variable in the I-function as an analytic function as opposed to a merely formal parameter, they relate the two I-functions via analytic continuation.

Why, though, without making reference either to mirror symmetry or to the physical justification, might one expect the FJRW theory of $W = x_1^5 + \cdots + x_5^5$ and the Gromov-Witten theory of the quintic hypersurface to coincide? As a first approximation toward explaining this fact mathematically, let us study only the cohomological correspondence.

Consider a quotient

$$\frac{\mathbb{C}^5 \times \mathbb{C}}{\mathbb{C}^*},$$

where the coordinates are denoted $x_1, \ldots, x_5, p$ and $\mathbb{C}^*$ acts by

$$\lambda(x_1, \ldots, x_5, p) = (\lambda x_1, \ldots, \lambda x_5, \lambda^{-5} p).$$

Then the polynomial

$$\overline{W}(x_1, \ldots, x_5, p) = p \cdot (x_1^5 + \cdots + x_5^5)$$

gives a well-defined map out of this quotient. The quotient itself, however, is geometrically bad; it is not Deligne-Mumford and not even separated. To make good geometric sense of the quotient, one should consider instead a GIT quotient

$$(\mathbb{C}^5 \times \mathbb{C}) /\!/_\theta \, \mathbb{C}^*$$

associated to a choice of character $\theta \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{C}^*, \mathbb{C}^*) \cong \mathbb{Z}$ of $\mathbb{C}^*$. The sign of θ leads to two distinct possible quotients:

$\underline{\theta > 0}$: The only unstable points are those with $\mathbf{x} = 0$, so

$$(\mathbb{C}^5 \times \mathbb{C}) /\!/_\theta \, \mathbb{C}^* = \left[\frac{(\mathbb{C}^5 \setminus \{0\}) \times \mathbb{C}}{\mathbb{C}^*} \right] \cong \mathcal{O}_{\mathbb{P}^4}(-5).$$

$\underline{\theta < 0}$: The only unstable points are those with $p = 0$, so

$$(\mathbb{C}^5 \times \mathbb{C}) /\!/_\theta \, \mathbb{C}^* = \left[\frac{\mathbb{C}^5 \times (\mathbb{C} \setminus \{0\})}{\mathbb{C}^*} \right] \cong [\mathbb{C}^5/\mathbb{Z}_5].$$

The polynomial $\overline{W}$ descends to give a map out of either of these quotients. When $\theta < 0$, the relative cohomology

$$H^*_{CR}([\mathbb{C}^5/\mathbb{Z}_5], \overline{W}^{+\infty}; \mathbb{C})$$

is precisely the state space of FJRW theory with respect to the group $J \cong \mathbb{Z}_5$, where as before, $\overline{W}^{+\infty} = (\mathrm{Re}\overline{W})^{-1}(M, \infty)$ for $M \gg 0$. On the other hand, when $\theta > 0$, one can compute that

$$H^*(\mathcal{O}_{\mathbb{P}^4}(-5), \overline{W}^{+\infty}; \mathbb{C}) \cong H^*(\mathbb{P}^4, \mathbb{P}^4 \setminus X_5; \mathbb{C}) \cong H^*(X_5; \mathbb{C})$$

up to a degree shift, by deformation retraction and the Thom isomorphism. The latter is the state space for the Gromov-Witten theory of the quintic, the vector space from which insertions to Gromov-Witten invariants are drawn.

Thus, by varying the parameter θ, the state spaces for FJRW theory and Gromov-Witten theory arise in entirely analogous ways. This observation can in fact be leveraged to prove the state space isomorphism [10], as we will explain in Chapter II. Furthermore, an analogous observation can be made in families to motivate the connection between the moduli spaces $\mathscr{W}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}(X_W, \beta)$ on the Landau-Ginzburg and Calabi-Yau sides. We will hold off on explaining this moduli-level dichotomy for the moment, however, returning to it in greater generality after expanding the picture to allow for complete intersections.

1.4 The hybrid model

The work of theoretical physicists [50] suggests that a Landau-Ginzburg/Calabi-Yau correspondence should apply not only to hypersurfaces in weighted projective space but to more general complete intersections. In physical language, the Landau-Ginzburg side of the correspondence should be given by the gauged linear sigma model (GLSM), which generalizes the Witten equation to the *gauged Witten equation*

$$\begin{aligned} \overline{\partial}_A + \overline{\frac{\partial W}{\partial u_i}} &= 0 \\ *F_A &= \mu, \end{aligned}$$

in which A is a connection on a certain principal bundle naturally associated to the weighted projective space and μ is the moment map that arises out of viewing the weighted projective space as a symplectic quotient.

In fact, under an appropriate mathematical theory of the gauged linear sigma model, both the Gromov-Witten theory of the complete intersection and the corresponding Landau-Ginzburg theory can be viewed as GLSMs, and the Landau-

Ginzburg/Calabi-Yau correspondence can be understood as a variation of the moment map [24]. This shows up in FJRW theory and the hybrid model as the variation of GIT mentioned in Section 1.3.

The general theory of GLSMs from a mathematical perspective remains a work-in-progress by Fan-Jarvis-Ruan [24]. The content of this thesis can be understood as a development of their model in the very special case of a complete intersection of hypersurfaces of the same degree in weighted projective space. We will also make a further restriction to the narrow sectors, which implies a decoupling of the gauged Witten equation analogous to (1.3). Thus, as in the r-spin case, we will not ultimately need to make reference to the gauged Witten equation in order to define the theory.

A first step toward understanding the hybrid model associated to a collection of polynomials is to mimic the ideas discussed at the end of Section 1.3, constructing a GIT quotient out of which the collection of polynomials defines a map. As in the case of the quintic, this will dictate the state space on the Landau-Ginzburg side and illuminate its connection to the Gromov-Witten state space.

Let $W_1, \ldots, W_r \in \mathbb{C}[x_1, \ldots, x_N]$ be a collection of quasihomogeneous polynomials, all of weights $c_1, \ldots, c_N$ and degree d, defining a nonsingular hypersurface $X_{\overline{W}} \subset \mathbb{P}(c_1, \ldots, c_N)$. These polynomials can be combined into

$$\overline{W}(x_1, \ldots, x_N, p_1, \ldots, p_r) = p_1 W_1(x_1, \ldots, x_N) + \cdots + p_r W_r(x_1, \ldots, x_N),$$

which gives a map out of the quotient

$$\frac{\mathbb{C}^N \times \mathbb{C}^r}{\mathbb{C}^*}$$

if $\mathbb{C}^*$ acts by

$$\lambda(x_1, \ldots, x_N, p_1, \ldots, p_r) = (\lambda^{c_1} x_1, \ldots, \lambda^{c_N} x_N, \lambda^{-d} p, \ldots, \lambda^{-d} p).$$

As in Section 1.3, there are two distinct ways to choose a character θ of $\mathbb{C}^*$ to interpret the above as a GIT quotient. If $\theta > 0$, then the quotient is

$$\frac{(\mathbb{C}^N \setminus \{0\}) \times \mathbb{C}^r}{\mathbb{C}^*} = \mathcal{O}_{\mathbb{P}^{N-1}}(-d)^{\oplus r},$$

where we identify the bundle geometrically with its total space. If $\theta < 0$, then the quotient is

$$\frac{\mathbb{C}^N \times (\mathbb{C}^r \setminus \{0\})}{\mathbb{C}^*} = \mathcal{O}_{\mathbb{P}(d,\ldots,d)}(-1)^{\oplus r},$$

where $\mathbb{P}(d,\ldots,d)$ is the weighted projective space in which $\mathbb{C}^*$ acts with weight r in each of the r factors; in other words, it is a (nontrivial) $\mathbb{Z}_d$ gerbe over the ordinary projective space $\mathbb{P}^{d-1}$.

One can check ([10], or Proposition 2.6.1 below in the special cases of interest in this thesis) that

$$H^*(\mathcal{O}_{\mathbb{P}^{N-1}}(-d)^{\oplus r}, \overline{W}^{+\infty}; \mathbb{C}) \cong H^*(X_{\overline{W}}; \mathbb{C}),$$

after an appropriate degree shift; here, as above, $\overline{W}^{+\infty} = (\mathrm{Re}\overline{W})^{-1}(M, +\infty)$ for $M \gg 0$. By analogy, then, the state space for the hybrid theory should be

$$\mathcal{H}_{hyb}(W_1, \ldots, W_r) := H^*_{CR}(\mathcal{O}_{\mathbb{P}(d,\ldots,d)}(-1)^{\oplus N}, \overline{W}^{+\infty}; \mathbb{C}).$$

More explicitly, the Chen-Ruan cohomology of $\mathcal{O}_{\mathbb{P}(d,\ldots,d)}(-1)^{\oplus N}$ is defined as the cohomology of the inertia stack, and thus its components are indexed by $\lambda \in \mathbb{C}^*$ with nontrivial fixed point sets. The only such elements are dth roots of unity. If, furthermore, $\lambda \in \mathbb{C}^*$ has $\lambda^{c_i} \neq 1$ for all i, then the corresponding component of the inertia stack is simply

$$\frac{\{0\} \times (\mathbb{C}^r \setminus \{0\})}{\mathbb{C}^*} = \mathbb{P}(d, \ldots, d).$$

Since this is disjoint from $\overline{W}^{+\infty}$, each such λ yields a component of the hybrid model state space isomorphic to $H^*(\mathbb{P}(d,\ldots,d)) = H^*(\mathbb{P}^{r-1})$. These fairly simple components of the state space are called the *narrow sectors*, and play the most important role in the cases considered in this thesis. For example, one case we will study is the complete intersection $X_{3,3}$ of two degree-3 hypersurfaces in $\mathbb{P}^5$, for which one can check that

$$\mathcal{H}_{hyb}(W_1, W_2) \cong H^*(\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 6}, \overline{W}^{+\infty}) \oplus H^*(\mathbb{P}^1) \oplus H^*(\mathbb{P}^1).$$

Here, the last two summands are the narrow sectors.

Once the state space for the hybrid model is constructed, one must define a moduli space over which insertions from the state space can be integrated. To put it roughly, the moduli space associated to a collection of quasihomogeneous polynomials $W_1, \ldots, W_r$ as above will be defined as

$$\widetilde{\mathcal{M}}^d_{g,n}(\mathbb{P}^{r-1}, \beta) = \{(f : C \to \mathbb{P}^{r-1}; x_1, \ldots, x_n; L; \varphi)\},$$

in which $(C; x_1, \ldots, x_n)$ is a marked orbifold curve, f is an orbifold stable map of degree β, L is an orbifold line bundle, and φ is an isomorphism

$$\varphi : L^{\otimes d} \xrightarrow{\sim} f^*\big(\mathcal{O}_{\mathbb{P}^{r-1}}(-1)\big) \otimes \omega_{C,\log}.$$

See Section 2.6.1 below for a more careful definition. Note that, just like the moduli space of stable maps in Gromov-Witten theory, the hybrid moduli space has ψ classes and evaluation maps $\mathrm{ev}_i : \widetilde{\mathcal{M}}^d_{g,n}(\mathbb{P}^{r-1}, \beta) \to \mathbb{P}^{r-1}$ at each marked point. Furthermore, just like the moduli space of W-structures, $\widetilde{\mathcal{M}}^d_{g,n}(\mathbb{P}^{r-1}, \beta)$ has a decomposition according to the weights of the actions of the isotropy groups G_{x_i} on the fibers of L. This decomposition indexes components of the moduli space by elements of $\mathbb{Z}_d$, which also index the summands of the state space.

Thus, given a collection of elements $\alpha_1, \ldots, \alpha_n$ chosen from the narrow sectors of $\mathcal{H}_{hyb}(W_1, \ldots, W_r)$ and nonnegative integers $l_1, \ldots, l_n$, one obtains correlators in the hybrid model by viewing $\alpha_i \in H^*(\mathbb{P}^{r-1})$ and integrating:

(1.6)

$$\langle \tau_{l_1}(\alpha_1)\cdots\tau_{l_n}(\alpha_n)\rangle^{hyb}_{g,n,\beta} = c_{g,\mathbf{m}} \int_{[\widetilde{\mathcal{M}}^d_{g,(m_1,\ldots,m_n)}(\mathbb{P}^{r-1},\beta)]^{\mathrm{vir}}} \mathrm{ev}_1^*(\alpha_1)\psi_1^{l_1}\cdots \mathrm{ev}_n^*(\alpha_n)\psi_n^{l_n},$$

where $c_{g,\mathbf{m}}$ is a constant (defined explicitly in Definition 3.5.3) and m_i is defined as the element of $\mathbb{Z}_d$ indexing the sector of the state space from which α_i is chosen. To extend this definition to arbitrary insertions in $\mathcal{H}_{hyb}(W_1,\ldots,W_r)$, a correlator is set to zero if any of its insertions is not narrow.

Of course, to make sense of (1.6), we will need to construct a virtual fundamental cycle on the narrow components against which to integrate. This will be carried out in detail in Section 3.4, using the cosection construction developed by Kiem-Li-Chang [39] [4]. The basic idea, which was first put into practice by Chang-Li [4] for the Gromov-Witten theory of the quintic and by Chang-Li-Li [5] for the corresponding FJRW theory, is to view $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^{r-1},\beta)$ as a substack of a certain noncompact moduli space

$$\mathfrak{S}_{g,\mathbf{m},\beta} = \left\{ (C;x_1,\ldots,x_n;L;s_1,\ldots,s_N;p_1,\ldots,p_r) \;\middle|\; \begin{array}{c} s_i\in H^0(L),\ p_j\in H^0(P) \\ P^{\otimes 3}\otimes\omega_{C,\log} \text{ is ample} \\ p_1,\ldots,p_r \text{ have no common zeroes} \end{array} \right\},$$

where

$$P = L^{\otimes -d}\otimes \omega_{C,\log}$$

and $\mathbf{m} = (m_1,\ldots,m_n)$ denotes the multiplicities of L. In particular, the conditions on $p_1,\ldots,p_r$ show that they collectively define a map $f : C \to \mathbb{P}^{r-1}$, from which perspective one has $L^{\otimes d} \cong f^*\mathcal{O}(-1)\otimes\omega_{C,\log}$. An obstruction theory for $\mathfrak{S}_{g,\mathbf{m},\beta}$ is already known from Chang-Li's work on moduli of sections [4]. From this obstruction theory on the larger moduli space, the cosection construction yields a virtual cycle supported only on the locus where $s_1 = \cdots = s_N = 0$, which is precisely $\widetilde{\mathcal{M}}^d_{g,(m_1,\ldots,m_n)}(\mathbb{P}^{r-1},\beta)$.

In addition to providing a construction of a virtual cycle on the hybrid moduli space, the cosection technique is particularly useful for understanding why the hybrid moduli space is defined as it is. Indeed, just as the state spaces for Gromov-Witten theory and the hybrid model arise by choosing characters that give two different stability conditions on a certain GIT quotient, the moduli spaces can be constructed by choosing two different stability conditions on the moduli space $\mathfrak{S}_{g,\mathbf{m},\beta}$.

Specifically, had we demanded that $s_1, \ldots, s_N$ rather than $p_1, \ldots, p_r$ define a stable map to projective space, then $\mathfrak{S}_{g,\mathbf{m},\beta}$ would have parameterized maps $f : C \to \mathbb{P}^{N-1}$ together with the extra data of section $p_j \in f^*\mathcal{O}(-1) \otimes \omega_{C,\log}$. Applying the cosection construction yields a virtual cycle supported on a compact locus inside this moduli space, and in this case, that locus consists of stable maps whose image lands in the complete intersection $X_{\mathbf{W}} \subset \mathbb{P}^{N-1}$ and for which the sections p_j all vanish; that is, it coincides precisely with the moduli space of stable maps to $X_{\mathbf{W}}$.

Thus, the moduli spaces over which Gromov-Witten and hybrid invariants are defined can also be viewed as arising from variation of a stability parameter. However, we should mention that this dichotomy does not, at least at the current moment, provide any way to *prove* the Landau-Ginzburg/Calabi-Yau correspondence. One problem is that the virtual cycle on the Gromov-Witten side obtained by the cosection construction does not obviously agree with the usual definition of the virtual cycle for stable maps; Chang-Li proved in [4] that invariants for the quintic threefold defined by way of the two competing virtual cycles agree up to a sign, but no such result has yet been formulated for complete intersections.

Even with such a result, there is no obvious notion of compatibility between the cosection-localized virtual cycles obtained from different stability conditions that would imply a correspondence between the resulting invariants. Some progress toward understanding the LG/CY correspondence via variation of stability condition has been made by Ross-Ruan [45] and Clader-Marcus-Ruan-Shoemaker [17] by placing both theories in the context of Ciocan-Fontanine-Kim's stable quasimaps, but a full understanding of this picture, especially the role of analytic continuation, is yet to be achieved.

1.5 Statement of results

Let $W_1, \ldots, W_r \in \mathbb{C}[x_1, \ldots, x_N]$ be a collection of quasihomogeneous polynomials, all of weights $c_1, \ldots, c_N$ and degree d. Assume that the hypersurface X_W cut out by these polynomials is nonsingular and satisfies the Calabi-Yau condition

$$dr = \sum_{i=1}^{N} c_i.$$

The genus-zero Landau-Ginzburg/Calabi-Yau correspondence for the hybrid model is the assertion that there is a degree-preserving isomorphism between the state spaces $\mathcal{H}_{hyb}(W_1, \ldots, W_r)$ and $H^*(X_W)$, and that the Lagrangian cones $\mathcal{L}_{hyb}$ and $\mathcal{L}_{GW}$ encoding the two genus-zero theories are related by a linear transformation and analytic continuation.

In this thesis, we prove that the correspondence holds whenever X_W is a Calabi-Yau threefold in ordinary, rather than weighted, projective space. This leaves only three possibilities for X_W: the quintic hypersurface $X_5 \subset \mathbb{P}^4$, the intersection of two cubic hypersurfaces $X_{3,3} \subset \mathbb{P}^5$, and the intersection of four quadrics $X_{2,2,2,2} \subset \mathbb{P}^7$. The first of these is the content of [11], while $X_{3,3}$ and $X_{2,2,2,2}$ represent new results.

After verifying the state space isomorphism in these cases (Proposition 2.6.1), the strategy for proving the relationship between the two Lagrangian cones is the same as in [11]: the Lagrangian cones are determined by the small J-functions $J_{GW}(t,z)$ and $J_{hyb}(t,z)$, and each of these is related to an I-function. On the Gromov-Witten side, the definition of I_{GW} and its relationship to J_{GW} were shown in [28]. Explicitly, I_{GW} is a hypergeometric series in the variable $q = \exp(t_0^1)$, where $t_0 = \sum t_0^\alpha \varphi_\alpha$ and $\varphi_1 \in H^2(X)$. Expanded in the variable $H \in H^*(X_W)$ corresponding to the hyperplane class, I_{GW} assembles the solutions to a Picard-Fuchs equation. In our two cases of interest, the Picard-Fuchs equations are:

$$\left[D_q^4 - 3^6 q \left(D_q + \frac{1}{3}\right)^2 \left(D_q + \frac{2}{3}\right)^2\right] I_{GW} = 0$$

and

$$\left[D_q^4 - 2^8 q \left(D_q + \frac{1}{2}\right)^4\right] I_{GW} = 0$$

for the cubic and quadric complete intersections, respectively, where $D_q = q\frac{\partial}{\partial q}$. There is a "mirror map"— that is, an explicit change of variables

$$q' = \frac{g_{GW}(q)}{f_{GW}(q)}$$

for $\mathbb{C}$-valued functions g_{GW} and f_{GW}— under which the small J-function J_{GW} matches I_{GW}:

$$\frac{I_{GW}(q,z)}{f_{GW}(q)} = J_{GW}(q',z).$$

We provide an analogous story on the Landau-Ginzburg side for each of the examples mentioned above. Using the machinery of twisted invariants developed in [20], we construct a hybrid I-function in each case. These are:

$$I_{hyb}(t,z) = \sum_{\substack{d\geq 0 \\ d\not\equiv -1 \mod 3}} \frac{ze^{(d+1+\frac{H^{(d+1)}}{z})t}}{3^{6\lfloor \frac{d}{3}\rfloor}} \frac{\prod_{\substack{1\leq b\leq d \\ b\equiv d+1 \mod 3}} (H^{(d+1)}+bz)^4}{\prod_{\substack{1\leq b\leq d \\ b\not\equiv d+1 \mod 3}} (H^{(d+1)}+bz)^2} \tag{1.7}$$

for the cubic and

(1.8)
$$I_{hyb}(t,z) = \sum_{\substack{d\geq 0\\ d\not\equiv -1 \mod 2}} \frac{ze^{(d+1+\frac{H^{(d+1)}}{z})t}}{2^{8\lfloor\frac{d}{2}\rfloor}} \frac{\prod\limits_{\substack{1\leq b\leq d\\ b\equiv d+1 \mod 2}} (H^{(d+1)}+bz)^4}{\prod\limits_{\substack{1\leq b\leq d\\ b\not\equiv d+1 \mod 2}} (H^{(d+1)}+bz)^4}$$

for the quadric, where $t = t + 0z + 0z^2 + \cdots$ lies in the degree-2 part of the Landau-Ginzburg state space.

These I-functions are shown in Theorem IV.1 to lie on the Lagrangian cones $\mathcal{L}_{hyb}$ for their respective hybrid theories. As in Gromov-Witten theory, the Lagrangian cone has a special geometric property that allows any function lying on it to be determined from only the first two coefficients in its expansion in powers of z. Using the expressions (1.7) or (1.8), one can write

$$I_{hyb}(t,z) = \omega_1^{hyb}(t)\cdot 1^{(1)}\cdot z + \omega_2^{hyb}(t) + O(z^{-1})$$

for explicit $\mathbb{C}$-valued functions $\omega_1^{hyb}(t)$ and $\omega_2^{hyb}(t)$ in either case. We therefore obtain the following theorem:

Theorem I.1. *Consider the hybrid model I-function (1.7) associated to a generic collection of two homogeneous cubic polynomials in six variables, whose coefficients when expanded in powers of $H^{(i)}$ span the solution space of the Picard-Fuchs equation*

$$\left[D_\psi^4 - 3^2\psi^{-1}\left(D_\psi - \frac{1}{3}\right)^2\left(D_\psi - \frac{2}{3}\right)^2\right] I_{hyb} = 0$$

for $D_\psi = \psi\frac{\partial}{\partial\psi}$ and $\psi = e^{3t}$. This I-function and the hybrid J-function J_{hyb} associated to the same collection of polynomials are related by an explicit change of variables (mirror map)

$$\frac{I_{hyb}(t,-z)}{\omega_1^{hyb}(t)} = J_{hyb}(t',-z), \quad \textit{where}\;\; t' = \frac{\omega_2^{hyb}(t)}{\omega_1^{hyb}(t)}.$$

The analogous statement holds for the hybrid model I-function (1.8) associated to a generic collection of four homogeneous quadric polynomials in eight variables, for which the coefficients span the solution space of the Picard-Fuchs equation

$$\left[D_\psi^4 - 2^4\psi^{-1}\left(D_\psi - \frac{1}{2}\right)^4\right] I_{hyb} = 0$$

with $D_\psi = \psi\frac{\partial}{\partial\psi}$ *and* $\psi = e^{2t}$*.*

The fact that the hybrid I-functions assemble the solutions to the specified Picard-Fuchs equations is an easy consequence of the explicit expressions for these functions. These equations are the same as the Picard-Fuchs equations for the corresponding Calabi-Yau complete intersections after setting $q = 3^{-4}\psi^{-1}$ or $q = 2^{-4}\psi^{-1}$, respectively. It follows that, if we use the state space isomorphism to identify the state spaces in which the I-functions take values, then I_{hyb} and the analytic continuation of I_{GW} to the ψ-coordinate patch are both comprised of bases of solutions to the same differential equation, and hence are related by a linear isomorphism performing the change of basis.

