NOTES ON CHEN-RUAN COHOMOLOGY

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The Orbifolds Involved

For an orbifold groupoid $\mathcal{G}$, we define

$$\boxed{S_{\mathcal{G}} = \{g \in G_1 \mid s(g) = t(g)\}.}$$

One can check that $S_{\mathcal{G}}$ is a manifold. In fact, it is a left $\mathcal{G}$-space, in which the anchor is the map

$$\beta : S_{\mathcal{G}} \to G_0$$

given by sending an arrow $g \in G_x$ to x, and in which the action of an arrow $h : x \to y$ in G_1 sends $g \in G_x$ to $hgh^{-1} \in G_y$.

The inertia groupoid $\wedge\mathcal{G}$ is the action groupoid $\mathcal{G} \ltimes S_{\mathcal{G}}$. That is,

$$\boxed{\begin{aligned} \mathrm{obj}\,(\wedge\mathcal{G}) &= \{g \in G_1 \mid s(g) = t(g)\}, \\ \mathrm{mor}\,(\wedge\mathcal{G}) &= \{(g,h) \in G_1 \times G_1 \mid s(g) = t(g) = s(h)\}. \end{aligned}}$$

Here, (g,h) is an arrow from g to hgh^{-1}. It is helpful to observe that since the arrows identify $g \in G_x$ with $hgh^{-1} \in G_{hx}$, we have

$$|\wedge\mathcal{G}| = \{([x], (g)_{G_x}) \mid [x] \in |\mathcal{G}|\},$$

where $(g)_{G_x}$ denotes the conjugacy class of the element $g \in G_x$.

An important interpretation of the inertia groupoid is[1]

$$\boxed{|\wedge\mathcal{G}| = \{\text{equivalence classes of constant orbifold morphisms } S^1 \to \mathcal{G}\}.}$$

To see that this agrees with the above definition of $\wedge\mathcal{G}$, notice first that a constant morphism whose image is $x \in G_0$ factors through $\bullet^{G_x}$, and hence is determined by a diagram as follows:

$$S^1 \leftarrow \mathcal{K} \to \bullet^{G_x}.$$

Moreover, the equivalence relation identifies two such diagrams if we replace G_x by G_{gx}, so the equivalence class of the morphism determines x only up to its class in $|\mathcal{G}|$. This is the only identification that occurs on the right arrow, so we may focus on the left.

[1] It seems that what was intended was not exactly this, but an identification of $\wedge\mathcal{G}$ itself with the groupoid whose objects are constant orbifold morphisms $S^1 \to \mathcal{G}$ and where there is an arrow $S^1 \leftarrow K \to \bullet^{G_x}$ to $S^1 \leftarrow K \to \bullet^{G_{gx}}$. But there's not a great way to do this with our current definitions; we'd have to separate off equivalence on the left-hand arrow of the orbifold morphism from equivalence on the right-hand arrow, and absorb the former sort of equivalence into our objects while having the latter be enacted by the groupoid arrows. This seems to be what's done in the case of $\overline{\mathcal{M}}_k(\mathcal{G})$ below.

The above is an orbifold morphism from S^1 to a global quotient, which is representable since S^1 has trivial isotropy; we have seen previously that equivalence classes of such morphisms are in bijection with equivalence classes of diagrams

$$S^1 \xleftarrow{p} E \to \bullet,$$

where p is a principal G_x-bundle and the right-hand arrow is (trivially) a G_x-equivariant map[2]. But principal G_x-bundles over S^1, in turn, are classified by conjugacy classes of homomorphisms[3]

$$\pi_1^{orb}(S^1) = \pi_1(S^1) \to G_x.$$

Since $\pi_1(S^1)$ is generated by a single element, conjugacy classes of homomorphisms are given by conjugacy classes of elements. We conclude that an equivalence class of constant orbifold morphism from S^1 to $\mathcal{G}$ is determined by a pair $([x], (g)_{G_x})$ with $[x] \in |\mathcal{G}|$. Such pairs are precisely the elements of $| \wedge \mathcal{G}|$.