A simple dimension count shows that all of the hybrid model correlators defining $\mathcal{L}_{hyb}$ can be computed via the string equation from the correlators appearing in the small J-function. It follows from the relationship between I_{hyb} and J_{hyb} that I_{hyb} also determines the entire cone. Thus, we arrive at the following corollary:

Corollary 1.5.1. *For either the complete intersection* $X_{3,3} \subset \mathbb{P}^5$ *or* $X_{2,2,2,2} \subset \mathbb{P}^7$*, there is a* $\mathbb{C}[z, z^{-1}]$*-valued degree-preserving linear transformation mapping* $\mathcal{L}_{hyb}$ *to the analytic continuation of* $\mathcal{L}_{GW}$ *near* $t = 0$*. That is, the genus-zero Landau-Ginzburg/Calabi-Yau correspondence holds in these cases.*

1.6 Outline

We begin, in Chapter II, by establishing the necessary terminology on singularities and defining the state space for the hybrid model. We then prove that in the two cases of interest, this state space is isomorphic to the cohomology of the corresponding complete intersection. In Chapter III, the quantum theory of the Landau-Ginzburg model is developed for arbitrary complete intersections of the same weights and degree in weighted projective space; that is, we define a moduli space and construct a virtual cycle in order to specify correlators as integrals over the moduli space. Here, the crucial ingredient is the cosection construction of Kiem-Li-Chang, so we include a slight detour to explain their ideas. At the end of Chapter III, we specialize to the two examples of interest, and in Chapter IV, we place those two examples in the context of Givental's quantization formalism, proving that the Lagrangian cone encoding the hybrid theory can be obtained from the Lagrangian cone encoding the genus-zero Gromov-Witten theory of projective space. This leads to the definition of the I-function and the proof of the LG/CY correspondence for these two examples.

CHAPTER II

The Landau-Ginzburg state space

We begin with a fairly terse overview of some terminology related to singularities, which can be found in greater detail in [27].

2.1 Quasihomogeneous singularities

The type of singularities for which FJRW theory, and more generally the hybrid model, is defined are as follows.

Definition 2.1.1. A polynomial $W \in \mathbb{C}[x_1, \ldots, x_N]$ is *quasihomogeneous* if there exist positive integers $c_1, \ldots, c_N$ (known as *weights*) and d (the *degree*) such that

$$W(\lambda^{c_1} x_1, \ldots, \lambda^{c_N} x_N) = \lambda^d W(x_1, \ldots, x_N)$$

for all $\lambda \in \mathbb{C}$ and $(x_1, \ldots, x_N) \in \mathbb{C}^N$.

Let $W_1, \ldots, W_r \in \mathbb{C}[x_1, \ldots, x_N]$ be a collection of quasihomogeneous polynomials in N complex variables all having the same weights and degree.

Definition 2.1.2. The collection $W_1, \ldots, W_r$ is called *nondegenerate* if

1. the *charges* $q_i := c_i/d$ are uniquely determined by each W_j;
2. the only $\mathbf{x} \in \mathbb{C}^N$ for which all of the polynomials W_j and all of their partial derivatives vanish is $\mathbf{x} = 0$.

The second condition implies that the hypersurface $X_{\overline{W}} = \{W_1 = \cdots = W_r = 0\} \subset \mathbb{P}(c_1, \ldots, c_N)$ cut out by the polynomials is nonsingular. Furthermore, $X_{\overline{W}}$ is Calabi-Yau if

$$dr = \sum_{i=1}^{N} c_i.$$

All of the collections of quasihomogeneous polynomials considered in what follows will be assumed nondegenerate and Calabi-Yau.

Associated to such a collection is a group of symmetries. In order to define this group, we will prefer to think of the W_i as together defining a polynomial

$$\overline{W}(\mathbf{x}, \mathbf{p}) = p_1 W_1(\mathbf{x}) + \cdots + p_r W_r(\mathbf{x}) \in \mathbb{C}[x_1, \ldots, x_N, p_1, \ldots, p_r].$$

From this perspective, symmetries of the collection of polynomials are simply symmetries of $\overline{W}$ in the sense of FJRW theory. Explicitly:

Definition 2.1.3. The *group $G_{W_1,\ldots,W_r}$ of diagonal symmetries* of a collection of quasihomogeneous polynomials of charges $c_1, \ldots, c_N$ and degree d is

$$G_{W_1,\ldots,W_r} = \{(\alpha, \beta) \in (\mathbb{C}^*)^N \times (\mathbb{C}^*)^r \mid \overline{W}(\alpha\mathbf{x}, \beta\mathbf{p}) = \overline{W}(\mathbf{x}, \mathbf{p}) \text{ for all } (\mathbf{x}, \mathbf{p}) \in \mathbb{C}^N \times \mathbb{C}^r\}.$$

The group of diagonal symmetries always contains the subgroup

$$J = \{(t^{c_1}, \ldots, t^{c_N}, t^{-d}, \ldots, t^{-d}) \mid t \in \mathbb{C}^*\}.$$

This is the analogue of the group denoted $\langle J \rangle$ in FJRW theory.

As mentioned in Chapter I, there is an extra datum in the definition of FJRW theory that will not be present in the current work: a subgroup G of the group of diagonal symmetries containing J. The theory developed here corresponds to the choice $G = J$. This, in particular, explains why the moduli space defined in

Chapter III will parameterize powers of a single line bundle rather than allowing N separate choices.

2.2 State space

Let $W_1, \dots, W_r \in \mathbb{C}[x_1, \dots, x_N]$ be a collection of quasihomogeneous polynomials of the same weights and degree. Associated to such a collection, the *state space* of the hybrid theory is the following vector space:

$$\mathcal{H}_{hyb}(W_1, \dots, W_r) = H^*_{CR}\left(\frac{\mathbb{C}^N \times (\mathbb{C}^r \setminus \{0\})}{J}, \overline{W}^{+\infty}; \mathbb{C}\right), \tag{2.1}$$

where $\overline{W}^{+\infty} = (\mathrm{Re}\overline{W})^{-1}(M, +\infty)$ for $M \gg 0$ and J acts by multiplication in each factor.

As a vector space, Chen-Ruan cohomology is the cohomology of the inertia stack, whose objects are pairs $((\mathbf{x}, \mathbf{p}), \gamma)$, where $\gamma \in J$, $(\mathbf{x}, \mathbf{p}) \in \mathbb{C}^N \times (\mathbb{C}^r \setminus \{0\})$, and $\gamma(\mathbf{x}, \mathbf{p}) = (\mathbf{x}, \mathbf{p})$. The only elements of J with nontrivial fixed-point sets are those of the form

$$(t^{c_1}, \dots, t^{c_N}, 1, \dots, 1),$$

where t is a dth root of unity, so such elements index the components of the inertia stack. These components are known as *twisted sectors*, and the component corresponding to $t = 1$ is called the *nontwisted sector*.

2.3 Degree shifting

As is usual in Chen-Ruan cohomology, we should shift the degree. The grading on the state space, however, will be shifted somewhat differently from the ordinary degree shift in Chen-Ruan cohomology.

Definition 2.3.1. Let $\gamma = (e^{2\pi i \Theta_1^\gamma}, \dots, e^{2\pi i \Theta_N^\gamma}, 1, \dots, 1) \in J$ be an element with nontrivial fixed-point set, where $\Theta_i^\gamma \in \{0, \frac{1}{d}, \dots, \frac{d-1}{d}\}$. The *degree-shifting number* or *age shift* for γ is

$$\iota(\gamma) = \sum_{j=1}^{N} (\Theta_j^\gamma - q_j),$$

where q_j are the charges defined in Definition 2.1.1.

Now, given $\alpha \in \mathcal{H}_{hyb}(W_1, \ldots, W_r)$ from the twisted sector indexed by γ, we set

$$\deg_{\overline{W}}(\alpha) = \deg(\alpha) + 2\iota(\gamma),$$

where $\deg(\alpha)$ denotes the ordinary degree of α as an element of the cohomology of the inertia stack. This gives a grading on $\mathcal{H}_{hyb}(W_1, \ldots, W_r)$.

2.4 Broad and narrow sectors

A twisted sector indexed by an element $(t^{c_1}, \ldots, t^{c_N}, 1, \ldots, 1) \in J$ will be called *narrow* if there is no i with $t^{c_i} = 1$. This condition ensures that the sector is supported on the suborbifold

$$\frac{\{0\} \times (\mathbb{C}^r \setminus \{0\})}{J} \subset \frac{\mathbb{C}^N \times (\mathbb{C}^r \setminus \{0\})}{J},$$

whose coarse underlying space is $\mathbb{P}^{r-1}$. Since the above is disjoint from $\overline{W}^{+\infty}$, the relative cohomology on these sectors is an absolute cohomology group, and indeed, each narrow sector is isomorphic to $H^*(\mathbb{P}^{r-1})$. A sector that is not narrow will be called *broad*.

2.5 Cases of interest

For most of what follows, we will restrict to the cases mentioned in the introduction, in which $W_1, \ldots, W_r$ define a Calabi-Yau threefold complete intersection in ordinary, rather than weighted, projective space. This leaves the following three possibilities:

1. $r = 1, d = 5, N = 5$ (quintic hypersurface in $\mathbb{P}^4$);

2. $r = 2, d = 3, N = 6$ (intersection of two cubics in $\mathbb{P}^5$);

3. $r = 4, d = 2, N = 8$ (intersection of four quadrics in $\mathbb{P}^7$).

The first case was handled in [11], while the second and third are considered in this work.

In case (2), the state space is

$$H^*_{CR}\left(\frac{\mathbb{C}^6 \times (\mathbb{C}^3 \setminus \{0\})}{\mathbb{C}^*}, \overline{W}^{+\infty}; \mathbb{C}\right),$$

where $\mathbb{C}^*$ acts via

$$\lambda(x_1, \dots, x_6, p_1, p_2, p_3) = (\lambda, \dots, \lambda, \lambda^{-3}, \lambda^{-3}, \lambda^{-3}). \tag{2.2}$$

The orbifold in question[1], then, is the total space of the orbifold vector bundle $\mathcal{O}_{\mathbb{P}(3,3)}(-1)^{\oplus 6}$, where

$$\mathcal{O}_{\mathbb{P}(3,3)}(-1) = \frac{(\mathbb{C}^3 \setminus \{0\}) \times \mathbb{C}}{\mathbb{C}^*}$$

with $\mathbb{C}^*$ acting with weights $(3, 3, 3, -1)$. The only broad sector is the nontwisted sector, while the twisted (narrow) sectors each contribute $H^*(\mathbb{P}^1)$. Thus, the decomposition of the state space into sectors is:

$$H^*(\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 6}, \overline{W}^{+\infty}) \oplus H^*(\mathbb{P}^1) \oplus H^*(\mathbb{P}^1).$$

A similar analysis shows that the state space in case (3) is

$$H^*(\mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus 8}, \overline{W}^{+\infty}) \oplus H^*(\mathbb{P}^3).$$

[1] If one considers $\mathbb{P}(3,3)$ as arising via the root construction applied to $\mathbb{P}^1$ with its line bundle $\mathcal{O}(-1)$, this is the natural third root of the pullback of $\mathcal{O}(-1)$ (see Section 2.1.5 of [37]).

2.6 Cohomological LG/CY correspondence

In the two new cases mentioned above, we verify that the state space isomorphism, or cohomological LG/CY correspondence, holds. This is only a simple special case of a general state space isomorphism for Calabi-Yau complete intersections that will be proved in upcoming work of Chiodo and Nagel [10]. It was discussed in a talk by J. Nagel at the Workshop on Recent Developments on Orbifolds at the Chern Institute of Mathematics in July 2011 and communicated to the author by A. Chiodo.

Proposition 2.6.1. *Let $W_1(x_1, \ldots, x_6)$ and $W_2(x_1, \ldots, x_6)$ be homogeneous cubic polynomials defining a complete intersection $X_{3,3} \subset \mathbb{P}^6$. Then the hybrid state space associated to these polynomials is isomorphic to the Gromov-Witten state space of $X_{3,3}$; that is,*

$$H^*(\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 6}, \overline{W}^{+\infty}) \oplus H^*(\mathbb{P}^1) \oplus H^*(\mathbb{P}^1) \cong H^*(X_{3,3}). \tag{2.3}$$

Moreover, this isomorphism is degree-preserving under the degree shift (2.3.1) for the left-hand side.

Similarly, there is a degree-preserving state space isomorphism for a collection of eight quadrics defining a complete intersection $X_{2,2,2,2} \subset \mathbb{P}^7$:

$$H^*(\mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus 8}, \overline{W}^{+\infty}) \oplus H^*(\mathbb{P}^3) \cong H^*(X_{2,2,2,2}).$$

Proof. The three summands on the left-hand side of (2.3) have degree shifts $-2, 0,$ and 2, respectively. Thus, the narrow sectors contribute one-dimensional summands in degrees 0, 2, 4, and 6. By the Lefschetz hyperplane principle, this matches the primitive cohomology of $X_{3,3}$, so all that remains in the cubic case is to prove that

$$H^k(\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 6}, \overline{W}^{+\infty}) \cong \begin{cases} H^3(X_{3,3}) & k = 7 \\ 0 & \text{otherwise.} \end{cases} \tag{2.4}$$

Similarly, in the quadric case, the only statement that is not immediate is

$$H^k(\mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus 8}, \overline{W}^{+\infty}) \cong \begin{cases} H^3(X_{2,2,2,2}) & k = 11 \\ 0 & \text{otherwise.} \end{cases}$$

The two arguments are entirely analogous and both elementary, so we describe only the cubic case.

It is useful to replace $\overline{W}^{+\infty}$ with a general fiber F of $\overline{W}$; this is called a Milnor fiber of $\overline{W}$, and is homotopy equivalent to $\overline{W}^{+\infty}$. Furthermore, for our convenience, we will write

$$\mathcal{O}_1 := \mathcal{O}_{\mathbb{P}(3,3)}(-1)^{\oplus 6}$$

and let $\mathcal{O}_1^{\times}$ denote the complement of the zero section in this bundle. Since we will be working with ordinary cohomology and not Chen-Ruan cohomology for orbifolds, we will identify $\mathcal{O}_1$ with its coarse underlying space $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 6}$.

The basic observation is that there is another bundle,

$$\mathcal{O}_3 := \mathcal{O}_{\mathbb{P}^5}(-3)^{\oplus 2},$$

and the complement $\mathcal{O}_3^{\times}$ of the zero section in $\mathcal{O}_3$ is isomorphic to $\mathcal{O}_1^{\times}$; indeed, they are precisely the same subset of the quotient

$$\frac{\mathbb{C}^6 \times \mathbb{C}^2}{\mathbb{C}^*}$$

(where $\mathbb{C}^*$ acts by $\lambda \cdot (\mathbf{x}, \mathbf{p}) = (\lambda \mathbf{x}, \lambda^{-3}\mathbf{p})$) on which neither $\mathbf{x}$ nor $\mathbf{p}$ vanishes. In particular, we will sometimes think of F as lying inside $\mathcal{O}_1^{\times}$ and sometimes think of it as lying inside $\mathcal{O}_3^{\times}$, and this interplay will yield the claim.

Having established the notation, consider first the case where $k \geq 4$ in (2.4). Then the long exact sequence of the pair $(\mathcal{O}_1, F)$ is

$$H^{k-1}(\mathcal{O}_1) \to H^{k-1}(F) \to H^k(\mathcal{O}_1, F) \to H^k(\mathcal{O}_1).$$

The outer two terms are isomorphic to $H^{k-1}(\mathbb{P}^1)$ and $H^k(\mathbb{P}^1)$, respectively, so they vanish for dimension reasons. It follows that

$$H^k(\mathcal{O}_1, F) \cong H^{k-1}(F). \tag{2.5}$$

Now, switch perspectives: consider F as a subset of $\mathcal{O}_3$. If $\pi : \mathcal{O}_3 \to \mathbb{P}^5$ is the projection map, then $\pi^{-1}(x)$ intersects F in an affine hyperplane if $x \notin X_{3,3}$, and it is empty if $x \in X_{3,3}$. Since the affine hyperplane has trivial cohomology, it follows that

$$H^*(F) \cong H^*(\mathbb{P}^5 \setminus X_{3,3}).$$

To compute the latter, consider the long exact sequence of the pair $(\mathbb{P}^5, X_{3,3})$:

(2.6)

$$H^k(\mathbb{P}^5, \mathbb{P}^5 \setminus X_{3,3}) \xrightarrow{i_*} H^k(\mathbb{P}^5) \to H^k(\mathbb{P}^5 \setminus X_{3,3}) \to H^{k+1}(\mathbb{P}^5, \mathbb{P}^5 \setminus X_{3,3}) \to H^{k+1}(\mathbb{P}^5).$$

The Thom isomorphism and Poincaré duality together imply that

$$H^k(\mathbb{P}^5, \mathbb{P}^5 \setminus X_{3,3}) \cong H^{k-4}(X_{3,3}) \cong H_{10-k}(X_{3,3}),$$

and Poincaré duality also implies that $H^k(\mathbb{P}^5) \cong H_{10-k}(\mathbb{P}^5)$. Under these isomorphisms, the map marked i_* in (2.6) is induced by the inclusion $X_{3,3} \hookrightarrow \mathbb{P}^5$. In particular, by the Lefschetz hyperplane principle, i_* is an isomorphism unless $k = 7$. The same reasoning shows that the rightmost map in (2.6) is an isomorphism unless $k = 6$. At this point, an easy case analysis yields

$$H^{k-1}(F) \cong \begin{cases} H^3(X_{3,3}) & \text{if } k = 6 \\ 0 & \text{otherwise.} \end{cases}$$

Combined with equation (2.5), this proves the claim for $k \geq 4$.

Finally, suppose that $k \leq 3$. Then the long exact sequence of the triple $F \subset \mathcal{O}_1^\times \subset \mathcal{O}_1$ gives

$$H^k(\mathcal{O}_1, \mathcal{O}_1^\times) \to H^k(\mathcal{O}_1, F) \to H^k(\mathcal{O}_1^\times, F). \tag{2.7}$$

The first term is

$$H^k(\mathcal{O}_1, \mathcal{O}_1^\times) \cong H^{k-12}(\mathbb{P}^1) = 0$$

by the Thom isomorphism. The third term is the same as $H^k(\mathcal{O}_3^\times, F)$, and the long exact sequence of the triple $F \subset \mathcal{O}_3^\times \subset \mathcal{O}_3$ gives

(2.8)

$$H^k(\mathcal{O}_3, \mathcal{O}_3^\times) \to H^k(\mathcal{O}_3, F) \to H^k(\mathcal{O}_3^\times, F) \to H^{k+1}(\mathcal{O}_3, \mathcal{O}_3^\times) \to H^{k+1}(\mathcal{O}_3, F).$$

One has

$$H^k(\mathcal{O}_3, \mathcal{O}_3^\times) \cong H^{k-4}(\mathbb{P}^1)$$

by the Thom isomorphism, and

$$H^k(\mathcal{O}_3, F) \cong H^{k-4}(X_{3,3})$$

by the computation above. Thus, we can use the Five Lemma to compare (2.8) to the long exact sequence of the pair $(\mathbb{P}^1, X_{3,3})$, and this gives:

$$H^k(\mathcal{O}_3^\times, F) \cong H^{k-4}(\mathbb{P}^1, X_{3,3}) = 0.$$

Returning to (2.7), we have shown that both of the outer terms vanish when $k \leq 3$, so $H^k(\mathcal{O}_1, F) = 0$ in this case. This completes the proof of the Proposition. $\square$

CHAPTER III

Quantum theory for the Landau-Ginzburg model

Similarly to FJRW theory, the hybrid model concerns curves equipped with a collection of line bundles whose tensor powers satisfy certain conditions. However, the moduli problem of roots of line bundles is better-behaved with respect to orbifold curves than smooth curves. For example, a line bundle may have no rth roots at all on a smooth curve, if its degree is not a multiple of r on each component; even worse, the number of roots may change within flat families of smooth curves (see Section 1.2 of [7] for an example). Thus, the underlying curves of our theory should be allowed limited orbifold structure.

3.1 Orbifold curves and their line bundles

We will follow the definition of orbifold curve given in [2].

Definition 3.1.1. An *orbifold curve* (or "balanced twisted curve") is a one-dimensional Deligne-Mumford stack with a finite ordered collection of marked points and at worst nodal singularities such that

1. the only points with nontrivial stabilizers are marked points and nodes;
2. all nodes are *balanced*; i.e., in the local picture $\{xy = 0\}$ at a node, the action

of the isotropy group $\mathbb{Z}_k$ is given by

$$(x, y) \mapsto (\zeta_k x, \zeta_k^{-1} y)$$

with ζ_k a primitive kth root of unity.

The second condition is required to ensure that all nodal orbifold curves arise as degenerations of non-nodal curves.

The notions of stable maps and line bundles generalize from smooth curves to their orbifold analogues; these definitions can be found, for example, in [2] and [7]. An orbifold curve C has a coarse underlying curve $|C|$, which, roughly speaking, is the smooth curve obtained from C by forgetting the orbifold structure at special points. There is a "coarsening" map

$$\epsilon : C \to |C|.$$

This is a flat morphism, so in particular, if L is a line bundle on C, one obtains a coarse underlying bundle $|L| := \epsilon_* L$ via pushforward.

3.1.2 Multiplicities of orbifold line bundles

Let C be an orbifold curve and let L an orbifold line bundle on C. Choose a node n of C with isotropy group $\mathbb{Z}_\ell$ and a distinguished branch of n, so that the local picture can be expressed as $\{xy = 0\}$ with x being the coordinate on the distinguished branch. Let g be a generator of the isotropy group $\mathbb{Z}_\ell$ at the node acting on these local coordinates by $g \cdot (x, y) = (\zeta_\ell x, \zeta_\ell^{-1} y)$.

Definition 3.1.3. The *multiplicity* of L at (the distinguished branch of) the node n is the integer $m \in \{0, \ldots, \ell - 1\}$ such that, in local coordinates (x, y, λ) on the total space of L, the action of g is given by

$$g \cdot (x, y, \lambda) = (\zeta_\ell x, \zeta_\ell^{-1} y, \zeta_\ell^m \lambda).$$

In the same way, one can define the multiplicity of L at a marked point by the action of a generator of the isotropy group on the fiber.