Along similar lines, an important orbifold is

$$\boxed{\overline{\mathcal{M}}_k(\mathcal{G}) = \{\text{constant representable morphisms } \mathbb{S}^2 \to \mathcal{G}\}.}$$

Here, $\mathbb{S}^2$ is an orbifold Riemann sphere with k orbifold points. As we did above, let us determine more concretely what this space consists of. A constant orbifold morphism from $\mathbb{S}^2$ to $\mathcal{G}$ is determined by a choice of $x \in G_0$ together with a diagram

$$\mathbb{S}^2 \leftarrow \mathcal{K} \to \bullet^{G_x}$$

in which the left arrow is a principal G_x-bundle, and this in turn is determined by a conjugacy class of homomorphism $\rho : \pi_1^{orb}(\mathbb{S}^2) \to G_x$. Now,

$$\pi_1^{orb}(\mathbb{S}^2, x_0) = \langle \lambda_1, \ldots, \lambda_k \mid \lambda_i^{m_i} = 1, \lambda_1 \cdots \lambda_k = 1 \rangle,$$

and ρ is determined by the images of the λ_i. To say that the orbifold morphism is representable is to say that the local inverse near the ith marked point gives a morphism

$$\mathbb{Z}_{m_i} \ltimes \mathbb{C} \to G_x \ltimes \bullet$$

in which the corresponding homomorphism $\varphi : \mathbb{Z}_{m_i} \to G_x$ is injective. This is the case if and only if $\varphi(1_{m_i})$ has order m_i, and tracing through the definitions, one can check that $\varphi(1_{m_i}) = \rho(\lambda_i)$, so the representability requirement is tantamount to the assertion that $\rho(\lambda_i)$ has order m_i for all i.

The content of all of this is is simply that one does not need to include the orbifold structure on $\mathbb{S}^2$ as an extra datum when classifying constant representable morphisms $\mathbb{S}^2 \to \mathcal{G}$, because it is forced once we have chosen the conjugacy class of homomorphism $\rho : \pi_1^{orb}(\mathbb{S}^2) \to G_x$ that defines the morphism. We conclude that

$$\boxed{\overline{\mathcal{M}}_k(\mathcal{G}) = \{(g_1, \ldots, g_k)_{(G_x)} \mid g_i \in G_x, g_1 \cdots g_k = 1\}.}$$

[2]The correspondence, for reference, sends a representable morphism F from S^1 to $\bullet^{G_x}$ to the diagram $S^1 \leftarrow F^*G_x \to \bullet$, where G_x is the obvious G_x-bundle over a point. Conversely, it sends a diagram $S^1 \leftarrow E \to \bullet$ to $S^1 \leftarrow \mathcal{E} \to \bullet^{G_x}$, where $\mathcal{E}$ is the groupoid associated to the principal G_x-bundle E.

[3]This correspondence sends a principal G_x-bundle $p : E \to S^1$ to the homomorphism $\gamma \mapsto$ endpoint of a lift of γ to a path in E, where we use the identification $p^{-1}(x_0) \cong G_x$ from the principal G_x-bundle structure to view this endpoint as an element of G_x.

In the above, $(g_1, \ldots, g_x)_{(G_x)}$ denotes the equivalence class of the tuple under conjugation (of the entire tuple) by elements of G_x.

This is a $\mathcal{G}$-space, in which the anchor is the map

$$\overline{\mathcal{M}}_k(\mathcal{G}) \to G_0$$

sending $(g_1, \ldots, g_k)_{(G_x)}$ to x, and in which the action of an arrow $h : x \to y$ in G_1 sends

$$(g_1, \ldots, g_k)_{(G_x)} \mapsto (hg_1h^{-1}, \ldots, hg_kh^{-1})_{(G_{hk})}.$$

The associated action groupoid is sometimes also denoted $\overline{\mathcal{M}_k}(\mathcal{G})$.

The last orbifold we will need to consider is the orbifold $\mathcal{G}^k$ of k-multisectors. This has

$$\boxed{\begin{array}{c}\mathrm{Obj}(\mathcal{G}^k) = S^k_{\mathcal{G}} = \{(g_1, \ldots, g_k) \mid g_i \in G_1, s(g_1) = t(g_1) = \cdots = s(g_k) = t(g_k)\}, \\ \mathrm{Mor}(\mathcal{G}^k) = \{(g_1, \ldots, g_k) \to (hg_1h^{-1}, \ldots, hg_kh^{-1})\} \text{ for all } g \in G_1 \text{ with } s(h) = s(g_i).\end{array}}$$

In other words, $S^k_{\mathcal{G}}$ is a $\mathcal{G}$-space whose anchor sends $\mathbf{g} \in G^k_x$ to x and where the action of $h : x \to y$ sends $(g_1, \ldots, g_k)$ to $(hg_1h^{-1}, \ldots, hg_kh^{-1})$. The arrows in $\mathcal{G}^k$ are given by this $\mathcal{G}$-space.

We should notice that

$$|\mathcal{G}^k| = \{([x], (g_1, \ldots, g_k)_{G_x} \mid [x] \in |\mathcal{G}|\},$$

so in particular,

$$|\mathcal{G}^{k-1}| \cong \overline{\mathcal{M}}_k(\mathcal{G}).$$

The Morphisms Involved

For each subset $\{i_1, \ldots, i_\ell\} \subset \{1, \ldots, k\}$, there is an evaluation map

$$\boxed{\begin{array}{c} e_{i_1,\ldots,i_\ell} : S^k_{\mathcal{G}} \to S^\ell_{\mathcal{G}} \\ (g_1, \ldots, g_k) \mapsto (g_{i_1}, \ldots, g_{i_\ell}). \end{array}}$$

It is worth reflecting for a moment on why this is called evaluation. We have seen that $S^k_{\mathcal{G}}$ is the set of objects in the groupoid $\mathcal{G}^k$, and the quotient space $|\mathcal{G}^k|$ of this groupoid is the space $\overline{\mathcal{M}_{k+1}}$ of constant representable morphisms from an orbifold Riemann sphere with $k+1$ marked points to $\mathcal{G}$. Thus, we can think of an element of $S^k_{\mathcal{G}}$ as a morphism from such a sphere; specifically, the elements $g_1, g_2, \ldots, g_k$, and $(g_1 \cdots g_k)^{-1}$ give the endpoints in $p^{-1}(x_0) \cong G_x$ of lifts of loops around the marked points in $\mathbb{S}^2$ to paths in the principal G_x-bundle $p : E \to \mathbb{S}^2$ corresponding to the morphism. "Evaluation" can be thought of as sending one of these elements of G_x to the point x; but of course, this preserves no information, since the data of the point x is encoded in the other g_i's, anyway. So evaluation, in this setting, amounts to forgetting some of the elements of G_x.

The other important morphism is the involution

$$\boxed{\begin{array}{c} I : S^k_{\mathcal{G}} \to S^k_{\mathcal{G}} \\ (g_1, \ldots, g_k) \mapsto (g_1^{-1}, \ldots, g_k^{-1}). \end{array}}$$

It is easy to show that each evaluation map is a finite union of embeddings and I is an isomorphism. Therefore, if $\mathcal{G}$ is a complex or almost complex orbifold, one can use these embeddings to define induced complex or almost complex structures on each $S^k_{\mathcal{G}}$ and hence on each $\mathcal{G}^k$. Under these structures, $\mathcal{G}^k$ becomes (pseudo-)holomorphic. By the same token, if $\mathcal{G}$ has a symplectic or Riemannian structure, so does $\mathcal{G}^k$.

Twisted Sectors

Let us begin with an example to get an intuition. Suppose that $\mathcal{G} = X/G$ is a global quotient. Then $\mathcal{G}^k$ has objects

$$\bigsqcup_{(g_1,\dots,g_k)\in G^k} X^{g_1} \cap \cdots \cap X^{g_k} \times \{(g_1,\dots,g_k)\}$$

and has an arrow starting at each object for each element of G. On the other hand, when we pass to the quotient $|\mathcal{G}^k|$, many of these components get identified, so up to orbifold equivalence we can forget them. Specifically, the component indexed by $(g_1,\dots,g_k)$ is identified with the component indexed by $(hg_1h^{-1},\dots,hg_kh^{-1})$, and within a given component there are identifications given by $C(g_1)\cap\cdots\cap C(g_k)$. Thus:

If $\mathcal{G} = X/G$ is a global quotient:

$$\mathcal{G}^k \sim \bigsqcup_{(g_1,\dots,g_k)_G} \left(X^{g_1} \cap \cdots \cap X^{g_k} \times \{(g_1,\dots,g_k)\}\right)/C(g_1)\cap\cdots\cap C(g_k)$$

In particular:

$$\wedge(X/G) \sim \bigsqcup_{g\in G_*} X^g/C(g)$$

(Here, G_* denotes the set of conjugacy classes in G.)

We would like a decomposition like this for an arbitrary orbifold groupoid. To do so, we will need to define an equivalence relation on k-tuples $(\mathbf{g})_{G_p}$.

Suppose first that U_x/G_x is a local chart in $\mathcal{G}$, and choose preimages $\tilde{p}$ and $\tilde{q}$ of p and q in U_x. Since we can view $G_p = (G_x)_p$ and $G_q = (G_x)_q$, we can think of both G_p and G_q as subgroups of G_x. We will say that conjugacy classes $(\mathbf{g}_1)_{G_p}$ and $(\mathbf{g}_2)_{G_q}$ of k-tuples are equivalent if there exists $h \in G_x$ such that $\mathbf{g}_1 = h\mathbf{g_2}h^{-1}$.