One extremely important property of the multiplicity is that it allows one to determine the equation satisfied by the coarsening of L on each of its components [7] [11]. Suppose that

$$\nu : L^{\otimes \ell} \xrightarrow{\sim} \epsilon^* N$$

is an isomorphism between a power of L and a line bundle pulled back from the coarse curve $|C|$ and $Z \subset C$ is a non-nodal irreducible component of C. Let $m_1, \ldots, m_k$ be the multiplicities of L at the nodes where Z meets the rest of C, where in each case the distinguished branch is the one lying on Z. Let $\epsilon : C \to |C|$ be the coarsening map. If $|L| = \epsilon_* L$, then we have an isomorphism

$$\epsilon_* \nu : |L|^{\otimes \ell} \to N \otimes \mathcal{O}_{|Z|} \left(- \sum_{i=1}^{k} m_i [p_i] \right), \tag{3.1}$$

where $p_1, \ldots, p_k$ are the images in $|Z|$ of the points where Z meets the rest of C.

Since ϵ is flat, $|L|$ is a line bundle; in particular, the fact that it has integral degree can often be used to find constraints on the multiplicities of L. Conversely, the multiplicities at all of the marked points and nodes of C, together with the bundle $|L|$ on $|C|$, collectively determine L as an orbifold line bundle. See Lemma 2.2.5 of [11] for a precise statement to this effect.

3.2 Moduli space

Let $W_1, \ldots, W_r$ be a nondegenerate collection of quasihomogeneous polynomials, each having weights $c_1, \ldots, c_N$ and degree d. Let $\overline{d}$ denote the *exponent*[1] of the group $G_{W_1,\ldots,W_r}$; i.e., the smallest integer k for which $g^k = 1$ for all $g \in G_{W_1,\ldots,W_r}$.

[1] In the examples of interest in this thesis, we will have $\overline{d} = d$, but this is not necessarily the case in general.

For each i, set

$$\overline{c_i} = c_i \frac{\overline{d}}{d},$$

where as usual d is the degree of the polynomials W_i, and c_i are the weights.

Definition 3.2.1. A genus-g, degree β *Landau-Ginzburg stable map* with n marked points over a base T is given by the following objects:

$$\begin{array}{ccccc} \mathscr{L} & \longrightarrow & (\mathscr{C}, \{\mathscr{S}_i\}) & \xrightarrow{f} & \mathbb{P}^{r-1} \\ & & \downarrow{\scriptstyle \pi} & & \\ & & T, & & \end{array}$$

together with an isomorphism

$$\varphi : \mathscr{L}^{\otimes d} \xrightarrow{\sim} \omega_{\log} \otimes f^* \mathcal{O}(-1),$$

where

1. $\mathscr{C}/T$ is a genus-g, n-pointed orbifold curve;

2. For $i = 1, \ldots, n$, the substack $\mathscr{S}_i \subset \mathscr{C}$ is a (trivial) gerbe over T with a section $\sigma_i : T \to \mathscr{S}_i$ inducing an isomorphism between T and the coarse moduli of $\mathscr{S}_i$;

3. f is a morphism whose induced map between coarse moduli spaces is an n-pointed genus g stable map of degree β;

4. $\mathscr{L}$ is an orbifold line bundle on $\mathscr{C}$ and φ is an isomorphism of line bundles;

5. For any $p \in \mathscr{C}$, the representation $r_p : G_p \to \mathbb{Z}_d$ given by the action of the isotropy group on the fiber of $\mathscr{L}$ is faithful.

Definition 3.2.2. A morphism between two Landau-Ginzburg stable maps

$$(\mathscr{C}/T, \{\mathscr{S}_i\}, f, \mathscr{L}, \phi) \to (\mathscr{C}'/T', \{\mathscr{S}_i'\}, f', \mathscr{L}', \phi')$$

is a tuple of morphisms (τ, μ, α), where (τ, μ) forms a morphism of pointed orbifold stable maps:

$$\begin{array}{ccccc} T & \longleftarrow & \mathscr{C} & & \\ & & & \searrow & \\ \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \mu} & & \mathbb{P}^{r-1} \\ & & & \nearrow & \\ T' & \longleftarrow & \mathscr{C}' & & \end{array}$$

and $\alpha : \mu^*\mathscr{L}' \to \mathscr{L}$ is an isomorphism of line bundles such that

$$\phi \circ \alpha^{\otimes d} = \delta \circ \mu^* \phi',$$

where $\delta : \mu^*\omega_{\mathscr{C}'} \to \omega_{\mathscr{C}}$ is the natural map.

Definition 3.2.3. The *hybrid model moduli space* is the stack $\widetilde{\mathcal{M}}^d_{g,n}(\mathbb{P}^{r-1}, \beta)$ parameterizing n-pointed genus-g Landau-Ginzburg stable maps of degree β, up to isomorphism.

Before we prove that this is a proper Deligne-Mumford stack, a few remarks are in order.

Remark 3.2.4. Landau-Ginzburg stable maps can be viewed as tensor products of stable maps to $\mathbb{P}(d, \ldots, d)$ and spin structures. Indeed, the datum of a stable map to $\mathbb{P}(d, \ldots, d)$ is equivalent to a map $f : C \to \mathbb{P}^{r-1}$ together with a dth root of the line bundle $f^*\mathcal{O}(1)$, while a spin structure on C is a dth root of $\omega_{\log}$.

Remark 3.2.5. Given the variation of GIT perspective mentioned repeatedly above, it would in some sense be more natural to define Landau-Ginzburg stable maps as maps to a weighted projective space $\mathbb{P}(d, \ldots, d)$ instead of the coarse underlying $\mathbb{P}^{r-1}$. In fact, though, this is equivalent to what we have done, since if $f : C \to \mathbb{P}(d, \ldots, d)$ is an orbifold stable map and there exists a line bundle L

on C such that $L^{\otimes d} \cong f^*\mathcal{O}(-1) \otimes \omega_{\log}$, then $f^*\mathcal{O}(-1)$ is forced to have integral degree, which implies that f factors through a map to $\mathbb{P}^{r-1}$.

Remark 3.2.6. In the case where $r = 1$, the above is not exactly the same as the moduli space of W-structures in FJRW theory. However, Proposition 2.3.13 of [11] shows that the map

$$\widetilde{\mathcal{M}}^d_{g,n}(\mathbb{P}^0, 0) \to W_{g,n,\langle J \rangle}$$

$$(C, f, L, \varphi) \mapsto (C, (L^{\otimes c_1}, \varphi^{\overline{c_1}}), \ldots, (L^{\otimes c_N}, \varphi^{\overline{c_N}}))$$

is surjective and locally isomorphic to $B\mu_d \to B(\mu_{\overline{d}})^N$, so integrals over $W_{g,n,\langle J \rangle}$ can be expressed as integrals over $\widetilde{\mathcal{M}}^d_{g,n,}(\mathbb{P}^0, 0)$, and the correlators defined below agree with those in FJRW theory.

Forgetting the line bundle $\mathscr{L}$ and the orbifold structure gives a morphism

$$\rho : \widetilde{\mathcal{M}}^d_{g,n}(\mathbb{P}^{r-1}, \beta) \to \overline{\mathcal{M}}_{g,n}(\mathbb{P}^{r-1}, \beta).$$

This map is quasifinite (see Remark 2.1.20 of [27]). Indeed, for any orbifold stable map $f : C \to \mathbb{P}^{r-1}$, any two choices of L such that $L^{\otimes d} \cong \omega_{\log} \otimes f^*\mathcal{O}(-1)$ differ by a choice of a line bundle N with an isomorphism $\xi : N^{\otimes d} \cong \mathcal{O}_C$. The set of isomorphism classes of such pairs (N, ξ) is isomorphic to the finite group $H^1(C, \mathbb{Z}_d)$.

Proposition 3.2.7. *For any nondegenerate collection of quasihomogeneous polynomials* $\overline{W}$ *as above, the stack* $\widetilde{\mathcal{M}}^d_{g,n}(\mathbb{P}^{r-1}, \beta)$ *is a proper Deligne-Mumford stack with projective coarse moduli.*

Proof. The proof follows closely that of Theorem 2.2.6 of [27] and uses repeatedly the identification between orbifold line bundles on $\mathscr{C}$ and maps $\mathscr{C} \to B\mathbb{C}^*$. Given

$(C, \{\sigma_i\}, f) \in \overline{\mathcal{M}}_{g,n}(\mathbb{P}^{r-1}, \beta)$, an element of $\rho^{-1}(C, \{\sigma_i\}, f)$ is given by a map

$$\mathfrak{L} : C \to B\mathbb{C}^*$$

such that

$$\begin{array}{ccc} & & B\mathbb{C}^* \\ & \overset{\mathfrak{L}}{\nearrow} & \downarrow \\ C & \xrightarrow[\delta]{} & B\mathbb{C}^* \end{array}$$

commutes, where δ is the map corresponding to the line bundle $f^*\mathcal{O}(-1) \otimes \omega_{\log}$ and the vertical arrow is $x \mapsto x^d$.

Let $C_{\overline{\mathcal{M}}} \to \overline{\mathcal{M}}_{g,n}(\mathbb{P}^{r-1}, \beta)$ denote the universal family, and abbreviate $\overline{\mathcal{M}} = \overline{\mathcal{M}}_{g,n}(\mathbb{P}^{r-1}, \beta)$. Let $C_{\overline{W}}$ be the fiber product

$$\begin{array}{ccc} C_{\overline{W}} & \longrightarrow & B\mathbb{C}^* \\ \downarrow & & \downarrow \\ C_{\overline{\mathcal{M}}} & \xrightarrow{\delta} & B\mathbb{C}^*, \end{array}$$

with the right vertical arrow as before. Note that $C_{\overline{W}}$ is an étale gerbe over $C_{\overline{\mathcal{M}}}$ banded by $\mathbb{Z}_d$, so it is a Deligne-Mumford stack.

Any Landau-Ginzburg stable map $(\mathscr{C}/T, \{\mathscr{S}_i\}, f, \mathscr{L}, \phi)$ induces a representable morphism $\mathscr{C} \to C_{\overline{W}}$ which is a balanced twisted stable map for which the homology class of the image of the coarse curve C is the class F of a fiber of the universal curve $C_{\overline{\mathcal{M}}} \to \overline{\mathcal{M}}$. Furthermore, the family of coarse curves and maps $(C, \{\sigma_i\}, f) \to T$ gives rise to a morphism $T \to \overline{\mathcal{M}}$, and we have an isomorphism $C \cong T \times_{\overline{\mathcal{M}}} C_{\overline{\mathcal{M}}}$. Thus, there is a basepoint-preserving functor

$$\widetilde{\mathcal{M}}^d_{g,n}(\mathbb{P}^{r-1}, \beta) \to \mathscr{K}_{g,n}(C_{\overline{W}}/\overline{\mathcal{M}}, F),$$

where the latter denotes the stack of balanced, n-pointed twisted stable maps of genus g and class F into $C_{\overline{W}}$ relative to the base stack $\overline{\mathcal{M}}$ (see Section 8.3 of [2]).

The image lies in the closed substack where the markings of C line up over the markings of $C_{\overline{\mathcal{M}}}$, and the functor given by the restricting to this substack is an equivalence. Thus, the results of [2] imply that $\widetilde{\mathcal{M}}^d_{g,n}(\mathbb{P}^{r-1},\beta)$ is a proper Deligne-Mumford stack with projective coarse moduli. □

3.2.8 Decomposition by multiplicities

With the analogy to FJRW theory mentioned in Remark 3.2.6 in mind, one obtains a decomposition of the hybrid moduli space just as in Proposition 2.3.7 of [11]. In $W_{g,n,\langle J\rangle}$, let $\gamma_i \in \mathrm{Aut}(W)$ give the multiplicities of $L^{\otimes c_1},\ldots,L^{\otimes c_N}$ at the ith marked point. Then the condition that $\gamma_i \in J$ implies that there exists $e^{2\pi i\frac{m_i}{d}} \in \mathbb{Z}_d$ such that $e^{2\pi i\frac{m_i\overline{c_j}}{\overline{d}}} = e^{2\pi i\frac{m_{i,j}}{\overline{d}}}$ for all j, so γ_i is determined by $m_i \in \{0,1,\ldots,d-1\}$. Let

$$\widetilde{\mathcal{M}}^d_{g,n}(\mathbb{P}^{r-1},\beta) = \bigsqcup_{m_1,\ldots,m_n\in\mathbb{Z}_d} \widetilde{\mathcal{M}}^d_{g,(m_1,\ldots,m_n)}(\mathbb{P}^{r-1},\beta),$$

where $\widetilde{\mathcal{M}}^d_{g,(m_1,\ldots,m_n)}(\mathbb{P}^{r-1},\beta)$ is the substack in which the multiplicity of $L^{\otimes c_j}$ at the ith marked point is $m_{i,j} \equiv m_i\overline{c_j} \mod \overline{d}$, or equivalently, the multiplicity of L at the ith marked point is m_i. The following terminology will be used later:

Definition 3.2.9. A marking or node is called *narrow* if all of the line bundles $L^{\otimes c_1},\ldots,L^{\otimes c_N}$ have nonzero multiplicity $m_{i,j} \in \mathbb{Z}_{\overline{d}}$, and is called *broad* otherwise. (In the literature, these situations are sometimes referred to as *Neveu-Schwartz* and *Ramond*, respectively.)

Remark 3.2.10. It is no accident that this terminology coincides with that used for sectors of the state space in Section 2.4. Indeed, elements of J index both sectors of the state space and components of the moduli space, and the narrow sectors of the state space correspond to components of the moduli space in which every marked point is narrow.

3.3 Cosection construction

In order to define a virtual cycle on the hybrid moduli space, we will make use of the cosection technique developed in [39], [4], and [5]. Before turning to our specific situation, let us say a few words about the method in general.

Given a moduli space X for which one desires a virtual cycle, the idea of the cosection construction is to embed in X into a noncompact Deligne-Mumford stack $\mathcal{M}$ whose obstruction theory we can understand more easily. For example, when we apply the technique to the hybrid moduli space, $\mathcal{M}$ will parameterize curves with a bundle and a collection of sections, and Chang-Li [4] have described a simple, explicit relative perfect obstruction theory on any stack of this form. The goal, then, is to define a virtual cycle for $\mathcal{M}$ supported only on X.

3.3.1 Notation and statement of the cosection localization theorem

Let $\mathcal{M}$ be a Deligne-Mumford stack and let $\mathcal{S}$ be a smooth Artin stack with a map

$$\pi : \mathcal{M} \to \mathcal{S}.$$

Suppose that there exists a relative perfect obstruction theory

$$\phi : \mathbb{E}^\bullet \to \mathbb{L}^\bullet_{\mathcal{M}/\mathcal{S}}.$$

The relative obstruction sheaf, by definition, is $\mathcal{O}b_{\mathcal{M}/\mathcal{S}} = h^1((\mathbb{E}^\bullet)^\vee)$.

It is possible to define an absolute obstruction sheaf, as well. To do so, consider the distinguished triangle

$$\pi^*\mathbb{L}_\mathcal{S} \to \mathbb{L}_\mathcal{M} \to \mathbb{L}_{\mathcal{M}/\mathcal{S}} \xrightarrow{\delta} \pi^*\mathbb{L}_\mathcal{S}[1].$$

The connecting map δ induces

$$\phi^\vee \circ \delta^\vee : \pi^*\mathbb{T}_\mathcal{S} \to \mathbb{T}_{\mathcal{M}/\mathcal{S}}[1] \to \mathbb{E}[1]. \tag{3.2}$$

Let $\eta = h^0(\phi^\vee \circ \delta^\vee) : H^0(\pi^* \mathbb{T}_{\mathcal{S}}) \to \mathcal{O}b_{\mathcal{M}/\mathcal{S}}$. Then the absolute obstruction sheaf is defined as

$$\mathcal{O}b_{\mathcal{M}} = \text{coker}(\eta).$$

Equipped with this definition, we can state the cosection localization theorem:

Theorem III.1 (Kiem-Li [39])**.** *Suppose that we have a "cosection"— that is, a map*

$$\sigma : \mathcal{O}b_{\mathcal{M}}|_U \to \mathcal{O}_{\mathcal{M}}|_U$$

defined and surjective on some open $U \subset \mathcal{M}$. Let $\mathcal{D}(\sigma) = \mathcal{M} \setminus U$ denote the degeneracy locus of σ. Then there exists a "cosection-localized virtual cycle"

$$[\mathcal{M}]^{vir}_{\sigma,loc} \in A_*(\mathcal{D}(\sigma))$$

that pushes forward to the ordinary virtual cycle under the inclusion of the degeneracy locus.

We will sometimes speak loosely of a homomorphism

$$\sigma : \mathcal{O}b_{\mathcal{M}/\mathcal{S}} \to \mathcal{O}_{\mathcal{M}}$$

as a "cosection". To ensure that such a map actually defines a cosection in the sense of Theorem III.1, one must verify that $\sigma \circ \eta = 0$, so that σ lifts to a homomorphism $\overline{\sigma} : \mathcal{O}b_{\mathcal{M}} \to \mathcal{O}_{\mathcal{M}}$.

3.3.2 Motivation

To understand the intuition behind the cosection construction, it is helpful to compare it to the analogous notion of a localized Euler class. If $V \to M$ is a vector bundle on a variety, the usual definition of the Euler class is given by refined self-intersection of the zero section $Z \subset M$; that is, $e(V) = 0^!(Z)$. This definition falls apart when M is noncompact, but if $s : M \to V$ is a section with compact zero locus, then one can define a "localized Euler class" by

$$e_{s,loc}(V) = 0^!(\Gamma_s) \in A_*(\{s = 0\}),$$

where Γ_s is the graph of s.

The cosection construction generalizes this idea from vector bundles to "bundle stacks" of the form $\mathbf{E} = h^1/h^0(E^\bullet)$, where $E^\bullet$ is an object in the derived category of $\mathcal{M}$ quasi-isomorphic to a two-term complex $[E^0 \to E^1]$ of vector bundles. The role of the section is played in this context by a cosection

$$\sigma : h^1(E^\bullet) \to \mathcal{O}_{\mathcal{M}}$$

(possibly only defined over an open substack of $\mathcal{M}$). Furthermore, rather than trying to compute the refined intersection of the zero section with itself, as we did when making sense of the Euler class, the virtual cycle should be the refined intersection of the zero section with the intrinsic normal cone $\mathbf{c}_{\mathcal{M}} \subset \mathbf{E}$. The goal is to tweak this intersection so that the result lies in the cohomology of the degeneracy locus $\mathcal{D}(\sigma) \subset \mathcal{M}$.

Kiem-Li's definition of the cosection-localized virtual cycle proceeds in two main steps. First, they prove that if

$$\mathbf{E}(\sigma) = \mathbf{E}|_{\mathcal{D}(\sigma)} \cup \ker(\sigma|_U)$$

contains the entire fiber of $\mathbf{E}$ over the degeneracy locus of σ and only the kernel of σ in other fibers, then there is a "localized Gysin map"

$$s^!_{\mathbf{E},\sigma} : A_*(\mathbf{E}(\sigma)) \to A_*(\mathcal{D}(\sigma))$$

generalizing the usual Gysin map from the total space to the base of a bundle. The basic idea of the localized Gysin map is to mimic the situation in which the degeneracy locus is a divisor (or at least when $\mathcal{M}$ can be replaced by a blowup to make this the case), since in that situation, one can simply apply the ordinary Gysin map $A_*(\mathbf{E}) \to A_*(\mathcal{M})$ and then intersect with this divisor. The reduction of the general case to this one is quite complicated, however; see Sections 2 and 3 of [39].

The second step of Kiem-Li's definition is to prove that the intrinsic normal cone is represented by an element of $A_*(\mathbf{E}(\sigma))$. The cosection-localized virtual cycle is then defined as $s^!_{\mathbf{E},\sigma}([\mathbf{c}_{\mathcal{M}}])$.

3.4 Virtual cycle

Now, let us apply the cosection construction to the present situation. Since the hybrid model correlators will be defined as integrals over $\widetilde{\mathcal{M}}^d_{g,(m_1,\ldots,m_n)}(\mathbb{P}^{r-1},\beta)$, it suffices to define a virtual cycle on each of these substacks. In fact, we will only define the virtual cycle for the narrow components– that is, when $m_{i,j} \neq 0 \in \mathbb{Z}_{\overline{d}}$ for all i and j. This implies, in particular, that $m_i \geq 1$ for all i.

3.4.1 Construction of the virtual cycle

By passing to a partial coarsening, an element (C, f, L, φ) in one of the components $\widetilde{\mathcal{M}}^d_{g,(m_1,\ldots,m_n)}(\mathbb{P}^{r-1},\beta)$ of the hybrid moduli space is equivalent to a tuple (C, f, L, φ) in which $f : C \to \mathbb{P}^{r-1}$ is a stable map with orbifold structure only at the nodes of C and φ is an isomorphism

$$L^{\otimes d} \cong f^*\mathcal{O}(-1) \otimes \omega_{\log} \otimes \mathcal{O}\left(-\sum_{i=1}^{n} m_i[x_i]\right);$$

see (3.1) and Lemma 2.2.5 of [11]. In what follows, we will view elements of $\widetilde{\mathcal{M}}^d_{g,(m_1,\ldots,m_n)}(\mathbb{P}^{r-1},\beta)$ from this perspective.

Consider the stack $\mathcal{P}$ parameterizing tuples $(C, f, L, \varphi, s_1, \ldots, s_N)$, in which

$(C, f, L, \varphi) \in \widetilde{\mathcal{M}}^d_{g,(m_1,\ldots,m_n)}(\mathbb{P}^{r-1}, \beta)$ and $s_i \in H^0(C, L^{\otimes c_i})$. This is in general not proper; it should be viewed as the Landau-Ginzburg analogue of Chang and Li's moduli space of stable maps with p-fields [4]. In their paper, Chang and Li exhibit a relative perfect obstruction theory on $\mathcal{P}$ relative to the Artin stack $\mathcal{D}_g$ parameterizing genus-g curves with a line bundle of fixed degree. While the present situation also requires marked points, the same construction applies.

Namely, denote by $\mathcal{D}_{g,n}$ the moduli stack of genus-g, n-pointed curves with a line bundle of fixed degree. Let $\mathscr{L}_{\mathcal{D}_{g,n}}$ be the universal line bundle over $\mathcal{D}_{g,n}$, let $\pi_{\mathcal{D}_{g,n}} : \mathcal{C}_{\mathcal{D}_{g,n}} \to \mathcal{D}_{g,n}$ be the universal family, and let

$$\mathscr{P}_{\mathcal{D}_{g,n}} = \mathscr{L}_{\mathcal{D}_{g,n}}^{\otimes -d} \otimes \omega_{\mathcal{C}_{\mathcal{D}_{g,n}}/\mathcal{D}_{g,n}} \otimes \mathcal{O}\left(\sum_{i=1}^n (1-m_i)[x_i]\right).$$

Then $\mathcal{P}$ embeds into the moduli of sections of

$$\mathcal{Z} = \mathrm{Vb}(\mathscr{L}_{\mathcal{D}_{g,n}}^{\oplus N} \oplus \mathscr{P}_{\mathcal{D}_{g,n}}^{\oplus r})$$

over $\mathcal{D}_{g,n}$ (see Section 2.2 of [4]), where Vb denotes the total space of a vector bundle.