Now, extend this to an equivalence relation on any conjugacy classes of k-tuples by saying

$$(\mathbf{g})_{G_p} \approx (\mathbf{g}')_{G_q}$$

if there exists a finite sequence $(p_0,(\mathbf{g}_0)_{G_{p_0}}),\dots,(p_k,(\mathbf{g}_k)_{G_{p_k}})$ such that

(1) $(p_0,(\mathbf{g}_0)_{G_{p_0}}) = (p,(\mathbf{g})_{G_p})$
(2) $(p_k,(\mathbf{g}_k)_{G_{p_k}}) = (q,(\mathbf{g}')_{G_q})$
(3) For each $0 \le i \le k-1$, p_i and p_{i+1} are in the same linear chart and $(\mathbf{g}_i)_{G_p}$ and $(\mathbf{g}_{i+1})_{G_{p+1}}$ are equivalent in the above sense.

Warning: There could be monodromy! That is, it is possible that $(\mathbf{g})_{G_p} \approx (\mathbf{g}')_{G_p}$ although $(\mathbf{g})_{G_p} \neq (\mathbf{g}')_{G_p}$.

Having defined this equivalence relation, we obtain the desired decomposition of $\mathcal{G}^k$. If T_k denotes the set of equivalence classes of k-tuples under $\approx$, and if

$$|\mathcal{G}^k|_{(\mathbf{g})} = \{(p,(\mathbf{g}')_{G_p}) \mid p \in |\mathcal{G}|, (\mathbf{g}')_{G_p} \approx (\mathbf{g})\},$$

then the decomposition is

$$|\mathcal{G}^k| = \bigsqcup_{(\mathbf{g})\in T_k} |\mathcal{G}^k|_{(\mathbf{g})}.$$

This is a decomposition into connected components.

By the same token, we get a decomposition

$$\mathcal{G}^k = \bigsqcup_{(\mathbf{g})\in T_k} \mathcal{G}^k_{(\mathbf{g})},$$

where $\mathcal{G}^k_{(\mathbf{g})}$ denotes the full subgroupoid on the preimage of $|\mathcal{G}^k|_{(\mathbf{g})}$.

Finally, letting $T_k^0 \subset T_k$ denote the equivalence classes such that $g_1 \cdots g_k = 1$, we get a decomposition

$$\overline{\mathcal{M}_k}(\mathcal{G}) = \bigsqcup_{(\mathbf{g})\in T_k^0} \mathcal{G}^k_{(\mathbf{g})}.$$

Each $\mathcal{G}^1_{(g)} \subset \wedge\mathcal{G}$ for $g \neq 1$ is called a twisted sector, while $\mathcal{G}^1_{(1)}$ is called the non-twisted sector. Note that $\mathcal{G}^1_{(1)} \cong \mathcal{G}$.

Degree-Shifting Numbers

From now on, we will assume that $\mathcal{G}$ is an almost complex orbifold. Recall that the almost complex structure on $\mathcal{G}$ induces one on $\wedge\mathcal{G}$ and more generally on $\mathcal{G}^k$, and in this almost complex structure, $e_{i_1,\ldots,i_\ell}$ and I are pseudo-holomorphic. We will assume furthermore that $|\mathcal{G}|$ admits a finite good cover and hence $\mathcal{G}$ satisfies Poincaré duality. It follows that each $|\mathcal{G}^k|$ admits a finite good cover, so their connected components $\mathcal{G}^k_{(\mathbf{g})}$ all satisfy Poincaré duality, also.

For any $g \in \mathrm{Obj}(\wedge\mathcal{G}) = \{h \in G_1 \mid s(h) = t(h)\}$ with $s(g) = t(g) = p$, the action of G_p on T_pG_0 induces (by way of the almost complex structure) a homomorphism

$$\rho_p : G_p \to GL_n(\mathbb{C}),$$

and since g has finite order, the matrix $\rho_p(g)$ is diagonalizable. We can write the diagonalized matrix as

$$\begin{pmatrix} e^{2\pi i m_{1,g}/m_g} & & \\ & \ddots & \\ & & e^{2\pi i m_{n,g}/m_g} \end{pmatrix},$$

where m_g is the order of $\rho_p(g)$ and $0 \le m_{i,g} < m_g$. This matrix depends only on the conjugacy class of g in G_p, so it makes sense to define a function as follows:

$$\boxed{\begin{aligned} \iota : |\wedge\mathcal{G}| &\to \mathbb{Q} \\ (g)_{G_p} &\mapsto \sum_{i=1}^{n} \frac{m_{i,g}}{m_g}. \end{aligned}}$$

Lemma. The function ι defined above is locally constant. Its constant value on the component $|\wedge\mathcal{G}|_{(g)}$, denoted $\iota_{(g)}$, satisfies:

(1) $\iota_{(g)} \in \mathbb{Z}$ if and only if $\rho_p(g) \in SL(n,\mathbb{C})$
(2) $\iota_{(g)} + \iota_{(g^{-1})} = \mathrm{rank}(\rho_p(g) - I)$.