Similarly, over $\mathcal{P}$, let $\mathscr{L}$ be the universal line bundle, $\pi : \mathcal{C}_{\mathcal{P}} \to \mathcal{P}$ be the universal family, and $\mathscr{P} = f^*\mathcal{O}(1) = \mathscr{L}^{\otimes -d} \otimes \omega_{\mathcal{C}_{\mathcal{P}}/\mathcal{P}} \otimes \mathcal{O}(\sum_{i=1}^n (1-m_i)[x_i])$. The tautological

$$\mathfrak{s}_i \in \Gamma(\mathcal{C}_{\mathcal{P}}, \mathscr{L}^{\otimes c_i}) \quad \text{and} \quad \mathfrak{p}_j \in \Gamma(\mathcal{C}_{\mathcal{P}}, \mathscr{P}),$$

in which the latter are given by the pullbacks of coordinate sections of $\mathcal{O}_{\mathbb{P}^{r-1}}(1)$, combine to give a map $\mathcal{C}_{\mathcal{P}} \to \mathrm{Vb}\left(\bigoplus_{i=1}^N \mathscr{L}_{\mathcal{D}_{g,n}}^{\otimes c_i} \oplus \mathscr{P}_{\mathcal{D}_{g,n}}^{\oplus r}\right) \times_{\mathcal{C}_{\mathcal{D}_{g,n}}} \mathcal{C}_{\mathcal{P}}$ which is a section of the projection map. Composing this with the projection to the first factor yields an "evaluation map"

$$\mathfrak{e} : \mathcal{C}_{\mathcal{P}} \to \mathcal{Z}.$$

Using Proposition 2.5 of [4] and the canonical isomorphism

$$\mathfrak{e}^*\Omega^\vee_{\mathcal{Z}/\mathcal{C}_{\mathcal{D}_{g,n}}} \cong \bigoplus_{i=1}^{N} \mathscr{L}^{\otimes c_i} \oplus \mathscr{P}^{\oplus r}_{\mathcal{D}_{g,n}},$$

one finds that there is a relative perfect obstruction theory

$$\mathbb{E}_{\mathcal{P}/\mathcal{D}_{g,n}} = R^\bullet \pi_*(\mathscr{L}^{\otimes c_1} \oplus \cdots \oplus \mathscr{L}^{\otimes c_N} \oplus \mathscr{P}^{\oplus r}).$$

Thus, we have $\mathcal{O}b_{\mathcal{P}/\mathcal{D}_{g,n}} = R^1\pi_*(\bigoplus_{i=1}^N \mathscr{L}^{\otimes c_i} \oplus \mathscr{P}^{\oplus r})$. The polynomial $\overline{W}$ defines a homomorphism

$$\sigma : \mathcal{O}b_{\mathcal{P}/\mathcal{D}_{g,n}} \to \mathcal{O}_{\mathcal{P}}.$$

To define σ, fix an element $\xi = (C, f, L, \varphi, s_1, \ldots, s_N) \in \mathcal{P}$ and let $p_j = f^*x_j \in H^0(C, f^*\mathcal{O}(1))$, where $x_j \in H^0(\mathbb{P}^{r-1}, \mathcal{O}(1))$ are the coordinate functions. Take an étale chart $T \to \mathcal{P}$ around ξ with $\mathcal{C}_T = \mathcal{C}_{\mathcal{P}} \times_{\mathcal{P}} T$. Then σ is defined in these local coordinates as the map

$$H^1(\mathcal{C}_T, \mathscr{L}^{\otimes c_1} \oplus \cdots \oplus \mathscr{L}^{\otimes c_N}) \oplus H^1(\mathcal{C}_T, \mathscr{P}^{\oplus r}) \to \mathbb{C}$$

given by sending $(\tilde{s}_1, \ldots, \tilde{s}_N, \tilde{p}_1, \ldots, \tilde{p}_r)$ to

$$\sum_{i=1}^{N} \frac{c_i}{d} \frac{\partial \overline{W}}{\partial x_i}(s_1, \ldots, s_N, p_1, \ldots, p_r) \cdot \tilde{s}_i - \sum_{j=1}^{r} \frac{\partial \overline{W}}{\partial p_j}(s_1, \ldots, s_N, p_1, \ldots, p_r) \cdot \tilde{p}_j.$$

The fact that this is canonically an element of $\mathbb{C}$ relies crucially on the fact that $m_i \geq 1$ for all i. For example, $\frac{\partial \overline{W}}{\partial p_j}(s_1, \ldots, s_N, p_1, \ldots, p_r)$ lies in

$$H^0(\mathscr{L}^{\otimes d}) = H^0\left(\mathscr{P}^\vee \otimes \omega_{\mathscr{C}_{\mathcal{P}}/\mathcal{P}} \otimes \sum (m_i - 1)[x_i]\right) \hookrightarrow H^1(\mathscr{P})^\vee$$

by Serre duality.

The degeneracy locus of σ, which is the locus where the localized virtual cycle will be supported, is the substack $\mathcal{D}(\sigma)$ of $\mathcal{P}$ on which the fiber of σ is the zero homomorphism.

Lemma 3.4.2. *The degeneracy locus of σ is precisely $\widetilde{\mathcal{M}}^d_{g,(m_1,\ldots,m_n)}(\mathbb{P}^{r-1},\beta)$.*

Proof. The hybrid moduli space $\widetilde{\mathcal{M}}^d_{g,(m_1,\ldots,m_n)}(\mathbb{P}^{r-1},\beta)$ embeds in $\mathcal{P}$ as the locus where $s_1 = \cdots = s_N = 0$, and it is clear that the fiber of σ is identically zero on this locus. Conversely, if $(s_1,\ldots,s_N) \neq 0$, then either $(s_1,\ldots,s_N)$ does not lie in the common vanishing locus of the polynomials W_i, or there is some i for which not every $\frac{\partial W_j}{\partial x_i}(s_1,\ldots,s_N)$ vanishes. In the first case, if $W_j(s_1,\ldots,s_N) \neq 0$, then one can choose $\tilde{p}_j$ so that

$$W_j(s_1,\ldots,s_N)\cdot\tilde{p}_j = \frac{\partial\overline{W}}{\partial p_j}(s_1,\ldots,s_N,p_1,\ldots,p_r)\cdot\tilde{p}_j \neq 0,$$

so taking all other $\tilde{p}_i$'s and all $\tilde{s}_i$'s to be zero shows that the fiber of σ over ξ is not identically zero. In the second case, independence of the sections p_j shows that

$$\sum_{j=1}^r p_j\frac{\partial W_j}{\partial x_i}(s_1,\ldots,s_N) = \frac{\partial\overline{W}}{\partial x_i}(s_1,\ldots,s_N,p_1,\ldots,p_r) \neq 0.$$

Thus, there exists $\tilde{s}_i$ such that $\frac{\partial\overline{W}}{\partial x_i}(s_1,\ldots,s_N,p_1,\ldots,p_r)\cdot\tilde{s}_i \neq 0$, so again one can choose all other $\tilde{s}_j$'s and all $\tilde{p}_j$'s to be zero to see that the fiber of σ over ξ is not identically zero. □

Remark 3.4.3. By studying σ a bit more carefully, one notices that it descends to the obstruction theory of $\mathcal{P}$ relative to $\overline{\mathcal{M}}_{g,n}$ rather than $\mathcal{D}_{g,n}$.[2] To do so, consider the deformation exact sequence

$$T_{\mathcal{D}_{g,n}/\overline{\mathcal{M}}_{g,n}} \xrightarrow{\tau} \mathcal{O}b_{\mathcal{P}/\mathcal{D}_{g,n}} \to \mathcal{O}b_{\mathcal{P}/\overline{\mathcal{M}}_{g,n}} \to 0. \tag{3.3}$$

The deformation space $T_{\mathcal{D}_{g,n}/\overline{\mathcal{M}}_{g,n}}$ parameterizes deformations of a line bundle fixing the underlying curve, so its fiber over ξ is $H^1(C,\mathcal{O}_C)$. The map τ can be viewed fiberwise as

$$\tau = (\tau_1,\tau_2) : H^1(C,f^*\mathcal{O}_{\mathbb{P}^{r-1}}) \to \bigoplus_{i=1}^N H^1(L^{\otimes c_i}) \oplus H^1(f^*\mathcal{O}(1))^{\oplus r}.$$

[2]The following argument was suggested by H.-L. Chang in correspondence with Y. Ruan.

Here, τ_1 is the dual of the map $\bigoplus_{i=1}^{N} H^0(L^{\otimes -c_i} \otimes \omega) \to H^0(\omega)$ given by

$$(q_1, \ldots, q_N) \mapsto \sum_{i=1}^{N} q_i s_i, \tag{3.4}$$

and τ_2 is dual to the map $H^0(f^*\mathcal{O}(-1) \otimes \omega)^{\oplus r} \to H^0(\omega)$ given by

$$(u_1, \ldots, u_r) \mapsto \sum_{j=1}^{r} u_j t_j;$$

in other words, τ_2 arises via the Euler sequence on $\mathbb{P}^{r-1}$. Thus, with (3.3) in mind, we can view $\mathcal{O}b_{\mathcal{P}/\overline{\mathcal{M}}_{g,n}}$ as $\operatorname{coker}(\tau)$. A straightforward argument using the quasi-homogeneous Euler identity shows that the composition $\sigma \circ \tau$ vanishes, and therefore σ descends to a cosection $\mathcal{O}b_{\mathcal{P}/\overline{\mathcal{M}}_{g,n}} \to \mathcal{O}_{\mathcal{P}}$.

In order to apply Theorem III.1 to conclude the existence of a localized virtual cycle, one must verify that σ lifts to an honest cosection, which should be a homomorphism $\overline{\sigma} : \mathcal{O}b_{\mathcal{P}} \to \mathcal{O}_{\mathcal{P}}$. Recall from (3.2) and the subsequent discussion that the $\mathcal{O}b_{\mathcal{P}}$ is defined as the cokernel of the map

$$q^* T_{\mathcal{D}_{g,n}} \to H^1(\mathbb{T}_{\mathcal{P}/\mathcal{D}_{g,n}}) \to H^1(\mathbb{E}_{\mathcal{P}/\mathcal{D}_{g,n}}) = \mathcal{O}b_{\mathcal{P}/\mathcal{D}_{g,n}} \tag{3.5}$$

given by $h^0(\phi_{\mathcal{P}/\mathcal{D}_{g,n}} \circ \delta^\vee)$, in which $\phi_{\mathcal{P}/\mathcal{D}_{g,n}}$ is the relative perfect obstruction theory for $\mathcal{P}$ and δ is a connecting homomorphism.

Lemma 3.4.4. *The following composition is trivial:*

$$H^1(\mathbb{T}_{\mathcal{P}/\mathcal{D}_{g,n}}) \to \mathcal{O}b_{\mathcal{P}/\mathcal{D}_{g,n}} \xrightarrow{\sigma} \mathcal{O}_{\mathcal{P}}.$$

Therefore, σ lifts to $\overline{\sigma} : \mathcal{O}b_{\mathcal{P}} \to \mathcal{O}_{\mathcal{P}}$.

Proof. The proof of this fact follows closely that of Lemma 3.6 of [4]. First, we will need a slightly different description of σ. Note that the polynomial $\overline{W}$ defines a bundle homomorphism

$$h_1 : \mathcal{Z} = \operatorname{Vb}\left(\bigoplus_{i=1}^{N} \mathscr{L}_{\mathcal{D}_{g,n}}^{\otimes c_i} \oplus \mathscr{P}_{\mathcal{D}_{g,n}}^{\oplus r}\right) \to \operatorname{Vb}(\omega_{\mathcal{C}_{\mathcal{D}_{g,n}}/\mathcal{D}_{g,n}}).$$

On tangent complexes, h_1 induces

$$dh_1 : \Omega^\vee_{\mathcal{Z}/\mathcal{C}_{\mathcal{D}_{g,n}}} \to h_1^*\Omega^\vee_{\mathrm{Vb}(\omega_{\mathcal{C}_{\mathcal{D}_{g,n}}/\mathcal{D}_{g,n}})}.$$

Pulling back dh_1 via the evaluation map $\mathfrak{e}$ defined above, one obtains

$$\mathfrak{e}^*(dh_1) : \mathfrak{e}^*\Omega^\vee_{\mathcal{Z}/\mathcal{C}_{\mathcal{D}_{g,n}}} \to \mathfrak{e}^*h_1^*\Omega^\vee_{\mathrm{Vb}(\omega_{\mathcal{C}_{\mathcal{D}_{g,n}}/\mathcal{D}_{g,n}})},$$

so applying $R^\bullet\pi_{\mathcal{P}*}$ and taking first cohomology gives a map

$$\mathcal{O}b_{\mathcal{P}/\mathcal{D}_{g,n}} \to \mathcal{O}_{\mathcal{P}},$$

where we use the canonical isomorphisms

$$\mathfrak{e}^*\Omega^\vee_{\mathcal{Z}/\mathcal{C}_{\mathcal{D}_{g,n}}} \cong \bigoplus_{i=1}^{N} \mathscr{L}^{\otimes c_i} \oplus \mathscr{P}^{\oplus r},$$

$$\mathfrak{e}^*h_1^*\Omega^\vee_{\mathrm{Vb}(\omega_{\mathcal{C}_{\mathcal{D}_{g,n}}/\mathcal{D}_{g,n}})} \cong \omega_{\mathcal{C}_\mathcal{P}/\mathcal{P}}.$$

One can check explicitly in coordinates that this coincides with the homomorphism σ defined above.

Equipped with this description of σ, we are ready to prove the Lemma. Let $\mathfrak{C}_\omega = C(\pi_*\omega_{\mathcal{C}_{\mathcal{D}_{g,n}}/\mathcal{D}_{g,n}})$ be the direct image cone (see Definition 2.1 of [4]), which parameterizes sections of ω on curves in $\mathcal{D}_{g,n}$. This has a universal curve $\mathcal{C}_{\mathfrak{C}_\omega} = \mathcal{C}_{\mathcal{D}_{g,n}} \times_{\mathcal{D}_{g,n}} \mathfrak{C}_\omega$. Let

$$\epsilon = \overline{W}(\mathfrak{s}_1, \ldots, \mathfrak{s}_N, \mathfrak{p}_1, \ldots, \mathfrak{p}_r) \in \Gamma(\mathcal{C}_\mathcal{P}, \omega_{\mathcal{C}_\mathcal{P}/\mathcal{P}}),$$

which tautologically induces morphisms

$$\Phi_\epsilon : \mathcal{P} \to \mathfrak{C}_\omega$$

and

$$\tilde{\Phi}_\epsilon : \mathcal{C}_\mathcal{P} \to \mathcal{C}_{\mathfrak{C}_\omega}.$$

There are evaluation maps fitting into a commutative diagram of stacks of $\mathcal{C}_{\mathcal{D}_{g,n}}$ as follows:

$$\begin{array}{ccc} \mathcal{C}_{\mathcal{P}} & \xrightarrow{\ \mathfrak{e}\ } & \mathcal{Z} \\ \downarrow{\scriptstyle \tilde{\Phi}_\epsilon} & & \downarrow{\scriptstyle h_1} \\ \mathcal{C}_{\mathfrak{C}_\omega} & \xrightarrow{\ \mathfrak{e}'\ } & \mathrm{Vb}(\omega_{\mathcal{C}_{\mathcal{D}_{g,n}}/\mathcal{D}_{g,n}}). \end{array}$$

Therefore, the following diagram of cotangent complexes is also commutative:

$$\begin{array}{ccccc} \pi_{\mathcal{P}}^* \mathbb{T}_{\mathcal{P}/\mathcal{D}_{g,n}} & = & \mathbb{T}_{\mathcal{C}_{\mathcal{P}}/\mathcal{C}_{\mathcal{D}_{g,n}}} & \longrightarrow & \mathfrak{e}^* \Omega^\vee_{\mathcal{Z}/\mathcal{C}_{\mathcal{D}_{g,n}}} \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle dh_1} \\ \pi_{\mathcal{P}}^* \Phi_\epsilon^* \mathbb{T}_{\mathfrak{C}_\omega/\mathcal{D}_{g,n}} & = & \tilde{\Phi}_\epsilon^* \mathbb{T}_{\mathcal{C}_{\mathfrak{C}_\omega}/\mathcal{C}_{\mathcal{D}_{g,n}}} & \longrightarrow & \tilde{\Phi}_\epsilon^* \mathfrak{e}'^* \Omega^\vee_{\mathrm{Vb}(\omega_{\mathcal{C}_{\mathcal{D}_{g,n}}/\mathcal{D}_{g,n}})/\mathcal{C}_{\mathcal{D}_{g,n}}}. \end{array} \tag{3.6}$$

Applying $R^1\pi_{\mathcal{P}*}$ to the lower horizontal arrow yields the homomorphism

$$H^1(\Phi_\epsilon^* \mathbb{T}_{\mathfrak{C}_\omega/\mathcal{D}_{g,n}}) \to \Phi_\epsilon^* R^1 \pi_{\mathfrak{C}_\omega *} \omega_{\mathcal{C}_{\mathfrak{C}_\omega}/\mathfrak{C}_\omega},$$

which is the pullback via Φ_ϵ of the obstruction homomorphism in the perfect obstruction theory for $\mathfrak{C}_\omega$ over $\mathcal{D}_{g,n}$. As observed in Equation 3.13 of [4], this is trivial since $\mathcal{C}_{\mathfrak{C}_\omega} \to \mathcal{C}_{\mathcal{D}_{g,n}}$ is smooth.

Based on the new definition of σ given above, it is clear that the composite whose vanishing we wish to show is obtained by applying $R^1\pi_{\mathcal{P}*}$ to the composition from the upper left to the lower right of (3.6). Since we have now shown that the lower horizontal arrow becomes trivial, the proof is complete. □

Combining Lemmas 3.4.2 and 3.4.4 with Theorem 1.1 of [39], one finds that $\mathcal{P}$ admits a localized virtual cycle $[\mathcal{P}]^{\mathrm{vir}}_{\mathrm{loc}}$ supported on the degeneracy locus

$$\widetilde{\mathcal{M}}^d_{g,(m_1,\dots,m_n)}(\mathbb{P}^{r-1},\beta) \subset \mathcal{P}$$

of σ.

Definition 3.4.5. The *virtual cycle* of the stack $\widetilde{\mathcal{M}}^d_{g,(m_1,\ldots,m_n)}(\mathbb{P}^{r-1},\beta)$ is defined as

$$[\widetilde{\mathcal{M}}^d_{g,(m_1,\ldots,m_n)}(\mathbb{P}^{r-1},\beta)]^{\mathrm{vir}} := [\mathcal{P}]^{\mathrm{vir}}_{\mathrm{loc}}.$$

As was mentioned in the Introduction, it is helpful in understanding the LG/CY correspondence to examine more closely the observation that $\mathcal{P}$ embeds into the moduli space $\mathfrak{S}$ of sections associated to the diagram

$$\begin{array}{ccc} \mathrm{Vb}(\mathscr{L}^{\otimes c_1}_{\mathcal{D}_{g,n}} \oplus \cdots \mathscr{L}^{\otimes c_N}_{\mathcal{D}_{g,n}} \oplus \mathscr{P}^{\oplus r}_{\mathcal{D}_{g,n}}) & \longrightarrow & \mathcal{C}_{\mathcal{D}_{g,n}} \\ & & \downarrow \\ & & \mathcal{D}_{g,n}. \end{array}$$

Specifically, $\mathcal{P}$ can be viewed as the substack of $\mathfrak{S}$ in which the r sections of $\mathscr{P}$ parameterized by $\mathfrak{S}$ together define a stable map to $\mathbb{P}^{r-1}$.

If, on the other hand, we had considered the substack of $\mathfrak{S}$ in which the sections of $\mathscr{L}^{\otimes c_1},\ldots,\mathscr{L}^{\otimes c_N}$ together define a stable map to $\mathbb{P}(c_1,\ldots,c_N)$, then the resulting moduli space would parameterize stable maps to this weighted projective space together with sections

$$t_j \in H^0\left(f^*\mathcal{O}(-d)\otimes\omega\otimes\mathcal{O}\left(\sum(1-m_i)[x_i]\right)\right)$$

for $j = 1,\ldots,r$, assuming that the Gorenstein condition (3.8) is satisfied. The cosection σ is still defined on this new moduli space, and its degeneracy locus is the moduli space of stable maps to the complete intersection $X_{\overline{W}} \subset \mathbb{P}(c_1,\ldots,c_N)$, as Chang-Li prove in [4] for the case of the quintic.[3]

Remark 3.4.6. Because we have used the cosection construction as opposed to the Witten top Chern class construction of [6] and [44], it is not clear that our

[3] In fact, much more is proved in [4], since even after showing that the degeneracy locus of the cosection is the moduli space of stable maps to the quintic, it is not at all obvious that the cosection localized virtual cycle agrees with the usual virtual cycle on this moduli space. Chang-Li prove that, after integrating, the two virtual cycles yield the same invariants up to an explicit sign discrepancy.

correlators agree in the case of the quintic with those defined in [11]. However, the equivalence of all existing constructions of the FJRW virtual cycle is proved in [5].

3.4.7 Virtual dimension

Let $\xi = (C, f, L, \varphi, s_1, \ldots, s_N) \in \mathcal{P}$. The virtual dimension of $\mathcal{P}/\mathcal{D}_{g,n}$ at ξ is

$$h^0(L^{\otimes c_1} \oplus \cdots \oplus L^{\otimes c_N} \oplus f^*\mathcal{O}(1)^{\oplus r}) - h^1(L^{\otimes c_1} \oplus \cdots \oplus L^{\otimes c_N} \oplus f^*\mathcal{O}(1)^{\oplus r}),$$

and an easy Riemann-Roch computation using (3.1) shows that this equals

$$(N-r)(1-g) + rn - \sum_{i=1}^{n}\sum_{j=1}^{N} \frac{c_j m_{i,j}}{d}.$$

Since

$$\begin{aligned}
\mathrm{vdim}(\mathcal{D}_{g,n}) &= \mathrm{vdim}(\mathcal{D}_{g,n}/\overline{\mathcal{M}}_{g,n}) + \mathrm{vdim}(\overline{\mathcal{M}}_{g,n}) \\
&= (h^0(\mathcal{O}_C) - 1) + 3g - 3 + n \\
&= 4g - 4 + n,
\end{aligned}$$

we find that the virtual dimension of $\mathcal{P}/\overline{\mathcal{M}}_{g,n}$ at ξ equals

(3.7)
$$(N-r-4)(1-g) + (r+1)n - \sum_{i=1}^{n}\sum_{j=1}^{N} \frac{c_j m_{i,j}}{d} = \mathrm{vdim}(\overline{\mathcal{M}}_{g,n}(\mathbb{P}^{r-1}, \beta)) + \sum_{j=1}^{N} \chi(L^{\otimes c_j}).$$

3.4.8 Virtual cycle in genus zero

In genus zero, the definition of the virtual cycle simplifies substantially, under the Gorenstein condition

(3.8) $$c_j | d \quad \text{for all } j.$$

Indeed, if this hypothesis is satisfied and L is a line bundle satisfying the requirements of $\mathcal{P}$, then the bundles $L^{\otimes c_j}$ have no global sections. To see this, one simply must compute the degree of such a line bundle using the fact that on each irreducible component Z of the source curve C,

$$L^{\otimes c_j}|_Z^{\otimes d/c_j} \cong \omega_{\log} \otimes f^*\mathcal{O}(-1) \otimes \mathcal{O}\left(-\sum_{i=1}^{n} m_{i,j}[x_j]\right).$$

Here, the x_j are the special points on Z and the $m_{i,j}$ are the multiplicities of $L^{\otimes c_j}$ at those special points, and we are once again using that the multiplicities at all marked points are nonzero. This equation implies that the degree of $L^{\otimes c_j}|_Z$ is negative, so if C is itself irreducible, it follows that $L^{\otimes c_j}$ has no global sections. If C is reducible, the claim still follows by an easy inductive argument using the fact that $\deg(L^{\otimes c_j}|_Z) < k-1$, where k is the number of points at which Z meets the rest of C.