Proof. The fact that ι is locally constant is clearly a local statement, and the connected components of $|\wedge\mathcal{G}|$ look locally like

$$\{(p, (g')_{G_p}) \mid (g') = h(g)_{G_p}h^{-1}\}$$

for some fixed $(g)_{G_p} \in |\wedge\mathcal{G}|$; it is clear that all such elements have the same value of ι. For statement (1), notice that

$$\det(\rho_p(g)) = \prod e^{2\pi i m_{i,g}/m_g} = e^{2\pi i \sum m_{i,g}/m_g} = e^{2\pi i \iota_{(g)}},$$

so $\iota_{(g)} \in \mathbb{Z}$ if and only if $\det(\rho_p(g)) = 1$. For (2), we observe that $m_{i,g^{-1}} \equiv -m_{i,g} \bmod m_g$, so if $m_{i,g} \neq 0$, then $m_{i,g} + m_{i,g^{-1}} = m_{i,g}$ and hence

$$\frac{m_{i,g}}{m_g} + \frac{m_{i,g^{-1}}}{m_g} = 1.$$

If $m_{i,g} = 0$, however, then $m_{i,g^{-1}} = 0$, and so

$$\frac{m_{i,g}}{m_g} + \frac{m_{i,g^{-1}}}{m_g} = 0.$$

It follows that $\iota_{(g)} + \iota_{(g^{-1})}$ is equal to the number of entries not equal to 1 in $\rho_p(g)$, which is precisely the rank of $\rho_p(g) - I$. $\square$

The number $\iota_{(g)}$ is called the degree-shifting number of $|\wedge\mathcal{G}|_{(g)}$, and it plays a vital role in the definition of Chen-Ruan cohomology. The significance of these numbers lies in the virtual dimension of the components of $\overline{\mathcal{M}}_3(\mathcal{G})$. Specifically, in the decomposition

$$\overline{\mathcal{M}}_3(\mathcal{G}) = \bigsqcup_{(g_1,g_2,g_3)\in T_k^0 3} |\mathcal{G}^3|_{(g_1,g_2,g_3)},$$

the virtual dimension of the component $|\mathcal{G}^3|_{(g_1,g_2,g_3)}$ is as follows[4]:

$$\boxed{\operatorname{virdim}(|\mathcal{G}^3|_{(g_1,g_2,g_3)}) = 2n - 2\iota_{(g_1)} - 2\iota_{(g_2)} - 2\iota_{(g_3)}.}$$

Therefore, if $\alpha_1, \alpha_2, \alpha_3 \in H^*(\wedge\mathcal{G})$ and $e_i : \overline{\mathcal{M}}_3(\mathcal{G}) \to \wedge\mathcal{G}$ denote the three evaluation maps, and if specifically we have $\alpha_i \in H^*(\mathcal{G}^1_{(g_i)})$, then in order to formally carry out an integral

$$\int_{\overline{\mathcal{M}}_3(\mathcal{G})} e_1^*\alpha_1 \wedge e_2^*\alpha_2 \wedge e_3^*\alpha_3,$$

we must have

$$\deg(\alpha_1) + \deg(\alpha_2) + \deg(\alpha_3) = 2n - 2\iota_{(g_1)} - 2\iota_{(g_2)} - 2\iota_{(g_3)}.$$

This rearranges to

$$(\deg(\alpha_1) + 2\iota_{(g_1)}) + (\deg(\alpha_2) + 2\iota_{(g_2)}) + (\deg(\alpha_3) + 2\iota_{(g_3)}) = 2n.$$

Hence, we will think of $\alpha_i \in H^k(\mathcal{G}^1_{(g_i)})$ as actually lying in degree $k + 2\iota_{(g_i)}$, so that the constraint is simply that the degrees add to the dimension.

For example, consider the action groupoid $\mathcal{G} = \mathbb{Z}/2 \ltimes S^2$, where $\mathbb{Z}/2$ acts on S^2 by rotation. In addition to the nontwisted sector, there are 0-dimensional twisted sectors $\bullet^{\mathbb{Z}/2}$, each of which has degree-shifting number $\frac{1}{2}$. A typical triple $(g_1, g_2, g_3) \in T_3^0$ is $(a, 1, a)$ with $a \in \mathbb{Z}/2$ the unique nontrivial element; thus, we have $2\iota_{(g_1)} + 2\iota_{(g_2)} + 2\iota_{(g_3)} = 2$. This verifies that the (virtual) dimension of $|\mathcal{G}|^3_{(g_1,g_2,g_3)} = \bullet^{\mathbb{Z}/2}$ is indeed $2n - 2\iota_{(g_1)} - 2\iota_{(g_2} - 2\iota_{(g_3)} = 2 - 2 = 0$ in this case.

Having explored the degree-shifting numbers, we are ready to define the Chen-Ruan cohomology groups:

$$\boxed{H^d_{CR}(\mathcal{G}) = \bigoplus_{(g)\in T_1} H^{d-2\iota_{(g)}}(\mathcal{G}^1_{(g)}).}$$

Note that these are in general rationally-graded. In fact, by the lemma above, the Chen-Ruan cohomology groups are integrally-graded if and only if $\mathcal{G}$ is Gorenstein.

[4]This isn't actually what's written in the book, so maybe it's not correct. What's written is $\operatorname{virdim}(\overline{\mathcal{M}}_3(\mathcal{G})) = 2n - 2\iota_{(g_1)} - 2\iota_{(g_2)} - 2\iota_{(g_3)}$, which seems like it can't be right. My guess is that what's meant is either what I wrote above, or possibly $\operatorname{virdim}(\overline{\mathcal{M}}_3(\mathcal{G})) = \dim(\mathcal{G}^3_{(g_1,g_2,g_3)}) - 2\iota_{(g_1)} - 2\iota_{(g_2)} - 2\iota_{(g_3)}$ for any component $\mathcal{G}^3_{(g_1,g_2,g_3)}$.

Poincaré Pairing

Proposition. Let $\dim_{\mathbb{R}}(\mathcal{G}) = 2n$. Define a pairing as follows:

$$\langle\ ,\ \rangle_{CR} : H^d_{CR}(\mathcal{G}) \times H^{2n-d}_{CR,c}(\mathcal{G}) \to \mathbb{R}$$

is the direct sum of the pairings

$$\langle\ ,\ \rangle_{(g)} : H^{d-2\iota_{(g)}}(\mathcal{G}^1_{(g)}) \times H^{2n-d-\iota_{(g^{-1})}}(\mathcal{G}^1_{(g^{-1})}) \to \mathbb{R}$$
$$\langle \alpha, \beta \rangle_{(g)} = \int_{\mathcal{G}^1_{(g)}} \alpha \wedge I^*\beta$$

Then $\langle\ ,\ \rangle_{CR}$ is nondegenerate.

Proof. We saw in the proof of the lemma in the last section that

$$\iota_{(g)} + \iota_{(g^{-1})} = \operatorname{rank}(\rho_p(g) - I),$$

which is the number of eigenvalues not equal to 1 of g, considered as a linear endomorphism of T_pG_0. We claim that this is also equal to the "complex codimension"

$$n - \dim_{\mathbb{C}} \mathcal{G}_{(g)},$$

of $\mathcal{G}_{(g)}$ in $\mathcal{G}$. This is equivalent to the claim that the dimension of the fixed locus in T_pG_0 under the action of g is equal to the dimension of $\mathcal{G}_{(g)}$. And indeed, if $g \in G_x$ and U_x/G_x is a chart around x, then

$$\mathcal{G}_{(g)} \cap U_x/G_x = U_x^g/G_x,$$

so the fixed locus in U_x/G_x under the action of g is precisely $\mathcal{G}_{(g)} \cap U_x/G_x$, which has the same dimension as $\mathcal{G}_{(g)}$ since U_x is open. Since we may assume that g acts linearly, the same is true of the fixed locus of its derivative in T_pG_0.

Using this, we have

$$\begin{aligned} 2n - d - 2\iota_{(g^{-1})} &= 2(\dim_{\mathbb{C}}(\mathcal{G}) - \iota_{(g^{-1})}) - d \\ &= 2(\dim_{\mathbb{C}}(\mathcal{G}^1_{(g)}) + \iota_{(g)}) - d \\ &= \dim_{\mathbb{R}}(\mathcal{G}^1_{(g)}) - (d - 2\iota_{(g)}). \end{aligned}$$