Because of this observation, $\mathcal{P} = \widetilde{\mathcal{M}}^d_{0,(m_1,\ldots,m_n)}(\mathbb{P}^{r-1},\beta)$, and the cosection localized virtual cycle is the same as the ordinary virtual cycle of $\widetilde{\mathcal{M}}^d_{0,(m_1,\ldots,m_n)}(\mathbb{P}^{r-1},\beta)$ defined by way of the perfect obstruction theory indicated above. Furthermore, abbreviating $\widetilde{\mathcal{M}} = \widetilde{\mathcal{M}}^d_{0,(m_1,\ldots,m_n)}(\mathbb{P}^{r-1},\beta)$ and $Y = \overline{\mathcal{M}}_{0,n}(\mathbb{P}^{r-1},\beta)$, the smoothness of the moduli space in this case implies that

$$[\widetilde{\mathcal{M}}]^{\mathrm{vir}} = c_{top}(\mathcal{O}b_{\widetilde{\mathcal{M}}/Y}) \cap [\overline{\mathcal{M}}_{0,n}(\mathbb{P}^{r-1},\beta)],$$

where $[\overline{\mathcal{M}}_{0,n}(\mathbb{P}^{r-1},\beta)]$ denotes the pullback of the fundamental class on Y to $\widetilde{\mathcal{M}}$ under the map that forgets L. Using the exact sequence

$$T_{\widetilde{\mathcal{M}}/\mathcal{D}_{g,n}} \xrightarrow{\sim} T_{Y/\mathcal{D}_{g,n}} \to \mathcal{O}b_{\widetilde{\mathcal{M}}/Y} \to \mathcal{O}b_{\widetilde{\mathcal{M}}/\mathcal{D}_{g,n}} \to 0,$$

one finds that $\mathcal{O}b_{\widetilde{\mathcal{M}}/Y} = R^1\pi_*(\mathcal{T}^{\otimes c_1} \oplus \cdots \oplus \mathcal{T}^{\otimes c_N})$, in which $\mathcal{T}$ is the universal line bundle on $\widetilde{\mathcal{M}}$. Thus, we obtain the formula

$$[\widetilde{\mathcal{M}}]^{\mathrm{vir}} = c_{top}(R^1\pi_*(\mathcal{T}^{\otimes c_1} \oplus \cdots \oplus \mathcal{T}^{\otimes c_N})) \cap [\overline{\mathcal{M}}_{0,n}(\mathbb{P}^{r-1},\beta)]$$

for the virtual cycle in genus zero.

3.5 Correlators

In analogy to Gromov-Witten theory, correlators will be defined as integrals over the moduli space against the virtual cycle.

3.5.1 Evaluation maps and psi classes

The classes that we integrate will come from two places. First, there are evaluation maps

$$\mathrm{ev}_i : \widetilde{\mathcal{M}}^d_{g,n}(\mathbb{P}^{r-1},\beta) \to \mathbb{P}^{r-1} \qquad\qquad i = 1,\ldots,n,$$

given by $(C, f, L, \varphi) \mapsto f(x_i)$, where $x_i \in C$ is the ith marked point. Therefore, we can pull back cohomology classes on $\mathbb{P}^{r-1}$ to obtain classes on the hybrid moduli space.

Second, there are classes

$$\psi_i \in H^2(\widetilde{\mathcal{M}}^d_{g,n}(\mathbb{P}^{r-1},\beta))$$

for $i = 1,\ldots,n$, defined in the same way as in Gromov-Witten theory. Namely, ψ_i is the first Chern class of the (orbifold) line bundle whose fiber at a point of the moduli space is the cotangent line to the orbifold curve at the ith marked point. Note that this differs from the definition of ψ_i used in [11], in which the cotangent line was always taken to the underlying curve; we will denote these "coarse" psi classes by $\overline{\psi}_i$. The two are related by

$$\overline{\psi}_i = d\psi_i.$$

3.5.2 Definition of correlators in the narrow case

We will only define correlators when all insertions are drawn from the narrow sectors of the state space.

Definition 3.5.3. Choose $\alpha_1, \ldots, \alpha_n \in \mathcal{H}_{hyb}(W_1, \ldots, W_r)$ from the narrow sectors and $l_1, \ldots, l_n \geq 0$. As explained in Section 2.4, each α_i can be viewed as an element of $H^*(\mathbb{P}^{r-1})$. Each also defines an element $\gamma_i \in J$ indicating the twisted sector from which it is drawn, and we let $m_i \in \{0, 1, \ldots, d-1\}$ be such that

$$\gamma_i = (e^{2\pi i \frac{m_i c_1}{d}}, \ldots, e^{2\pi i \frac{m_i c_N}{d}}, 1, \ldots, 1).$$

Define the associated *hybrid model correlator* $\langle \tau_{l_1}(\alpha_1) \cdots \tau_{l_n}(\alpha_n) \rangle^{hyb}_{g,n,\beta}$ to be

$$\frac{d(-1)^D}{\deg(\rho)} \int_{[\widetilde{\mathcal{M}}^d_{g,(m_1,\ldots,m_n)}(\mathbb{P}^{r-1},\beta)]^{vir}} \mathrm{ev}_1^*(\alpha_1)\psi_1^{l_1} \cdots \mathrm{ev}_n^*(\alpha_n)\psi_n^{l_n}, \tag{3.9}$$

where

$$D = \sum_{i=1}^N \left(h^1(L^{\otimes c_i}) - h^0(L^{\otimes c_i}) \right)$$

and $\deg(\rho)$ denotes the degree of the map

$$\rho : \widetilde{\mathcal{M}}^d_{g,(m_1,\ldots,m_n)}(\mathbb{P}^{r-1},\beta) \to \overline{\mathcal{M}}_{g,n}(\mathbb{P}^{r-1},\beta)$$

given by forgetting the $\overline{W}$-structure and passing to the coarse underlying source curve.

The strange-looking sign choice in this definition is a matter of convenience, following equation (50) of [27]. In genus zero under the Gorenstein condition (3.8), D is precisely the rank of the obstruction bundle and $\deg(\rho) = \frac{1}{d}$ whenever the substratum over which we are integrating is nonempty (see Equation (26) of [27]). Thus, the above is equivalent in such cases to

$$d^2 \int_{[\widetilde{\mathcal{M}}^d_{0,\mathbf{m}}(\mathbb{P}^{r-1},\beta)]^{\mathrm{vir}}} \mathrm{ev}_1^*(\alpha_1)\psi_1^{l_1} \cdots \mathrm{ev}_n^*(\alpha_n)\psi_n^{l_n} c_{top}((R^1\pi_*(\mathcal{T}^{\otimes c_1} \cdots \mathcal{T}^{\otimes c_N}))^\vee),$$

where $\mathbf{m} = (m_1, \ldots, m_n)$.

3.5.4 Broad insertions

The easiest way to extend the above theory to allow for broad insertions is to set a correlator to zero if any of its insertions comes from a broad sector. In order to ensure that the resulting theory satisfies the decomposition property required of Cohomological Field Theories, though, it is necessary to verify a Ramond vanishing property. This holds whenever the Gorenstein condition (3.8) is satisfied.[4]

Proposition 3.5.5 (Ramond vanishing)**.** *Suppose that for all i and j, $c_j|d$ and $m_{i,j} \neq 0$. Let $D \subset \widetilde{\mathcal{M}}^d_{0,\mathbf{m}}(\mathbb{P}^{r-1},\beta)$ be a boundary stratum whose general point is a source curve with a single, broad node. Then*

$$(3.10) \qquad \int_D ev_1^*(\alpha_1)\psi_1^{l_1}\cdots ev_n^*(\alpha_n)\psi_n^{l_n}c_{top}((R^1\pi_*(\mathcal{T}^{\otimes c_1}\oplus\cdots\oplus\mathcal{T}^{\otimes c_N}))^\vee) = 0$$

for any $a_1,\ldots,a_n \in \mathbb{Z}^{\geq 0}$ and any $\phi_1,\ldots,\phi_n \in H^*(\mathbb{P}^{r-1})$.

Proof. Let $C = C_1 \sqcup C_2$ be the decomposition of a fiber of π in D into irreducible components, and let n be the node at which the components meet. Consider the normalization exact sequence

$$0 \to \mathcal{O}_C \to \mathcal{O}_{C_1} \oplus \mathcal{O}_{C_2} \to \mathcal{O}_n \to 0.$$

Tensor with $L^{\otimes c_1}\oplus\cdots\oplus L^{\otimes c_N}$ and take the associated long exact sequence in cohomology to obtain the following equality in K-theory:

$$R^1\pi_*\left(\oplus\mathcal{T}^{\otimes c_j}\right) = R^0\pi_*(\oplus\mathcal{T}^{\otimes c_j}|_n) + R^1\pi_*(\oplus\mathcal{T}^{\otimes c_j}|_{C_1}) + R^1\pi_*(\oplus\mathcal{T}^{\otimes c_j}|_{C_2}).$$

Here, we use that $H^0(L^{\otimes c_j}|_{C_i}) = 0$ for all i and j, as an easy degree computation shows.

[4]The argument below was substantially simplified by a suggestion of A. Chiodo.

The key point is that, since n is broad, orbifold sections of $L^{\otimes c_j}$ over n are the same as ordinary sections over the coarse underlying curve. More precisely, let $N : D \to \mathscr{C}$ be the section of the universal curve defined by the node n. Then

$$c_{top}(R^0\pi_*(\oplus\mathcal{T}^{\otimes c_j}|_n)) = N^* c_{top}(\oplus\mathcal{T}^{\otimes c_j}),$$

and since $(\mathcal{T}^{\otimes c_j})^{\otimes d} \cong \omega_{\log}^{\otimes c_j} \otimes f^*\mathcal{O}(-c_j)$ for each j, the above equals

$$\prod_{j=1}^{N} \frac{1}{d}\left(c_{top}(\omega_{\log}^{\otimes c_j}|_n) + c_{top}(N^* f^*\mathcal{O}(-c_j))\right).$$

In this expression, $c_{top}(\omega_{\log}^{\otimes c_j}|_n) = 0$, since the restriction of $\omega_{\log}$ to the locus of nodes is trivial. Furthermore, $f \circ N = \mathrm{ev}_n$, so we can rewrite the above as

$$\prod_{j=1}^{N} \frac{1}{d} ev_n^* c_{top}(\mathcal{O}(-c_j)) = \frac{1}{d^N}\mathrm{ev}_n^* c_{top}(\mathcal{O}(-c_j)^{\oplus N}),$$

which is zero because $\mathcal{O}(-c_j)^{\oplus N}$ is an N-dimensional bundle on an r-dimensional space and $N > r$.

It follows that one of the summands in the expression for $R^1\pi_*(\oplus\mathcal{T}^{\otimes c_j})$ has trivial top Chern class, so the integral in (3.10) vanishes. □

Remark 3.5.6. This definition of the broad correlators seems initially ad hoc. However, analogously to Proposition 2.4.5 of [11], it is possible to unify the broad and narrow cases in genus 0 into a single geometric definition by slightly modifying the moduli space.

3.5.7 Multiplicity conditions

Certain tuples of multiplicities correspond to empty components of the moduli space, so the resulting correlators clearly vanish. Indeed, (3.1) and the subsequent discussion imply that if $m_1, \ldots, m_n$ are as in Definition 3.5.3, then the correlator

$\langle \tau_1(\phi_1) \cdots \tau_n(\phi_n) \rangle^{hyb}_{g,n,\beta}$ vanishes unless

$$2g - 2 + n - \beta - \sum_{i=1}^{n} m_i \equiv 0 \mod d. \tag{3.11}$$

This selection rule will be useful later.

CHAPTER IV

Proof of the correspondence in genus zero

In both Gromov-Witten theory and the hybrid model, the genus-zero theory can be realized as a Lagrangian cone in a certain symplectic vector space. Because the genus-zero hybrid invariants are described via a top Chern class, they fit into the framework of twisted invariants described in [20], and Givental's quantization formalism provides a tool for realizing them in terms of the corresponding *untwisted* theory, which is essentially the Gromov-Witten theory of projective space. The following section describes this process in detail and uses it to prove the LG/CY correspondence in the two cases of interest.

4.1 Givental's formalism

For the sake of expository clarity, we will describe the setup in the case of the cubic singularities first, commenting briefly on the requisite modifications for the quadric case at the end.

4.1.1 The symplectic vector spaces

It is convenient to modify the state space slightly, replacing the broad sector with another copy of $H^*(\mathbb{P}^1)$ to obtain

$$H_{hyb} = H_0^*(\mathbb{P}^1) \oplus H_1^*(\mathbb{P}^1) \oplus H_2^*(\mathbb{P}^1).$$

The subscripts denote the multiplicities to which the summands correspond. This modification does not affect the correlators, since they vanish when any insertion is broad. We will write $\phi^{(h)}$ for an element $\phi \in H^*(\mathbb{P}^1)$ coming from the summand $H^*_h(\mathbb{P}^1)$.

This vector space is equipped with a nondegenerate inner product (or *Poincaré pairing*), denoted $(\ ,\ )_{hyb}$ and defined as

$$(\Theta_1, \Theta_2)_{hyb} = \langle \tau_0(\Theta_1)\ \tau_0(\Theta_2)\ 1^{(1)} \rangle^{hyb}_{0,3,0}.$$

The symplectic vector space we will consider is

$$\mathcal{V}_{hyb} = H_{hyb} \otimes \mathbb{C}((z^{-1})),$$

with the symplectic form Ω_{hyb} given by

$$\Omega_{hyb}(f, g) = \text{Res}_{z=0}\Big((f(-z), g(z))_{hyb} \Big).$$

This induces a polarization $\mathcal{V}_{hyb} = \mathcal{V}^+_{hyb} \oplus \mathcal{V}^-_{hyb}$, where $\mathcal{V}^+_{hyb} = H_{hyb} \otimes \mathbb{C}[z]$ and $\mathcal{V}^-_{hyb} = z^{-1} H_{hyb} \otimes \mathbb{C}[[z^{-1}]]$. Thus, we can identify $\mathcal{V}_{hyb}$ as a symplectic manifold with the cotangent bundle to $\mathcal{V}^+_{hyb}$. An element of $\mathcal{V}_{hyb}$ can be expressed in Darboux coordinates as $\sum_{k\geq 0} q^\alpha_k \phi_\alpha z^k + \sum_{\ell \geq 0} p_{\ell,\beta} \phi^\beta (-z)^{-\ell-1}$, where $\{\phi_\alpha\}$ is a basis for H_{hyb}.

Analogously, there is a symplectic vector space on the Gromov-Witten side [11] [20]. The restriction to narrow states is mirrored in that setting by the restriction to cohomology classes pulled back from the ambient projective space, which are the only ones that give nonzero correlators. Let H_{GW} denote the vector space of such classes:

$$H_{GW} = H^{even}(X_{3,3}) = \bigoplus_{h=0}^{3} [H^h]\mathbb{C},$$

where H is the restriction to $X_{3,3}$ of the hyperplane class on the ambient projective space.[1] The symplectic vector space $\mathcal{V}_{GW}$ on the Gromov-Witten side is defined as above, and the usual Poincaré pairing on H_{GW} induces a symplectic form in the same way.

4.1.2 The potentials

Defining the correlators in the hybrid theory as above, the generating function for the genus-g invariants is

$$\mathcal{F}^g_{hyb}(\mathbf{t}) = \sum_{n,d} \frac{Q^d}{n!} \langle \mathbf{t}(\overline{\psi}), \ldots, \mathbf{t}(\overline{\psi}) \rangle^{hyb}_{g,n,d},$$

where $\mathbf{t} = t_0 + t_1 z + t_2 z^2 + \cdots \in H_{hyb}[[z]]$. These generating functions fit together into a total-genus descendent potential

$$\mathcal{D}_{hyb} = \exp\left(\sum_{g \geq 0} \hbar^{g-1} \mathcal{F}^g_{hyb} \right).$$

In the same way, one can define a generating function for the genus-g Gromov-Witten invariants of the corresponding complete intersection,

$$\mathcal{F}^g_{GW}(\mathbf{t}) = \sum_{n,d} \frac{Q^d}{n!} \langle \mathbf{t}(\psi), \ldots, \mathbf{t}(\psi) \rangle^{GW}_{g,n,d},$$

where $\mathbf{t} = t_0 + t_1 z + t_2 z^2 + \cdots \in H_{GW}[[z]]$. These, too, fit together into a total-genus descendent potential $\mathcal{D}_{GW}$.

4.1.3 The Lagrangian cones

In the Gromov-Witten setting, the dilaton shift

$$q^\alpha_k = t^\alpha_k - 1 \cdot z$$

[1]Of course, to be completely symmetric, we might want to add an additional two-dimensional summand to H_{GW}, as we did for H_{hyb}, and define the correlators to vanish if any insertion comes from this summand. Since we will not be doing any computations on the Gromov-Witten side, we will ignore this asymmetry and leave H_{GW} as above.

makes $\mathcal{F}^0_{GW}$ into a power series in the Darboux coordinates q^α_k, where 1 denotes the constant function 1 in H^0. In this way, the genus-zero Gromov-Witten theory is encoded by a Lagrangian cone

$$\mathcal{L}_{GW} = \{(\mathbf{q}, \mathbf{p}) \mid \mathbf{p} = d_{\mathbf{q}}\mathcal{F}^0_{GW}\} \subset \mathcal{V}_{GW},$$

where we use the Darboux coordinates $(\mathbf{q}, \mathbf{p})$ defined above to identify $\mathcal{V}_{GW}$ with the cotangent bundle to its Lagrangian subspace $\mathcal{V}^+_{GW}$. As proved in [20], $\mathcal{L}_{GW}$ is a Lagrangian cone whose tangent spaces satisfy the geometric condition

$$zT_f\mathcal{L}_{GW} = \mathcal{L}_{GW} \cap T_f\mathcal{L}_{GW} \tag{4.1}$$

at any point.

The same story holds in the hybrid model, but it is important to note that in the dilaton shift

$$q^\alpha_k = t^\alpha_k - 1^{(1)} \cdot z,$$

the *unit is the constant function* 1 *from the summand of the state space corresponding to multiplicity*-1 *insertions*. Under this dilaton shift, we again have that $\mathcal{F}^0_{hyb}$ is a function of $\mathbf{q} \in \mathcal{V}^+_{hyb}$ and hence we can define

$$\mathcal{L}_{hyb} = \{(\mathbf{q}, \mathbf{p}) \mid \mathbf{p} = d_{\mathbf{q}}\mathcal{F}^0_{hyb}\} \subset \mathcal{V}_{hyb}.$$

Since the hybrid theory also satisfies the string equation, dilaton equation, and topological recursion relations, the same geometric condition holds for this cone as for the Lagrangian cone of Gromov-Witten theory.

On either the Gromov-Witten or the hybrid side, we define the J-function

$$J_{hyb/GW}(\mathbf{t}, z) = 1z + \mathbf{t} + \sum_{n,d} \frac{1}{n!} \left\langle \mathbf{t}, \ldots, \mathbf{t}, \frac{\phi_\alpha}{z - \psi} \right\rangle^{hyb/GW}_{0,n+1,d} \phi^\alpha,$$

where ϕ_α ranges over a basis for $H_{hyb/GW}$ with dual basis ϕ^α. In other words, $J(\mathbf{t}, -z)$ is the intersection of the Lagrangian cone with the slice $\{-1z + \mathbf{t} + \mathcal{V}^-\} \subset \mathcal{V}_{hyb/GW}$. It is a well-known consequence of (4.1) that this slice determines the rest of the Lagrangian cone, so the J-function specifies the entire genus-zero theory.

4.1.4 Twisted theory

The strategy for determining J_{hyb} is to introduce parameters that will interpolate between the hybrid invariants and the ordinary Gromov-Witten invariants of projective space. One can always define a multiplicative characteristic class $K_0(X) \to H^*(X;\mathbb{C})$ by

$$x \mapsto \exp\left(\sum_{k\geq 0} s_k \mathrm{ch}_k(x)\right).$$

When $s_k = 0$ for all $k \geq 0$, the result is a constant map sending every K-class to the fundamental class, while if we set

$$s_k = \begin{cases} -6\ln(\lambda) & k = 0 \\ \\ \dfrac{6(k-1)!}{\lambda^k} & k > 0, \end{cases} \tag{4.2}$$

then the resulting class satisfies

$$\exp\left(\sum_{k\geq 0} s_k \mathrm{ch}_k(-[V])\right) = e_{\mathbb{C}^*}(V^\vee)^6$$

for any vector bundle V equipped with the natural $\mathbb{C}^*$ action scaling the fibers. (The reason for passing to equivariant cohomology is to ensure that the above is invertible.) We will typically denote

$$c(x) = \exp\left(\sum_{k\geq 0} s_k \mathrm{ch}_k(x)\right)$$

when the parameters s_k are taking unspecified values.

Extend the hybrid model state space to

$$H_{tw} = \left(H_0^*(\mathbb{P}^1) \oplus H_1^*(\mathbb{P}^1) \oplus H_2^*(\mathbb{P}^1)\right) \otimes R,$$

where

$$R = \mathbb{C}[\lambda][[s_0, s_1, \ldots]].$$

Then, for any $\phi_1, \ldots, \phi_n \in H_{tw}$ and $a_1, \ldots, a_n \in \mathbb{Z}^{\geq 0}$, define the corresponding *twisted hybrid invariant* $\langle \tau_{a_1}(\phi_1), \ldots, \tau_{a_n}(\phi_n) \rangle^{tw}_{g,n,d}$ by

$$\frac{3}{\deg(\rho)} \int_{\rho^*[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1,d)]^{\mathrm{vir}}} \mathrm{ev}_1^*(\phi_1)\psi_1^{a_1} \cdots \mathrm{ev}_n^*(\phi_n)\psi_n^{a_n}\, c(R\pi_*\mathcal{T}),$$

where $\mathcal{T}$ denotes the universal line bundle on the universal curve over $\widetilde{\mathcal{M}}^3_{g,n}(\mathbb{P}^1, d)$, $\rho : \widetilde{\mathcal{M}}^3_{g,\mathbf{m}}(\mathbb{P}^1, d) \to \overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, d)$ is as in Section 3.5.3. We will sometimes adopt the notation of [20] and write the above as

$$\langle \tau_{a_1}(\phi_1), \ldots, \tau_{a_n}(\phi_n); c(R\pi_*\mathcal{T}) \rangle_{g,n,d},$$

or more generally, write a cohomology class on the universal curve after a semicolon to indicate that it is part of the integrand but is neither a ψ class nor pulled back from the target space.