This shows that the dimensions match up correctly to be a Poincaré pairing. More specifically, there is a commutative diagram

$$\begin{tikzcd} H^{d-2\iota_{(g)}}(\mathcal{G}^1_{(g)}) \times H^{2n-d-2\iota_{(g^{-1})}}(\mathcal{G}^1_{(g^{-1})}) \arrow[dr, "\langle\ ,\ \rangle_{(g)}"] \arrow[dd, "\cong"', "\mathrm{id}\times I^*"] & \\ & \mathbb{R}. \\ H^{d-2\iota_{(g)}}(\mathcal{G}^1_{(g)}) \times H^{\dim_{\mathbb{R}}(\mathcal{G}^1_{(g)})-(d-2\iota_{(g)})}(\mathcal{G}^1_{(g)}) \arrow[ur, "P.D."'] & \end{tikzcd}$$

So the nondegeneracy of the Chen-Ruan pairing follows from the nondegeneracy of the ordinary cup product. $\square$

The Obstruction Bundle

The definition of the cup product in Chen-Ruan cohomology involves the so-called "obstruction bundle", a vector bundle on $\overline{\mathcal{M}}_3(\mathcal{G})$. We will provide two definitions.

First Definition: Choose an element $f_y \in \overline{\mathcal{M}}_3(\mathcal{G})$, which is a constant morphism from $\mathbb{S}^2$ to $\mathcal{G}$. Suppose, in particular, that $f_y \in \mathcal{G}^3_{(g_1,g_2,g_3)}$. Consider the complex

$$\overline{\partial}_y : \Omega^0(f_y^* T\mathcal{G}) \to \Omega^{0,1}(f_y^* T\mathcal{G}).$$

Then

$$\text{index}(\overline{\partial}_y) = 2n - 2\iota_{(g_1)} - 2\iota_{(g_2)} - 2\iota_{(g_3)},$$

so in particular, if we vary f_y, then the index of $\overline{\partial}_y$ is constant on each component of $\overline{\mathcal{M}}_3(\mathcal{G})$. On the other hand, index theory shows that $\ker(\overline{\partial}_y) \cong T_{f_y}\mathcal{G}^3_{(\mathbf{g})}$, which implies that $\dim(\ker(\overline{\partial}_y))$ is also constant on each component of $\overline{\mathcal{M}}_3(\mathcal{G})$. Putting these two facts together, we find that $\dim(\text{coker}(\overline{\partial}_y))$ is constant on each component of $\overline{\mathcal{M}}_3(\mathcal{G})$, so there is an orbifold vector bundle

$$E_3 \to \overline{\mathcal{M}}_3(\mathcal{G})$$

whose fiber over f_y is $\text{coker}(\overline{\partial}_y)$. This is the obstruction bundle.

Second Definition: Again, choose $f_y \in \mathcal{G}^3_{(\mathbf{g})} \subset \overline{\mathcal{M}}_3(\mathcal{G})$. Consider the subgroup

$$N = \langle g_1, g_2, g_3 \rangle \subset G_{e(\mathbf{g})},$$

which is independent (up to isomorphism) of $\mathbf{g}$ within the same component of $\overline{\mathcal{M}}_3(\mathcal{G})$. The element $f_y \in \mathcal{G}^3_{(\mathbf{g})}$ corresponds to a morphism to $\mathcal{G}$ from an orbifold sphere $\mathbb{S}^2$ with three marked points x_i whose orders match the orders of g_1, g_2, and g_3. So if

$$\pi_1^{orb}(\mathbb{S}^2) = \langle \lambda_1, \lambda_2, \lambda_3 \mid \lambda_i^{m_i} = 1, \lambda_1\lambda_2\lambda_3 = 1 \rangle,$$

then there is a surjective homomorphism

$$\pi : \pi_1^{orb}(\mathbb{S}^2) \to N$$

sending λ_i to g_i.

Let $\tilde{\Sigma}$ denote the orbifold universal cover of $\mathbb{S}^2$; it is a fact that $\tilde{\Sigma}$ is smooth. Then $\Sigma = \tilde{\Sigma}/\ker(\pi)$ is also smooth, and

$$\Sigma/N \cong \tilde{\Sigma}/(N/\ker(\pi)) \cong \tilde{\Sigma}/\pi_1(\mathbb{S}^2) \cong \mathbb{S}^2,$$

so the quotient by N gives a cover $\Sigma \to \mathbb{S}^2$.