Via these invariants, H_{tw} is equipped with a pairing extending the pairing on H_{hyb}:

$$(\Theta_1, \Theta_2)_{tw} = \langle \Theta_1, \Theta_2, 1^{(1)} \rangle^{tw}_{0,3,0}.$$

We can then set $\mathcal{V}_{tw} = H_{tw} \otimes \mathbb{C}((z^{-1}))$, and this is a symplectic vector space under the symplectic form induced by the twisted pairing. The definitions of the genus-g potential, total descendent potential, and Lagrangian cone all generalize directly, and we thus obtain the twisted Lagrangian cone $\mathcal{L}_{tw} \subset \mathcal{V}_{tw}$. It is no longer obvious that this is indeed a Lagrangian cone, but this will follow from Proposition 4.2.1.

4.1.5 Untwisted theory

Let $\mathcal{V}_{un}$ denote the symplectic vector space obtained by setting $s_k = 0$ for all $k \geq 0$, and similarly H_{un} and $\mathcal{L}_{un}$. Note that $\mathcal{L}_{un}$ encodes the correlators $\langle \tau_{a_1}(\phi_1), \ldots, \tau_{a_n}(\phi_n) \rangle^{un}_{0,n,d}$, which are given by

$$\frac{3}{\deg(\rho)} \int_{\rho^*[\overline{\mathcal{M}}_{0,n}(\mathbb{P}^1,d)]^{\mathrm{vir}}} \mathrm{ev}_1^*(\phi_1)\psi_1^{a_1} \cdots \mathrm{ev}_n^*(\phi_n)\psi_n^{a_n}.$$

When the selection rule (3.11) is satisfied so that the component of the hybrid moduli space over which we are integrating is nonempty, these are simply three times the Gromov-Witten invariants of $\mathbb{P}^1$. In particular, the untwisted J-function is known explicitly.

We will use the untwisted Lagrangian cone to determine the cone $\mathcal{L}_{hyb}$. This can be viewed as a two-step procedure. First, $\mathcal{L}_{hyb}$ can be obtained from $\mathcal{L}_{tw}$ by taking a limit $\lambda \to 0$ and setting the parameters s_k to the values in (4.2), so that

$$c(R\pi_*\mathcal{T}) = c(-R^1\pi_*\mathcal{T}) = c_{top}((R^1\pi_*\mathcal{T})^\vee)^6,$$

which is what appears in the hybrid model correlators. Then, Proposition 4.2.1 demonstrates that $\mathcal{L}_{tw}$ can in turn be recovered from $\mathcal{L}_{un}$.

4.1.6 The quadric singularities

All of the above is defined analogously in the other example of interest. In that case,

$$H_{hyb} = H_0^*(\mathbb{P}^3) \oplus H_1^*(\mathbb{P}^3).$$

The hybrid Poincaré pairing is defined by the exact same formula, and we obtain a symplectic vector space $\mathcal{V}_{hyb} = H_{hyb} \otimes \mathbb{C}((z^{-1}))$. The symplectic vector space on the Gromov-Witten side is now $\mathcal{V}_{GW} = H_{GW} \otimes \mathbb{C}((z^{-1}))$, where

$$H_{GW} = H^{even}(X_{2,2,2,2}) = \bigoplus_{h=0}^{3} [H^h]\mathbb{C}$$

and H is the restriction to $X_{2,2,2,2}$ of the hyperplane class on $\mathbb{P}^7$. The genus-g generating functions and total-genus descendent potentials on both the hybrid and the Gromov-Witten side are defined just as before, and again the genus-0 theory on each side is encoded by a Lagrangian cone which is determined by the slice cut out by a J-function.

A twisted theory is again introduced, though now the values of s_k that give the hybrid theory are

$$s_k = \begin{cases} -8\ln(\lambda) & k = 0 \\ \\ \dfrac{8(k-1)!}{\lambda^k} & k > 0, \end{cases} \tag{4.3}$$

since the virtual class in genus 0 is $c_{top}((R^1\pi_*\mathcal{T})^\vee)^8$ in this case. The state space is extended to

$$H_{tw} = (H_0^*(\mathbb{P}^3) \oplus H_1^*(\mathbb{P}^3)) \otimes R$$

for $R = \mathbb{C}[\lambda][[s_0, s_1, \ldots]]$, and twisted hybrid invariants are defined as

$$\frac{2}{\deg(\rho)} \int_{\rho^*[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^3,d)]^{\mathrm{vir}}} \mathrm{ev}_1^*(\phi_1)\psi_1^{a_1} \cdots \mathrm{ev}_n^*(\phi_n)\psi_n^{a_n}\, c(R\pi_*\mathcal{T}),$$

for $\phi_1, \ldots, \phi_n \in H_{tw}$ and $a_1, \ldots, a_n \in \mathbb{Z}^{\geq 0}$. These permit the definition of the twisted Poincaré pairing and hence the twisted symplectic vector space. When $\lambda \to 0$ and the parameters s_k are set to the values in (4.3), we obtain the hybrid theory for the quadric singularity, while the untwisted theory (when $s_k = 0$ for all k) gives two times the Gromov-Witten theory of $\mathbb{P}^3$.

4.2 Lagrangian cone for the Landau-Ginzburg theory

Recall that the Bernoulli polynomials $B_n(x)$ are defined by the generating function

$$\sum_{n=0}^{\infty} B_n(x)\frac{z^n}{n!} = \frac{ze^{zx}}{e^z-1}.$$

Proposition 4.2.1. *(a) Let $\mathcal{V}_{tw}$ denote the symplectic vector space associated to the cubic singularities $W_1(x_1,\ldots,x_6),\ldots,W_3(x_1,\ldots,x_6)$, and let $\Delta:\mathcal{V}_{un}\to\mathcal{V}_{tw}$ be the symplectic transformation*

$$\Delta = \bigoplus_{\ell=0}^{2}\exp\left(\sum_{\substack{k\geq 0\\ m\geq 0}} s_k\frac{B_m(\frac{\ell}{3})}{m!}\exp(-\tfrac{H^{(\ell)}}{3})_{k+1-m}z^{m-1}\right).$$

Then $\mathcal{L}_{tw}=\Delta(\mathcal{L}_{un})$.

(b) Let $\mathcal{V}_{tw}$ denote the symplectic vector space associated to the quadric singularities $V_1(x_1,\ldots,x_8),\ldots,V_4(x_1,\ldots,x_8)$, and let $\Delta:\mathcal{V}_{un}\to\mathcal{V}_{tw}$ be the symplectic transformation

$$\Delta = \bigoplus_{\ell=0}^{1}\exp\left(\sum_{\substack{k\geq 0\\ m\geq 0}} s_k\frac{B_m(\frac{\ell}{2})}{m!}\exp(-\tfrac{H^{(\ell)}}{2})_{k+1-m}z^{m-1}\right).$$

Then $\mathcal{L}_{tw}=\Delta(\mathcal{L}_{un})$.

Proof. We will prove part (a) of the Proposition; the proof of part (b) is almost identical, so we will omit it. Our proof is modeled closely after that of Theorem 4.2.1 of [47], which in turn uses the main idea of Theorem 1′ of [20].

Let us begin by reducing the statement to something more concrete. According to the theory of Givental quantization, the desired statement $\mathcal{L}_{tw}=\Delta(\mathcal{L}_{un})$ will be implied if we can demonstrate that $\mathcal{D}_{tw}=\widehat{\Delta}(\mathcal{D}_{un})$. In fact, it suffces to show that $\mathcal{D}_{tw}\approx\Delta(\mathcal{D}_{un})$, where the symbol $\approx$ denotes equality up to a scalar factor in R, since $\mathcal{L}_{tw}$ is a cone and hence is unaffected by scalar multiplication. Furthermore, $\mathcal{D}_{tw} \approx \Delta(\mathcal{D}_{un})$ if and only if this holds after differentiating both sides with respect to s_k for all k. If $C_k : \mathcal{V}_{un} \to \mathcal{V}_{tw}$ denotes the infinitesimal symplectic transformation[2]

$$C_k = \bigoplus_{\ell=0}^{2} \left(\sum_{m\geq 0} \frac{B_m(\frac{\ell}{3})}{m!} \exp(-\tfrac{H^{(\ell)}}{3})_{k+1-m} z^{m-1} \right),$$

then we have $\Delta = \exp(\sum_{k\geq 0} s_k C_k)$, so $\mathcal{D}_{tw} \approx \Delta(\mathcal{D}_{un})$ is equivalent to the system of differential equations

$$\frac{\partial \mathcal{D}_{tw}}{\partial s_k} \approx \widehat{C_k}\mathcal{D}_{tw} + \mathcal{C}\mathcal{D}_{un}$$

for all k, where $\mathcal{C}$ is the cocycle coming from commuting the $\hat{z}$ terms of $\widehat{\Delta}$ past the $\widehat{1/z}$ term of $\widehat{C_k}$; see the discussion in Section 2 of [20]. Since we only seek equality up to a scalar factor, we can absorb the cocycle into the definition of C_k and prove that $\partial\mathcal{D}_{tw}/\partial s_k \approx \widehat{C_k}\mathcal{D}_{tw}$. We will use the orbifold Grothendieck-Riemann-Roch (oGRR) formula[3] (see Appendix A of [47] for the statement) to determine $\partial\mathcal{D}_{tw}/\partial s_k$ and identify it with an explicit expression for $\widehat{C_k}$.

Specifically, we have

$$\frac{\partial \mathcal{D}_{tw}}{\partial s_k} = \sum_{g,n,d} \frac{Q^d \hbar^{g-1}}{n!} \langle \mathbf{t}, \ldots, \mathbf{t}; \mathrm{ch}_k(R\pi_*\mathcal{T})\; c(R\pi_*\mathcal{T}) \rangle_{g,n,d} \mathcal{D}_{tw}, \tag{4.4}$$

and oGRR will be used to compute the contribution from $\mathrm{ch}_k(R\pi_*\mathcal{T})$. As remarked in Section 7.3 of [47], the moduli stack $\widetilde{\mathcal{M}}^3_{g,n}(\mathbb{P}^1, d)$ can be embedded in a smooth stack $\overline{\mathcal{M}}$ over which there exists a family $\mathcal{U}$ of orbicurves pulling back to the universal family $\mathscr{C}$ over $\widetilde{\mathcal{M}}^3_{g,n}(\mathbb{P}^1, d)$. Therefore, we lose no information if we assume that the moduli stack itself is smooth, in which case $\mathrm{ch}(R\pi_*\mathcal{T}) =$

[2]The fact that this transformation is infinitesimal symplectic is required for the quantization to be defined; it follows from the same argument as in Lemma 4.1.3 of [47].

[3]An alternative, and perhaps shorter, proof can be obtained by passing to the coarse underlying curve and applying the usual GRR formula, as in [12].

$\widetilde{\mathrm{ch}}(R\pi_*\mathcal{T})$ and oGRR states that

$$\mathrm{ch}(R\pi_*\mathcal{T}) = I\pi_*(\widetilde{\mathrm{ch}}(\mathcal{T})\widetilde{\mathrm{Td}}(T_\pi)). \tag{4.5}$$

This splits into several terms according to the decomposition of $I\mathscr{C}$ into twisted sectors:

$$I\mathscr{C} = \mathscr{C} \sqcup \coprod_{i=1}^{n} (\mathscr{S}_i^{(1)} \sqcup \mathscr{S}_i^{(2)}) \sqcup (\mathscr{Z}^{(1)} \sqcup \mathscr{Z}^{(2)}).$$

Here, $\mathscr{S}_i^{(h)}$ is the sector corresponding to the element $h \in \mathbb{Z}_3 = \{0,1,2\}$ of the isotropy group at the ith marked point and $\mathscr{Z}^{(h)}$ is the sector corresponding to the element h of the isotropy group at the substratum of nodes. Applying this decomposition to the right-hand side of (4.5) shows that $\mathrm{ch}(R\pi_*\mathcal{T})$ equals

$$\pi_*(\mathrm{ch}(\mathcal{T})\mathrm{Td}(T_\pi)) + \sum_{i=1}^{n}\sum_{\ell}^{2} \pi_*(\widetilde{\mathrm{ch}}(\mathcal{T})\widetilde{\mathrm{Td}}(T_\pi)|_{\mathscr{S}_i^{(\ell)}}) + \sum_{\ell=1}^{2} \pi_*(\widetilde{\mathrm{ch}}(\mathcal{T})\widetilde{\mathrm{Td}}(T_\pi)|_{\mathscr{Z}^{(\ell)}}).$$

The contribution from the nontwisted sector can be calculated via a computation nearly identical to that of Theorem 1′ of [20]; the result is:

$$\pi_*\left(\mathrm{ch}(\mathcal{T})\left(\mathrm{Td}^\vee(\overline{L}_{n+1}) - \sum_{i=1}^{n} s_{i*}\left[\frac{\mathrm{Td}^\vee(N_i^\vee)}{c_1(N_i^\vee)}\right]_+ + \iota_*\left[\frac{1}{\psi_+\psi_-}\left(\frac{\mathrm{Td}^\vee(L_+)}{\psi_+} + \frac{\mathrm{Td}^\vee(L_-)}{\psi_-}\right)\right]_+\right)\right)_k.$$

We have identified the universal family with $\widetilde{\mathcal{M}}^3_{g,n+1}(\mathbb{P}^1,d)'$, in which the prime indicates that the last marked point has multiplicity 1. In the second term, s_i denotes the inclusion of the divisor Δ_i of the ith marked point and N_i denotes the normal bundle of Δ_i in $\mathscr{C}$. In the third term, $\iota : Z' \to \mathscr{C}$ is the composition of the inclusion $i : Z \to \mathscr{C}$ of the singular locus with the double cover $\gamma : Z' \to Z$ consisting of choices of a branch at each node; also, $L_\pm$ are the cotangent line bundles to the two branches of a node and $\psi_\pm$ are the first Chern classes of these line bundles.

Accordingly, we can split $\mathrm{ch}_k(R\pi_*\mathcal{T})$ into a codimension-0, codimension-1, and codimension-2 term, and we compute each separately.

4.2.2 Codimension 0

Since $\mathcal{T}^{\otimes 3} \cong \omega_{\log} \otimes f^*\mathcal{O}(-1)$, we have

$$\mathrm{ch}(\mathcal{T}) = \exp(\tfrac{K}{3})\exp(-\tfrac{f^*H}{3}),$$

where $K = c_1(\omega_{\log})$. Thus, the codimension 0 term of $\mathrm{ch}(R\pi_*\mathcal{T})$ is

$$\pi_*(\exp(\tfrac{K}{3})\exp(-\tfrac{f^*H}{3})\mathrm{Td}^\vee(\overline{L}_{n+1})).$$

The contribution from the codimension 0 term to (4.4), then, is $\mathcal{D}_{tw}$ times the following, in which the superscript $\bullet$ denotes invariants in which the last marked point has multiplicity 1:

$$\begin{aligned}
&\sum_{g,n,d} \frac{Q^d\hbar^{g-1}}{n!}\left\langle \mathbf{t},\ldots;\pi_*\left(\exp(\tfrac{K}{3})\exp(-\tfrac{f^*H}{3})\mathrm{Td}^\vee(\overline{L}_{n+1})\right)_{k+1}\ c(R\pi_*\mathcal{T})\right\rangle_{g,n,d}\\
=&\sum_{g,n,d} \frac{Q^d\hbar^{g-1}}{n!}\left\langle \pi^*\mathbf{t},\ldots,(\exp(\tfrac{K}{3})\exp(-\tfrac{H}{3})\mathrm{Td}^\vee(\overline{L}_{n+1}))_{k+1}\ c(R\pi_*\mathcal{T})\right\rangle^{\bullet}_{g,n+1,d}\\
=&\sum_{g,n,d} \frac{Q^d\hbar^{g-1}}{n!}\left\langle \mathbf{t}-\sigma_{1*}\left[\frac{\mathbf{t}}{\psi}\right]_+,\ldots,(\exp(\tfrac{K}{3})\exp(-\tfrac{H}{3})\mathrm{Td}^\vee(\overline{L}_{n+1}))_{k+1}\right\rangle^{tw,\bullet}_{g,n+1,d}.
\end{aligned}$$

Now, under the identification of the universal family with $\widetilde{\mathcal{M}}^3_{g,n}(\mathbb{P}^1,d)'$, K is identified with $\overline{\psi}_{n+1}$. Furthermore, $\overline{\psi}_{n+1}$ vanishes on the image of each σ_{i*} with $1 \leq i \leq n$, so the above is equal to

$$\sum_{g,n,d} \frac{Q^d\hbar^{g-1}}{n!}(\exp(\tfrac{\overline{\psi}_{n+1}}{3})\exp(-\tfrac{H}{3})\mathrm{Td}^\vee(\overline{L}_{n+1}))_{k+1}; c(R\pi_*\mathcal{T})\rangle^{\bullet}_{g,n+1,d}$$

$$-\sum_{g,n,d} \frac{Q^d\hbar^{g-1}}{(n-1)!}\langle\sigma_{1*}\left[\frac{\mathbf{t}}{\psi}\right]_+,\ldots,(\exp(-\tfrac{H}{3})_{k+1}; c(R\pi_*\mathcal{T})\rangle_{g,n+1,d}$$

$$
\begin{aligned}
&= \sum_{g,n,d} \frac{Q^d \hbar^{g-1}}{(n-1)!} \left\langle \mathbf{t}, \ldots, \left(\exp(\tfrac{\overline{\psi}}{3}) \exp(-\tfrac{H}{3}) \mathrm{Td}^\vee(\overline{L}_n)\right)_{k+1}; c(R\pi_* \mathcal{T}) \right\rangle^\bullet_{g,n,d} \\
&\quad - \sum_{g,n,d} \frac{Q^d \hbar^{g-1}}{(n-1)!} \left\langle \mathbf{t}, \ldots, \mathbf{t}, \exp(-\tfrac{H}{3})_{k+1} \left[\tfrac{\mathbf{t}(\overline{\psi})}{\overline{\psi}}\right]_+; c(R\pi_* \mathcal{T}) \right\rangle_{g,n,d} \\
&\quad - \frac{1}{2\hbar} \left\langle \mathbf{t}, \mathbf{t}, (\exp(\tfrac{\overline{\psi}_3}{3}) \exp(-\tfrac{H}{3}) \mathrm{Td}^\vee(\overline{L}_3))_{k+1}; c(R\pi_* \mathcal{T}) \right\rangle^\bullet_{0,3,0} \\
&\quad - \left\langle (\exp(\tfrac{\overline{\psi}_1}{2})) \exp(-\tfrac{H}{3}) \mathrm{Td}^\vee(\overline{L}_1))_{k+1}; c(R\pi_* \mathcal{T}) \right\rangle^\bullet_{1,1,0}.
\end{aligned}
$$

The last two summands are known respectively as the genus-zero and the genus-one exceptional terms. Since $\overline{\psi}_3$ vanishes on $\widetilde{\mathcal{M}}^3_{0,3}(\mathbb{P}^1, 0)$, the genus zero exceptional term equals

$$
-\frac{1}{2\hbar}(\exp(-\tfrac{H}{3})_{k+1}\mathbf{q}, \mathbf{q})_{tw}.
$$

The rank of $R\pi_*\mathcal{T}$ is zero on $\widetilde{\mathcal{M}}^3_{1,1}(\mathbb{P}^1, 0)$, so the genus-one exceptional term does not depend on s_k. It is easily computed, but it will yield only a scalar factor and hence does not affect our present computation.

4.2.3 Codimension 1

Since K vanishes on the image of σ_{i*} for all i, we have $\mathrm{ch}(\mathcal{T}|_{\Delta_i}) = \exp(-f^*H/3)$. Thus, the untwisted contribution to $\mathrm{ch}_k(R\pi_*\mathcal{T})$ from the ith marked point is

$$
-\pi_* \left(\exp(-\tfrac{f^*H}{3})\ s_{i*} \left[\tfrac{\mathrm{Td}^\vee(N_i^\vee)}{c_1(N_i^\vee)}\right]_+\right)_k = -\pi_* s_{i*} \left(\exp(-\tfrac{f^*H}{3})\ \left[\tfrac{\mathrm{Td}^\vee(N_i^\vee)}{c_1(N_i^\vee)}\right]_+\right)_k.
$$

If $\sigma_i : \widetilde{\mathcal{M}}^3_{g,n}(\mathbb{P}^1, d) \to \Delta_i$ is the ith section, then we have $\sigma_{i*}\sigma_i^* = \mathrm{id}$ if the marked point is broad and $\sigma_{i*}\sigma_i^* = 3 \cdot \mathrm{id}$ if the marked point is narrow. Also, we have $f \circ \sigma_i = \mathrm{ev}_i$, and Lemma 7.3.6 of [47] shows that $\sigma_i^* N_i^\vee = L_i$. Since ev_i^* is zero away from the summand $H^*_{m_i}(\mathbb{P}^1) \otimes R$ where m_i is the multiplicity of the ith marked point, the above can be rewritten as

$$
-\frac{1}{r_i} \exp\left(-\frac{H^{(m_i)}}{3}\right) \left[\frac{\mathrm{Td}^\vee(L_i)}{\psi_i}\right]_+,
$$

where r_i is 1 if the marked point is broad and 3 if it is narrow. Note that the evaluation map in this expression has been suppressed as it will appear as an insertion in twisted invariants.

If the marked point is narrow, there are also twisted sectors, which together contribute

$$\sum_{m=1}^{2} \pi_*(\widetilde{\mathrm{ch}}(\mathcal{T})\widetilde{\mathrm{Td}}(T_\pi)|_{\mathscr{S}_i^{(m)}}) = \sum_{m=1}^{2} \pi_* s_{i*}\left(\frac{\sum_{0\le\ell\le 1} e^{2\pi i\frac{m\ell}{3}}\mathrm{ch}(\mathcal{T}^{(\ell)}|_{\Delta_i})}{1-e^{2\pi i\frac{-m}{3}}\mathrm{ch}(N_i^\vee)}\right),$$

where $\mathcal{T}^{(\ell)}$ is the subbundle of $\mathcal{T}$ in which the isotropy group acts by $e^{2\pi i\frac{\ell}{3}}$. This is either all of $\mathcal{T}$ or is rank zero, depending on whether $\ell = m_i$, so we can write the above as

$$\frac{1}{3}\exp\left(-\frac{H^{(m_i)}}{3}\right)\sum_{1\le m\le 2}\frac{e^{2\pi i\frac{mm_i}{3}}}{1-e^{2\pi i\frac{-m}{3}}e^{\psi_i}},$$

where we have used σ_i as above and again suppressed the evaluation. It is straightforward to check (see Section 7.3.5 of [47]) that for each ℓ,

$$\sum_{1\le m\le 2}\frac{\zeta^{m\ell}}{1-\zeta^{-m}e^{\psi_i}} = \frac{3e^{\ell\psi_i}}{1-e^{3\psi_i}} - \frac{1}{1-e^{\psi_i}}, \tag{4.6}$$

where $\zeta = e^{\frac{2\pi i}{3}}$. Applying this to the above twisted codimension-1 contribution and adding it to the untwisted part, we obtain

$$-\sum_{m\ge 1}\frac{\exp(-\frac{H^{(m_i)}}{3})B_m(\frac{m_i}{3})}{m!}\overline{\psi}_i^{m-1},$$

which is also the total contribution from a broad marked point. In other words, if A_m is the operator on H_{tw} given by

$$A_m = \bigoplus_{\ell=0}^{2}\exp(-\tfrac{H^{(\ell)}}{3})B_m(\tfrac{\ell}{3}),$$

then the total codimension-1 contribution to $\partial\mathcal{D}_{tw}/\partial s_k$ in either the broad or narrow case is

$$-\sum_{g,n,d}\frac{Q^d\hbar^{g-1}}{(n-1)!}\left\langle\left(\sum_{m\ge 1}\frac{A_m}{m!}\overline{\psi}^{m-1}\right)_k\mathbf{t},\ldots,\mathbf{t};c(R\pi_*\mathcal{T})\right\rangle_{g,n,d}\mathcal{D}_{tw}.$$

4.2.4 Codimension 2

The same exact proof as in [20] shows that the untwisted codimension-2 contribution to $\mathrm{ch}_k(R\pi_*\mathcal{T})$ can be expressed as

$$\frac{1}{2}\pi_*\iota_*\left(\frac{\mathrm{ch}(\mathcal{T}|_Z)}{\psi_+ + \psi_-}\left(\frac{1}{e^{\psi_+}-1} - \frac{1}{\psi_+} + \frac{1}{2} + \frac{1}{e^{\psi_-}-1} - \frac{1}{\psi_-} + \frac{1}{2}\right)\right).$$

To determine the twisted part, we must calculate the invariant and moving parts of $\iota^* T_\pi$. These can be computed by pulling back the Koszul resolution of the normal bundle of Z in $\mathscr{C}$ to the double cover Z' (Section 7.3.7 of [47]), yielding the exact sequence

$$(4.7) \qquad 0 \to L_+ \otimes L_- \to L_+ \oplus L_- \to \iota^* T_\pi \to \mathcal{O}_{Z'} \to \mathcal{O}_{Z'} \to 0.$$

Since the isotropy group acts by -1 on both L_+ and L_-, it acts trivially on their tensor product and nontrivially on their direct sum. Thus, in K-theory we have

$$\iota^* T_\pi^{\mathrm{inv}} = -(L_+ \otimes L_-)^\vee$$

and

$$\iota^* T_\pi^{\mathrm{mov}} = L_+^\vee \oplus L_-^\vee.$$

By oGRR, then, we compute the twisted codimension-2 contribution to $\mathrm{ch}_k(R\pi_*\mathcal{T})$ to be the degree-k part of the following:

$$\begin{aligned}
&\sum_{m=1}^{2} \pi_*(\widetilde{\mathrm{ch}}(\mathcal{T})\widetilde{\mathrm{Td}}(T_\pi)|_{\mathscr{Z}^{(m)}}) \\
=&\frac{1}{2}\sum_{m=1}^{2}\pi_*\iota_*\left(e^{2\pi i \frac{m m_{\mathrm{node}}}{3}} \frac{\exp(-\frac{H^{(m_{\mathrm{node}})}}{3})}{\psi_+ + \psi_-} \frac{e^{\psi_+ + \psi_-} - 1}{(1-\zeta^{-m}e^{\psi_+})(1-\zeta^m e^{\psi_+})}\right) \\
=&\frac{1}{2}\sum_{m=1}^{2}\pi_*\iota_*\left(\frac{\exp(-\frac{H^{(m_{\mathrm{node}})}}{3})}{\psi_+ + \psi_-}\left(\zeta^{m m_{\mathrm{node}}} + \frac{\zeta^{m m_{\mathrm{node}}}}{\zeta^{-m}e^{\psi_+} - 1} + \frac{\zeta^{m m_{\mathrm{node}}}}{\zeta^{m}e^{\psi_-} - 1}\right)\right)
\end{aligned}$$

Here, m_{node} is the locally constant function on Z' giving the action of the isotropy group at the node on $\mathcal{T}$. The identity (4.6) can again be applied to simplify this expression; if $m_{\text{node}} \neq 0$, then we obtain

$$\frac{1}{2}\pi_*\iota_*\left(\frac{\exp(-\frac{H^{(m_{\text{node}})}}{3})}{\psi_+ + \psi_-}\left(-1 + \frac{3e^{m_{\text{node}}}}{e^{3\psi_-}-1} - \frac{1}{\psi_+} + \frac{3e^{(3-m_{\text{node}})\psi_-}}{e^{3\psi_-}-1} - \frac{1}{\psi_-}\right)\right),$$

which when added to the untwisted codimension-2 contribution is

$$\frac{3}{2}\pi_*\iota_*\left(\frac{\exp(-\frac{H^{(m_{\text{node}})}}{3})}{\psi_+ + \psi_-}\left(\sum_{m\geq 2}\frac{B_m(\frac{m_{\text{node}}}{3})}{m!}\overline{\psi}_+^{m-1} + \frac{B_m(1-\frac{m_{\text{node}}}{3})}{m!}\overline{\psi}_-^{m-1}\right)\right).$$

In fact, the same holds, via a slightly different computation, when $m_{\text{node}} = 0$.

Adding this to the untwisted part and using the identity $B_m(1-x) = (-1)^m B_m(x)$, one finds that the codimension-2 contribution to $\partial\mathcal{D}_{tw}/\partial s_k$ is $\mathcal{D}_{tw}$ times

$$(4.8)\qquad \frac{1}{2}\sum_{g,n,d}\frac{Q^d\hbar^{g-1}}{n!}\left\langle \mathbf{t},\ldots;\pi_*\iota_*\left[\sum_{m\geq 2}\frac{3r_{\text{node}}A_m}{m!}\frac{\overline{\psi}_+^{m-1}+(-1)^m\overline{\psi}_-^{m-1}}{\overline{\psi}_+ + \overline{\psi}_-}\right]_{k-1}\right\rangle_{g,n,d}^{tw},$$

in which r_{node} is 1 if the node is broad and 3 if it is narrow.

The idea at this point is to apply the same argument as in Appendix 1 of [20] to decompose (4.8) into a sum over the moduli spaces corresponding to the two sides of the node. It is important to notice, however, that the relevant decomposition property in this setting is slightly different. Namely, if $\widetilde{D}$ denotes the locus in $\widetilde{\mathcal{M}}^3_{g,n}(\mathbb{P}^1, d)$ of curves with a separating node in which the two branches have genera g_i, n_i marked points, and degrees d_i (for $i = 1, 2$), then

$$3r_{\text{node}}\left(\frac{3}{\deg(\rho)}\int_{\widetilde{D}}\text{ev}_1^*(\phi_1)\psi_1^{a_1}\cdots\text{ev}_n^*(\phi_n)\psi_n^{a_n}c(R\pi_*\mathcal{T})\right)$$

$$= \left(\frac{3}{\deg(\rho)}\int_{\widetilde{\mathcal{M}}^3_{g_1,n_1+1}(\mathbb{P}^1,d_1)}\cdots c(R\pi_*\mathcal{T})\right)\left(\frac{3}{\deg(\rho)}\int_{\widetilde{\mathcal{M}}^3_{g_2,n_2+1}(\mathbb{P}^1,d_2)}\cdots c(R\pi_*\mathcal{T})\right),$$

where the integrands on the right-hand side depend on which marked points lie on which components in $\widetilde{D}$ and in all cases the integral is against the pullback of the virtual class under ρ. The proof of this equality is an application of the projection formula, using the fact that if $\rho_D = \rho|_{\widetilde{D}}$, then in the case where the node is narrow one has $\deg(\rho_D) = \frac{1}{3}\deg(\rho)$ due to the presence of an additional "ghost" automorphism acting locally around the node as $(x,y) \mapsto (\zeta x, y)$. An analogous computation shows the decomposition property for nonseparating nodes.

In particular, the factor of $3r_{\text{node}}$ appearing in (4.8) also appears in the decomposition property for twisted correlators, so (4.8) can be expressed as

$$\frac{1}{2}\sum_{\substack{g_1,g_2\\n_1,n_2\\d_1,d_2}} \frac{Q^{d_1+d_2}\hbar^{g_1+g_2-1}}{n_1!n_2!} \sum_{r,s,\alpha,\beta} \left\langle \mathbf{t},\ldots,\mathbf{t}, q_r^\alpha \phi_\alpha \overline{\psi}_+^r ; c(R\pi_*\mathcal{T}) \right\rangle_{g_1,n_1+1,d_1} \times$$
$$\left\langle q_s^\beta \phi_\beta \overline{\psi}_-^s, \mathbf{t},\ldots,\mathbf{t}; c(R\pi_*\mathcal{T}) \right\rangle_{g_2,n_2+1,d_2} \mathcal{D}_{tw}$$
$$+\frac{1}{2}\sum_{g,n,d} \frac{Q^d\hbar^{g-1}}{n!} \sum_{r,s,\alpha,\beta} \left\langle \mathbf{t},\ldots,\mathbf{t}, q_r^\alpha \phi_\alpha \overline{\psi}_+^r, q_s^\beta \phi_\beta \overline{\psi}_-^s ; c(R\pi_*\mathcal{T}) \right\rangle_{g-1,n,d} \mathcal{D}_{tw},$$

where the q's are determined by the requirement that $\sum_{r,s,\alpha,\beta} q_r^\alpha \phi_\alpha \overline{\psi}_+^r \otimes q_s^\beta \phi_\beta \overline{\psi}_-^s$ equals

$$\left(\sum_{m\geq 2} \frac{A_m}{m!} \frac{\overline{\psi}_+^{m-1} + (-1)^m \overline{\psi}_-^{m-1}}{\overline{\psi}_+ + \overline{\psi}_-} \right)_{k-1} \wedge (g^{\alpha\beta}\phi_\alpha \otimes \phi_\beta)$$

and $g^{\alpha\beta}$ is the inverse of the matrix for the twisted Poincaré pairing.

By Appendix C of [47], this equals

$$\frac{\hbar}{2}(\partial \otimes_{C_k} \partial)\mathcal{D}_{tw}$$

for

$$C_k = \bigoplus_{\ell=0}^{2} \sum_{m\geq 1} \frac{B_m(\frac{\ell}{3})}{m!} \exp(-\tfrac{H^{(\ell)}}{3})_{k+1-m} z^{m-1}.$$

4.2.5 Putting everything together

The sum of the codimension-1 and nonexceptional codimension-0 contributions is

$$\sum_{g,n,d} \frac{Q^d \hbar^{g-1}}{(n-1)!} \langle \mathbf{t}, \dots, \mathbf{t}, (\exp(\tfrac{\overline{\psi}_n}{3}) \exp(-\tfrac{H}{3}) \mathrm{Td}^\vee(L_n))_{k+1}; c(R\pi_* \mathcal{T}) \rangle^\bullet_{g,n,d} \mathcal{D}_{tw}$$

$$(4.9) \qquad - \sum_{g,n,d} \frac{Q^d \hbar^{g-1}}{(n-1)!} \langle \mathbf{t}, \dots, \mathbf{t}, \exp(-\tfrac{H}{3})_{k+1} \left[\tfrac{\mathbf{t}(\overline{\psi})}{\overline{\psi}}\right]_+ ; c(R\pi_* \mathcal{T}) \rangle_{g,n,d} \mathcal{D}_{tw}$$

$$- \sum_{g,n,d} \frac{Q^d \hbar^{g-1}}{(n-1)!} \left\langle \left(\sum_{m \geq 1} \frac{A_m}{m!} \, \overline{\psi}^{m-1} \right)_k \mathbf{t}, \dots, \mathbf{t}; c(R\pi_* \mathcal{T}) \right\rangle_{g,n,d} \mathcal{D}_{tw}.$$

Using that

$$\exp(-\tfrac{H}{3})_{k+1} \left[\tfrac{\mathbf{t}(\overline{\psi})}{\overline{\psi}}\right]_+ = \bigoplus_{\ell=0}^{2} \exp(-\tfrac{H^{(\ell)}}{3})_{k+1} \left(\frac{\mathbf{t}(\overline{\psi}) - t_0}{\overline{\psi}} \right)$$

and

$$\sum_{m \geq 1} \frac{A_m}{m!} z^{m-1} = \bigoplus_{\ell=0}^{2} \exp(-\tfrac{H^{(\ell)}}{3}) \left(\frac{e^{\frac{\ell}{3} z}}{e^z - 1} - \frac{1}{z} \right),$$

we find that the sum of the second two terms in (4.9) is $\mathcal{D}_{tw}$ times

$$- \sum_{g,n,d} \frac{Q^d \hbar^{g-1}}{(n-1)!} \left\langle \left[\left(\frac{\sum_{0 \leq \ell \leq 2} \exp(-\frac{H^{(\ell)}}{3}) e^{\frac{\ell}{3}\overline{\psi}}}{e^{\overline{\psi}} - 1} \right)_k \mathbf{t}(\overline{\psi}) \right]_+ , \dots ; c(R\pi_* \mathcal{T}) \right\rangle_{g,n,d}.$$

Also, keeping in mind that $1 \in H_1^*(\mathbb{P}^1)$, we find that the contribution from the remaining codimension-0 nonexceptional term is equal to

$$\begin{aligned}
(\exp(\overline{\psi}/3) \exp(-H/3) \mathrm{Td}^\vee(\overline{L_n}))_{k+1} &= \left(\frac{\exp(-\frac{H}{3}) e^{\frac{1}{3}\overline{\psi}}}{e^{\overline{\psi}} - 1} \overline{\psi} \right)_{k+1} \\
&= \left[\left(\frac{\exp(-\frac{H}{3}) e^{\frac{1}{3}\overline{\psi}}}{e^{\overline{\psi}} - 1} \right)_k 1\overline{\psi} \right]_+ \\
&= \left[\left(\frac{\sum_{0 \leq \ell \leq 2} \exp(-\frac{H^{(\ell)}}{3}) e^{\frac{\ell}{3}\overline{\psi}}}{e^{\overline{\psi}} - 1} \right)_k 1\overline{\psi} \right]_+ .
\end{aligned}$$

Therefore, the sum of the codimension-1 and nonexceptional codimension-0 terms is $\mathcal{D}_{tw}$ times

$$-\sum_{g,n,d}\frac{Q^d\hbar^{g-1}}{(n-1)!}\left\langle\left[\left(\frac{\sum_{0\leq\ell\leq 2}\exp(-\frac{H^{(\ell)}}{3})e^{\frac{\ell}{3}\overline{\psi}}}{e^{\overline{\psi}}-1}\right)_k\mathbf{q}(\overline{\psi})\right]_+,\ldots;c(R\pi_*\mathcal{T})\right\rangle_{g,n,d}$$

and the computation in Example 3.3 of [18] (reproducing a previous computation of Coates-Givental) shows that this equals $-\partial_{C_k}\mathcal{D}_{tw}$ for C_k as above.

Combining everything and using the explicit description of quantized operators that can be found, for example, in Section 3.2 of [18], we have proved that

$$\frac{\partial\mathcal{D}_{tw}}{\partial s_k}=\frac{1}{2\hbar}\Omega_{tw}((C_k\mathbf{q})(-z),\mathbf{q}(z))-\partial_{C_k}\mathcal{D}_{tw}+\frac{\hbar}{2}(\partial\otimes_{C_k}\partial)\mathcal{D}_{tw}=\widehat{C_k}\mathcal{D}_{tw},$$

which is part (a) of the proposition.

The proof of part (b) is nearly identical and somewhat simpler, since there is only one nontrivial twisted sector associated to each marked point and to the divisor of nodes, so we omit it. □

4.3 LG and GW I-functions

As in [11], [19], and [20], one can define a certain hypergeometric modification I_{tw} of the untwisted J-function in such a way that the family $\Delta^{-1}I_{tw}(t,-z)$ lies on the untwisted Lagrangian cone $\mathcal{L}_{un}$; in light of the above, it follows that $I_{tw}(t,-z)\in\mathcal{L}_{tw}$. When we take a nonequivariant limit $\lambda\to 0$ and set the parameters s_k as in (4.2), we will obtain a family lying on $\mathcal{L}_{hyb}$, and in fact, this family will determine the entire cone just as the hybrid J-function does.

As usual, we will define I_{tw} only in the case of the cubic singularities, commenting only briefly on how to apply the same procedure to define I_{tw} in the other case.

4.3.1 Setup in cubic case

First, decompose J_{un} according to topological types, as in [19]. The *topological type* of an element of some $\widetilde{\mathcal{M}}^3_{g,n}(\mathbb{P}^1, d)$ is the triple $\Theta = (g, d, \mathbf{i})$, where g is the genus of the source curve, d is the degree of the map, and $\mathbf{i} = (i_1, \ldots, i_n)$ gives the multiplicities of the line bundle at each of the marked points. Let J_Θ be the contribution to J_{un} from invariants of topological type Θ, and write

$$J_{un}(t,z) = \sum_{\Theta} J_\Theta(t,z),$$

where the sum is over all topological types.[4]

Let us also fix some notation, again mimicking [19]. Set

$$\mathbf{s}(x) = \sum_{k\geq 0} s_k \frac{x^k}{k!}$$

for any $x \in \mathcal{V}^+_{tw}$. For a multiplicity $h \in \{0, 1, 2\}$, let

$$D_{(h)} = \sum_{\alpha=0}^{1} t^\alpha_{0,(h)} \frac{\partial}{\partial t^\alpha_{0,(h)}}$$

denote the dilation vector field on $H^*_h(\mathbb{P}^1)$, where for $\mathbf{t} = t_0 + t_1 z + t_2 z^2 + \cdots \in H_{GW}[[z]]$ we write

$$t_i = \sum_{\substack{0\leq\alpha\leq 1\\ 0\leq h\leq 2}} t^\alpha_{i,(h)} \phi_i^{(h)}.$$

with $\{\phi_i^{(h)}\}$ denoting a basis for $H^*_h(\mathbb{P}^1)$. Also, set

$$G_y(x,z) = \sum_{k,m\geq 0} s_{k+m-1} \frac{B_m(y)}{m!} \frac{x^k}{k!} z^{m-1}$$

for $y \in \mathbb{Q}$ and $x \in H_{tw}$, where z denotes the variable in $\mathcal{V}_{tw}$, as usual.

For each topological type Θ, let $\overline{i_n}$ be the multiplicity that is equal modulo 3 to $-i_n$. Set

$$N_\Theta = \frac{-2 + n - d - \sum_{j=1}^{n-1} i_j}{3} + \frac{\overline{i_n}}{3}.$$

[4] The $z + t$ term in $J_{un}(t,z)$ should be understood as contributing to the unstable topological types corresponding to $(g, n, d) = (0, 1, 0)$ and $(0, 2, 0)$.

Note that this is an integer, since it equals either

$$\frac{-2+n-d-\sum_{j=1}^{n} i_j}{3} = \deg(|L|)$$

or $\deg(|L|)+1$ depending on whether i_n is zero or nonzero. Thus, we can set

$$M_\Theta = \frac{\prod_{-\infty<m\leq N_\Theta} \exp\left(\mathbf{s}(-\frac{H^{(\overline{i_n})}}{3} + (m - \frac{\overline{i_n}}{3})z)\right)}{\prod_{-\infty<m\leq 0} \exp\left(\mathbf{s}(-\frac{H^{(\overline{i_n})}}{3} + (m - \frac{\overline{i_n}}{3})z)\right)}$$

Note that these definitions of N_Θ and M_Θ are direct generalizations of those appearing in [19], and the same proof shows that the properties in Lemma 4.5 and equations (12) and (13) of that paper still hold.

4.3.2 Quadric case

The definitions of $\mathbf{s}(x)$ and $G_y(x,z)$ remain unchanged in the case of the quadric singularities, while the dilation vector field on $H_h^*(\mathbb{P}^3)$ changes only in that the summation runs over a basis for $H^*(\mathbb{P}^3)$, so $0 \leq \alpha \leq 3$. As for N_Θ, we should now take $\overline{i_n}$ to be equal to $-i_n$ modulo 2, which is the same as setting $\overline{i_n} = i_n$. The resulting definition is:

$$N_\Theta = \frac{-2+n-d-\sum_{j=1}^{n-1} i_j}{2} + \frac{i_n}{2}.$$

Similarly,

$$M_\Theta = \frac{\prod_{-\infty<m\leq N_\Theta} \exp\left(\mathbf{s}(-\frac{H^{(i_n)}}{2} + (m - \frac{i_n}{2})z)\right)}{\prod_{-\infty<m\leq 0} \exp\left(\mathbf{s}(-\frac{H^{(i_n)}}{2} + (m - \frac{i_n}{2})z)\right)}$$

Once again, the necessary properties of these expressions follow direct from the analogues in [19].

4.3.3 Twisted I-function

In either of the two cases under consideration, define

$$I_{tw}(\mathbf{t}, z) = \sum_{\Theta} M_{\Theta}(z)\; J_{\Theta}(\mathbf{t}, z).$$

The hybrid I-function will be defined by putting s_k to the values in (4.2), taking $\lambda \to 0$, specializing to multiplicity-1 divisor insertions with no ψ classes, and multiplying by a factor.

Theorem IV.1. *(a) For the cubic singularity, define*

$$I_{hyb}(t, z) = \sum_{\substack{d \geq 0 \\ d \not\equiv -1 \mod 3}} \frac{z e^{(d+1+\frac{H^{(d+1)}}{z})t}}{3^{6\lfloor \frac{d}{3} \rfloor}} \frac{\prod\limits_{\substack{1 \leq b \leq d \\ b \equiv d+1 \mod 3}} (H^{(d+1)} + bz)^4}{\prod\limits_{\substack{1 \leq b \leq d \\ b \not\equiv d+1 \mod 3}} (H^{(d+1)} + bz)^2},$$

where $t = t + 0z + 0z^2 + \cdots \in \mathcal{V}^+_{hyb}$ and $t \in H^2_1(\mathbb{P}^1)$. Then the family $I_{hyb}(t, -z)$ of elements of $\mathcal{V}_{hyb}$ lies on the Lagrangian cone $\mathcal{L}_{hyb}$.

(b) For the quadric singularity, define

$$I_{hyb}(t, z) = \sum_{\substack{d \geq 0 \\ d \not\equiv -1 \mod 2}} \frac{z e^{(d+1+\frac{H^{(d+1)}}{z})t}}{2^{8\lfloor \frac{d}{2} \rfloor}} \frac{\prod\limits_{\substack{1 \leq b \leq d \\ b \equiv d+1 \mod 2}} (H^{(d+1)} + bz)^4}{\prod\limits_{\substack{1 \leq b \leq d \\ b \not\equiv d+1 \mod 2}} (H^{(d+1)} + bz)^4},$$

where $t \in H^2_1(\mathbb{P}^3)$. Then the family $I_{hyb}(t, -z)$ of elements of $\mathcal{V}_{hyb}$ lies on the Lagrangian cone $\mathcal{L}_{hyb}$.

Remark 4.3.4. These I-functions have expressions in terms of the Γ function, which can be useful for computations– see Section 4.4.

Proof. The proof mimics that of Theorem 4.6 of [19]. We will begin by proving that $I_{tw}(\mathbf{t}, -z)$ lies on $\mathcal{L}_{tw}$ for the cubic singularity, and then show how to obtain I_{hyb}

from I_{tw}. As usual, everything we say will carry over to the quadric case with only minor modifications, so we omit the proof.

Using equations (12) and (13) of [19], it is easy to check that

$$\begin{aligned} M_\Theta(-z) &= \exp\left(G_0\left(-\tfrac{H^{(\overline{i_n})}}{3}+\tfrac{\overline{i_n}}{3}z,z\right)-G_0\left(-\tfrac{H^{(\overline{i_n})}}{3}+(\tfrac{\overline{i_n}}{3}-N_\Theta)z,z\right)\right)\\ &= \exp\left(G_{\frac{\overline{i_n}}{3}}\left(-\tfrac{H^{(\overline{i_n})}}{3},z\right)-G_0\left(-\tfrac{H^{(\overline{i_n})}}{3}+(\tfrac{\overline{i_n}}{3}-N_\Theta)z,z\right)\right). \end{aligned}$$

Furthermore,

$$\Delta=\bigoplus_{\ell=0}^{3}\exp\left(G_{\frac{\ell}{3}}\left(-\tfrac{H^{(\ell)}}{3},z\right)\right).$$

Given that $\Delta(\mathcal{L}_{un})=\mathcal{L}_{tw}$, the desired statement is equivalent to the statement $\Delta^{-1}I_{tw}(\mathbf{t},-z)\in\mathcal{L}_{un}$. Using Lemma 4.5(1) of [19] and the above expression for $M_\Theta(-z)$, this is equivalent to

$$\sum_\Theta \exp\left(-G_{\frac{1}{3}}\left(-\tfrac{H^{(\overline{i_n})}}{3}+\tfrac{d}{3}z-\tfrac{\sum_{j=1}^{n-1}(1-i_j)}{3}z,x\right)\right)J_\Theta(\mathbf{t},-z)\in\mathcal{L}_{un}.$$

Now, we can write

$$\frac{\sum_{j=1}^{n-1}(1-i_j)}{3}=\frac{n_0}{3}-\frac{n_2}{3},$$

and Lemma 4.5(2) of [19] says that n_0 and n_2 act on J_Θ in the same way, respectively, as $D_{(0)}$ and $D_{(2)}$. Furthermore, $-\frac{H^{(\overline{i_n})}}{3}+\frac{d}{3}z$ acts on J_Θ in the same way as does $-z\nabla_{-\frac{H^{(\overline{i_n})}}{3}}$. So if $D=\frac{1}{3}D_{(0)}-\frac{1}{3}D_{(2)}$, we can re-express the desired statement as

$$\exp\left(-G_{\frac{1}{3}}\left(z\nabla_{-\frac{H}{3}}-zD,z\right)\right)J_{un}(\mathbf{t},z)\in\mathcal{L}_{un}, \tag{4.10}$$

where $H=H^{(0)}+H^{(1)}+H^{(2)}$.

Denote the expression in (4.10) by $J_\mathbf{s}(\mathbf{t},-z)$. To prove that $J_\mathbf{s}(\mathbf{t},-z)\in\mathcal{L}_{un}$ is to show that

$$E_j(J_\mathbf{s}(\mathbf{t},-z))=0$$

for all j, where E_j are the functions $\mathcal{V}_{un} \to H_{un}$ given by

$$(\mathbf{p}, \mathbf{q}) \mapsto p_j - \sum_{n,d,\alpha,h} \frac{Q^d}{n!} \langle \mathbf{t}, \ldots, \mathbf{t}, \psi^j \phi_\alpha^{(h)} \rangle_{g,n+1,d}^{un} \phi^{\alpha,(h)}.$$

This is proved exactly as in [19]— namely, by induction on the degree of the terms in $E_j(J_\mathbf{s})(\mathbf{t}, -z)$ with respect to the variables s_k under the convention that s_k has degree $k+1$.

The terms of degree 0 vanish, as such terms are constant with respect to the s_k and vanish when all s_k are 0 because $J_\mathbf{0} = J_{un}$. Assume, then, that $E_j(J_\mathbf{s}(\mathbf{t}, -z))$ vanishes up to degree n in the variables s_k. To show that it vanishes up to degree $n+1$, we will show that $\frac{\partial}{\partial s_i} E_j(J_\mathbf{s}(\mathbf{t}, -z))$ vanishes up to degree n for all i. We have

$$\frac{\partial}{\partial s_i} E_j(J_\mathbf{s}(\mathbf{t}, -z)) = d_{J_\mathbf{s}(\mathbf{t}, -z)} E_j(z^{-1} P_i J_\mathbf{s}(\mathbf{t}, -z)),$$

where

$$P_i = -\sum_{m=0}^{i+1} \frac{1}{m!(i+1-m)!} z^m B_m(0)(-z\Delta_{-\frac{H}{3}} - zD)^{i+1-m}.$$

The inductive hypothesis implies the existence of an element $\widetilde{J}_\mathbf{s}(\mathbf{t}, -z) \in \mathcal{L}_{un}$ that agrees with $J_\mathbf{s}(\mathbf{t}, -z)$ up to degree n in the s_k, and hence satisfies

$$\frac{\partial}{\partial s_i} E_j J_\mathbf{s}(\mathbf{t}, -z) = d_{\widetilde{J}_s(\mathbf{t}, -z)} E_j(z^{-1} P_i \widetilde{J}_\mathbf{s}(\mathbf{t}, -z))$$

up to degree n in these variables. It suffices, then, to show that the right-hand side of this equation is identically zero, or in other words that

$$P_i \widetilde{J}_\mathbf{s}(\mathbf{t}, -z) \in z T_{\widetilde{J}_\mathbf{s}(\mathbf{t}, -z)} \mathcal{L}_{un} = \mathcal{L}_{un} \cap T_{\widetilde{J}_\mathbf{s}(\mathbf{t}, -z)} \mathcal{L}_{un}.$$

Let $T = T_{\widetilde{J}_\mathbf{s}(\mathbf{t}, -z)} \mathcal{L}_{un}$. Breaking P_i up into a sum of terms of the form

$$Cz^a (z\nabla_{-\frac{H}{3}})^b (zD)^c$$

for a coefficient C and exponents a, b, and c, it suffices to show that z, $z\nabla_{-\frac{H}{3}}$, and zD all preserve zT. In the first case, this is because $zT = \mathcal{L}_{un} \cap T \subset T$ and hence

$z(zT) \subset zT$. In the second case, the operator $\nabla_{-\frac{H}{3}}$ is a first-order derivative and hence takes $\mathcal{L}_{un}$ to T; it follows that $\nabla_{-\frac{H}{3}}$ takes $zT = \mathcal{L}_{un} \cap T \subset \mathcal{L}_{un}$ to T also, and hence $z\nabla_{-\frac{H}{3}}$ takes zT to zT. The same argument applies to the operator zD, so this completes the proof that $I_{tw}(\mathbf{t}, -z) \in \mathcal{L}_{tw}$ in the cubic case.

Now suppose we set s_k as in (4.2), so that $c(-V) = e_{\mathbb{C}^*}(V^\vee)^6$, and take a limit $\lambda \to 0$. It is easy to check via the Taylor expansion of the natural logarithm that in the cubic case, we get

$$M_\Theta(z) = \frac{\prod_{-\infty < m \leq 0} \left(\frac{H^{(\overline{i_n})}}{3} + (\frac{\overline{i_n}}{3} - m)z\right)^6}{\prod_{-\infty < m \leq N_\Theta} \left(\frac{H^{(\overline{i_n})}}{3} + (\frac{\overline{i_n}}{3} - m)z\right)^6}.$$

Restrict $\mathbf{t}$ to allow only those insertions in $H^2_1(\mathbb{P}^1)$ with no ψ classes; in this case,

$$N_\Theta = \frac{-d-1}{3} + \frac{\overline{i_n}}{3},$$

which is always nonpositive, and $\overline{i_n} \equiv d+1 \mod 3$. Thus, we obtain

$$M_\Theta(z) = \prod_{\substack{0 \leq b < \frac{d+1}{3} \\ \{b\} = \{\frac{d+1}{3}\}}} \left(\frac{H^{(d+1)}}{3} + bz\right)^6,$$

where we use the convention $H^{(h)} = H^{(h \mod 3)}$ if $h \geq 3$. Notice that if $d+1 \equiv 0 \mod 3$, then one of the factors in the above product is $b = 0$, in which case the product is 0 because $H^2 = 0$. Thus, $M_\Theta(z)$ vanishes in these cases.[5]

Set $t = t_0 + 0z + 0z^2 + \cdots$. Since untwisted invariants are essentially Gromov-Witten invariants of $\mathbb{P}^1$, we can compute $J_\Theta(t, z)$ explicitly in every case where Θ corresponds to a nonempty component of the moduli space. Indeed, Givental's Mirror Theorem for $\mathbb{P}^1$ states that

$$1 + \sum_{d,\alpha} Q^d \left\langle \frac{\phi_\alpha}{z - \psi}, 1 \right\rangle_{0,d} \phi^\alpha = \sum_d Q^d \frac{1}{((H+z)\cdots(H+dz))^2}.$$

[5]In fact, we already knew that this had to be the case, because the fact that $I_{tw}(t, -z) \in \mathcal{L}_{tw}$ implies that $I_{tw}(t, z)$ differs from the small hybrid J-function by a change of variables, and the hybrid invariants vanish if any insertion is broad.

Using the string and divisor equations, then, one can show that

$$\sum_{\Theta \text{ with degree } d} J_\Theta = \frac{3ze^{(\frac{H^{(d+1)}}{z}+d)t}}{((H^{(d+1)}+z)\cdots(H^{(d+1)}+dz))^2}.$$

Since all Θ with the same degree yield the same M_Θ, namely

$$M_\Theta = \frac{1}{3^{6\lfloor \frac{d}{3} \rfloor}} \prod_{\substack{1\le b\le d \\ b\equiv d+1 \mod 3}} (H^{(d+1)}+bz)^6,$$

we obtain a formula for $I_{tw}(t,z)$. Writing $\Theta = (0, d, (1, \ldots, 1))$ (with k 1's) and taking $Q \to 1$ as is done in the Gromov-Witten setting, the formula is

$$I_{tw}(t,z) = \sum_{\substack{d\ge 0 \\ d\not\equiv -1 \mod 3}} \frac{3ze^{(\frac{H^{(d+1)}}{z}+d)t}}{3^{6\lfloor \frac{d}{3} \rfloor}} \frac{\prod_{\substack{1\le b\le d \\ b\equiv d+1 \mod 3}} (H^{(d+1)}+bz)^4}{\prod_{\substack{1\le b\le d \\ b\not\equiv d+1 \mod \delta}} (H^{(d+1)}+bz)^2}.$$

Multiplying by $\frac{1}{3}e^t$, which preserves $\mathcal{L}_{hyb}$ because it is a cone, gives the function I_{hyb} of the statement. An analogous computation shows that the hybrid I-function in the quadric case is as stated. □

Equipped with an explicit expression for the hybrid I-functions and having proved that they lie on the respective Lagrangian cones $\mathcal{L}_{tw}$, we are finally ready to prove the main theorem of the thesis:

Proof of Theorem I.1. We have shown that $I_{hyb}(t,-z)$ lies on the Lagrangian cone $\mathcal{L}_{hyb}$. The property (4.1) implies that the J-function is characterized by the fact that $J_{hyb}(t,-z) \in \mathcal{L}_{hyb}$ together with the first two terms of its expansion in powers of z:

$$J_{hyb}(t,-z) = -1^{(1)}z + t + O(z^{-1}).$$

Using the formula for $I_{hyb}(t,z)$, it is not difficult to show that it can be expressed as

$$I_{hyb}(t,z) = \omega_1^{hyb}(t) \cdot 1^{(1)} \cdot z + \omega_2^{hyb}(t) + O(z^{-1})$$

for $\mathbb{C}$-valued functions ω_1^{hyb} and ω_2^{hyb}. These can be calculated explicitly, but the computation is tedious and not strictly necessary to prove the LG/CY correspondence, so we relegate it to a separate section (Section 4.4).

Having obtained such functions ω_i^{hyb}, we have

$$\frac{I_{hyb}(t,-z)}{\omega_1^{hyb}(t)} = -1^{(1)} \cdot z + \frac{\omega_2^{hyb}(t)}{\omega_1^{hyb}(t)} + O(z^{-1}),$$

and this still lies on $\mathcal{L}_{hyb}$ because it is a cone. So by the uniqueness property of J_{hyb}, we have

$$\frac{I_{hyb}(t,-z)}{\omega_1^{hyb}(t)} = J_{hyb}(t',-z), \text{ where } t' = \frac{\omega_2^{hyb}(t)}{\omega_1^{hyb}(t)}. \tag{4.11}$$

This is the change of variables relating the hybrid I-function and J-function.

As for the symplectic transformation matching I_{hyb} with the analytic continuation of I_{GW}, the comments in the Introduction show that it is sufficient to prove that the hybrid I-function assembles the solutions to the Picard-Fuchs equation

$$\left[\left(\psi\frac{\partial}{\partial\psi}\right)^4 - \psi^{-1}\left(\psi\frac{\partial}{\partial\psi} - \frac{1}{3}\right)^2\left(\psi\frac{\partial}{\partial\psi} - \frac{2}{3}\right)^2\right] F = 0 \tag{4.12}$$

for the cubic singularity, where $\psi = e^{3t}$, or

$$\left[\left(\psi\frac{\partial}{\partial\psi}\right)^4 - \psi^{-1}\left(\psi\frac{\partial}{\partial\psi} - \frac{1}{2}\right)^4\right] F = 0$$

for the quadric singularity, where $\psi = e^{2t}$. As usual, we prove only the first of these statements.

Split I_{hyb} into two parts corresponding to the two narrow summands of H_{tw}, changing the variable of summation in each:

$$I_{hyb}(t,z) = \sum_{d\geq 0} \frac{ze^{(3d+1+\frac{H^{(1)}}{z})t}}{3^{6d}} \frac{\prod_{\substack{1\leq b\leq 3d \\ b\equiv 1 \mod 3}} (H^{(1)}+bz)^4}{\prod_{\substack{1\leq b\leq 3d \\ b\equiv 0,2 \mod 3}} (H^{(1)}+bz)^2}$$

$$+ \sum_{d\geq 0} \frac{ze^{(3d+2+\frac{H^{(2)}}{z})t}}{3^{6d}} \frac{\prod_{\substack{1\leq b\leq 3d+1 \\ b\equiv 2 \mod 3}} (H^{(2)}+bz)^4}{\prod_{\substack{1\leq b\leq 3d+1 \\ b\equiv 0,1 \mod 3}} (H^{(2)}+bz)^2}.$$

The claim is that, when we set $\psi = e^{3t}$, each of these summands separately satisfies (4.12) as a cohomology-valued function. For the first summand, let Ψ_d be the contribution from d:

$$\Psi_d = z\frac{\psi^{d+\frac{1}{3}+\frac{H^{(1)}}{3z}}}{3^{6d}} \frac{\prod_{\substack{1\leq b\leq 3d \\ b\equiv 1 \mod 3}} (H^{(1)}+bz)^4}{\prod_{\substack{1\leq b\leq 3d \\ b\equiv 0,2 \mod 3}} (H^{(1)}+bz)^2}.$$

By computing the ratio Ψ_d/Ψ_{d-1}, it is easy to check that

$$\left(\tfrac{H^{(d+1)}}{3z}+d-\tfrac{2}{3}\right)^4 \Psi_{d-1} = 3^6\psi^{-1}\left(\tfrac{H^{(d+1)}}{3z}+d\right)^2\left(\tfrac{H^{(d+1)}}{3z}+d-\tfrac{1}{3}\right)^2\Psi_d.$$

But the operator $\psi\frac{\partial}{\partial\psi}$ acts on Ψ_d by multiplication by $\left(\frac{H^{(d+1)}}{3z}+d+\frac{1}{3}\right)$, so the above can be expressed as

$$\left(\psi\frac{\partial}{\partial\psi}\right)^4\Psi_{d-1} = 3^6\psi^{-1}\left(\psi\frac{\partial}{\partial\psi}-\frac{1}{3}\right)^2\left(\psi\frac{\partial}{\partial\psi}-\frac{2}{3}\right)^2\Psi_d.$$

It follows that if one applies the Picard-Fuchs operator in (4.12) to the first summand of $I_{hyb}(t,z)$, all but possibly the Ψ_0 summand will be annihilated. In fact, though, it is easy to see using the fact that $H^2 = 0$ that Ψ_0 is also killed. Thus, the Picard-Fuchs equation holds for this summand, and an analogous argument proves the same claim for the second summand. □

4.4 Explicit mirror map

In order to explicitly compute the change of variables (4.11), it is necessary to find the coefficients of z^1 and z^0 in $I_{hyb}(t,z)$. Let us do this first in the cubic case.

Using the identity

$$z^{\ell}\frac{\Gamma(1+\frac{x}{z}+\ell)}{\Gamma(1+\frac{x}{z})}=\prod_{k=1}^{\ell}(x+kz),$$

one can rewrite I_{hyb} as

$$z\sum_{\substack{d\geq 0\\ d\equiv -1 \bmod 3}} e^{(d+1+\frac{H^{(d+1)}}{z})t}z^{-6\langle\frac{d}{3}\rangle}\frac{\Gamma(\frac{H^{(d+1)}}{3z}+\frac{d}{3}+\frac{1}{3})^6}{\Gamma(\frac{H^{(d+1)}}{3z}+\langle\frac{d}{3}\rangle+\frac{1}{3})^6}\frac{\Gamma(\frac{H^{(d+1)}}{z}+1)^2}{\Gamma(\frac{H^{(d+1)}}{z}+d+1)^2}.$$

It is easy to see from here that the only terms that contribute to the coefficient of either z^1 or z^0 are those with $d\equiv 0 \mod 3$. In particular, if we expand the function

$$F(\eta)=\sum_{\substack{d\geq 0\\ d\equiv 0 \bmod 3}} e^{(d+1+\eta)t}\frac{\Gamma(\frac{\eta}{3}+\frac{d}{3}+\frac{1}{3})^6}{\Gamma(\frac{\eta}{3}+\langle\frac{d}{3}\rangle+\frac{1}{3})^6}\frac{\Gamma(\eta+1)^2}{\Gamma(\eta+d+1)^2}$$

in powers of η, then

$$\omega_1^{hyb}(t)=F(0)=\sum_{d\geq 0}e^{(3d+1)t}\frac{\Gamma(d+\frac{1}{3})^6}{\Gamma(\frac{1}{3})^6\Gamma(3d+1)^2}$$

and

$$\omega_2^{hyb}(t)=F'(0)=\sum_{d\geq 0}e^{(3d+1)t}\frac{\Gamma(d+\frac{1}{3})^5}{\Gamma(\frac{1}{3})^6\Gamma(3d+1)^2}\Big(2\Gamma'(d+\tfrac{1}{3})+2\Gamma(d+\tfrac{1}{3})\psi(1)$$
$$-2\Gamma(d+\tfrac{1}{3})\psi(\tfrac{1}{3})-2\Gamma(d+\tfrac{1}{3})\psi(3d+1)+t\Gamma(d+\tfrac{1}{3})\Big),$$

where ψ is the digamma function, the logarithmic derivative of Γ.

The same argument shows that in the quadric case, one has

$$\omega_1^{hyb}(t)=G(0)=\sum_{d\geq 0}e^{(d+1)t}\frac{(2d)!^8(2d+1)!^4}{4^{8d}d!^8}$$

and

$$\omega_2^{hyb}(t) = G'(0) = \sum_{d\geq 0} e^{(d+1)t} \frac{\Gamma(d+\frac{1}{2})^6}{\Gamma(\frac{1}{2})^8\Gamma(2d+1)^4}\Big(4\Gamma'(d+\tfrac{1}{2}) + 4\Gamma(d+\tfrac{1}{2})\psi(1) - 4\Gamma(d+\tfrac{1}{2})\psi(\tfrac{1}{2}) - 4\Gamma(d+\tfrac{1}{2})\psi(2d+1) + t\Gamma(d+\tfrac{1}{2})\Big),$$

where

$$G(\eta) = \sum_{\substack{d\geq 0\\ d\equiv 0 \bmod 2}} e^{(d+1+\eta)t} \frac{\Gamma(\frac{\eta}{2}+\frac{d}{2}+\frac{1}{2})^8}{\Gamma(\frac{\eta}{2}+\frac{1}{2})^8}\frac{\Gamma(\eta+1)^4}{\Gamma(\eta+d+1)^4}.$$

4.5 The genus-zero correspondence

It follows from Theorem I.1 that I_{hyb} and the analytic continuation of I_{GW} to the ψ-coordinate patch are both comprised of bases of solutions to the same differential equation, so they are related by a linear transformation $\mathbb{U}$ performing the change of basis. From here, the genus-zero LG/CY correspondence as stated in Corollary 1.5.1 needs just one more observation: all of the correlators defining the Lagrangian cone $\mathcal{L}_{hyb}$ are determined via the string equation, dilaton equation, and dimension constraints from the correlators appearing in the small J-function. This relies on the Calabi-Yau condition and the resulting simplification of the virtual dimension formula.

For example, in the quartic case, one must have $m_1 = \ldots = m_n = 1$ to obtain nonzero correlators, in which case

$$\mathrm{vdim}(\widetilde{\mathcal{M}}^2_{0,(1,\ldots,1)}(\mathbb{P}^3, d)) = n$$

by formula (3.7). Thus, a correlator $\langle \tau_{l_1}(\alpha_1)\cdots\tau_{l_n}(\alpha_n)\rangle^{hyb}_{0,n,d}$ can only be nonzero if $\deg_{\mathbb{C}}(\alpha_i) + l_i = 1$ for some i. If $\deg_{\mathbb{C}}(\alpha_i) = l_i = 0$, then one can apply the string equation to reduce the number of marked points. If $\deg_{\mathbb{C}}(\alpha_i) = 0$ and $l_i = 1$, then the dilaton equation will also reduce the number of marked points. The only remaining possibility is that $\deg_{\mathbb{C}}(\alpha_i) = 1$ and $l_i = 0$, so α_i comes from the (real) degree-two part of the state space. By repeatedly applying this argument, one can ensure that all but at most one of the insertions has degree two with no ψ classes, which is precisely the type of correlator appearing in the small J-function.

Thus, the entire Lagrangian cone $\mathcal{L}_{hyb}$ is determined by the small J-function, so Theorem I.1 implies that it is determined by the I-function. An analogous computation, using the fact that $\text{vdim}(\overline{\mathcal{M}}_{0,n}(X_{\overline{W}}, d)) = n$ for a Calabi-Yau complete intersection $X_{\overline{W}}$, shows that $\mathcal{L}_{GW}$ is determined by I_{GW}. Since $\mathbb{U}$ takes I_{hyb} to the analytic continuation of I_{GW}, it follows that $\mathbb{U}$ takes $\mathcal{L}_{hyb}$ to the analytic continuation of the Lagrangian cone $\mathcal{L}_{GW}$. This proves Corollary 1.5.1, establishing the genus-zero LG/CY correspondence.

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ABSTRACT

The Landau-Ginzburg/Calabi-Yau Correspondence for Certain Complete Intersections

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We define a generalization of Fan-Jarvis-Ruan-Witten theory, a “hybrid” model associated to a collection of quasihomogeneous polynomials of the same weights and degree, which is expected to match the Gromov-Witten theory of the Calabi-Yau complete intersection cut out by the polynomials. In genus zero, we prove that the correspondence holds for any such complete intersection of dimension three in ordinary, rather than weighted, projective space, generalizing the results of Chiodo-Ruan for the quintic threefold.