If U_y/G_y is an orbifold chart around the point $y \in G_0$ to which f_y maps, then f_y lifts to an ordinary constant morphism $\tilde{f}_y : \Sigma \to \{y\} \subset \mathcal{G}$. In particular, $\tilde{f}_y^*(T\mathcal{G}) \cong T_y\mathcal{G}$, a trivial bundle. Lifting the complex in the first definition to

$$\overline{\partial}_\Sigma : \Omega^0(\tilde{f}_y^* T\mathcal{G}) \to \Omega^{0,1}(\tilde{f}_y^* T\mathcal{G}),$$

we find that the old complex is simply the N-invariant part of the new one. In particular, the fact that

$$\text{coker}(\overline{\partial}_\Sigma) = H^{0,1}(\Sigma) \otimes e^*_{(\mathbf{g})}(T\mathcal{G})$$

implies that the fiber of E_3 over $f_y \in \mathcal{G}^3_{(\mathbf{g})}$ is

$$(H^{0,1}(\Sigma) \otimes e^*_{(\mathbf{g})}(T\mathcal{G}))^N.$$

Cup Product

The final ingredient necessary for defining the Chen-Ruan cup product is the three-point function:

> If $\alpha, \beta \in H^*_{CR}(\mathcal{G};\mathbb{C})$ and $\gamma \in H^*_{CR,c}(\mathcal{G};\mathbb{C})$, then
> $$\langle \alpha, \beta, \gamma \rangle = \sum_{(\mathbf{g}) \in T_3^0} \int_{\mathcal{G}_{(\mathbf{g})}} e_1^*\alpha \wedge e_2^*\beta \wedge e_3^*\gamma \wedge e(E_{(\mathbf{g})}).$$

Here, $E_{(\mathbf{g})}$ denotes the restriction of the obstruction bundle to the component $\mathcal{G}_{(\mathbf{g})}$ of $\overline{\mathcal{M}}_3(\mathcal{G})$.

We are finally ready to define the cup product itself:

> If $\alpha, \beta \in H^*_{CR}(\mathcal{G};\mathbb{C})$, then $\alpha \cup \beta$ is defined by the relation
> $$\langle \alpha \cup \beta, \gamma \rangle_{CR} = \langle \alpha, \beta, \gamma \rangle$$
> for all $\gamma \in H^*_{CR,c}(\mathcal{G};\mathbb{C})$.

Perhaps the first thing we should check about the cup product is that it respects the grading; that is, if $\alpha \in H^{d_1}_{CR}(\mathcal{G};\mathbb{C})$ and $\beta \in H^{d_2}_{CR}(\mathcal{G};\mathbb{C})$, then $\alpha \cup \beta \in H^{d_1+d_2}_{CR}(\mathcal{G};\mathbb{C})$. Equivalently, we must show that $\langle \alpha \cup \beta, \gamma \rangle_{CR}$ is nonzero when $\gamma \in H^{2n-d_1-d_2}_{CR,c}(\mathcal{G})$. It suffices to do this in the case where α and β each lie in a single summand of $H^*_{CR}(\mathcal{G};\mathbb{C})$, say $\alpha \in H^{d_1-2\iota_{(g_1)}}(\mathcal{G}^1_{(g_1)})$ and $\beta \in H^{d_2-2\iota_{(g_2)}}(\mathcal{G}^1_{(g_2)})$. Then if we are to get a nonzero result, we certainly must have $\gamma \in H^{2n-d_1-d_2-2\iota_{((g_1g_2)^{-1})}}_{CR,c}(\mathcal{G}^1_{((g_1g_2)^{-1})})$, and in this case

$$\langle \alpha \cup \beta, \gamma \rangle_{CR} = \langle \alpha, \beta, \gamma \rangle = \int_{\mathcal{G}_{(g_1,g_2,(g_1g_2)^{-1})}} e_1^*\alpha \wedge e_2^*\beta \wedge e_3^*\gamma \wedge e(E_{(\mathbf{g})}).$$

The total degree of the integrand is

$$\begin{aligned}
&d_1 - 2\iota_{(g_1)} + d_2 - 2\iota_{(g_2)} + 2n - d_1 - d_2 - 2\iota_{((g_1g_2)^{-1})} + \operatorname{rank}(E_{(\mathbf{g})}) \\
&= \operatorname{index}(\overline{\partial}_y) + \operatorname{rank}(E_{(\mathbf{g})}) \\
&= \dim(\ker(\overline{\partial}_y)) - \dim(\operatorname{coker}(\overline{\partial}_y)) + \operatorname{rank}(E_{(\mathbf{g})}) \\
&= \dim(\ker(\overline{\partial}_y)) \\
&= \dim(\mathcal{G}^3_{(\mathbf{g})}),
\end{aligned}$$

so we indeed get a nonzero integral only in this case.

Another important thing to check is that the restriction of the Chen-Ruan cup product to the nontwisted sector is the ordinary cup product on $\mathcal{G}$. Indeed, suppose that $\alpha, \beta \in H^*(\mathcal{G}^1_{(1)}) = H^*(\mathcal{G})$. We must show that

$$\langle \alpha \cup \beta, \gamma \rangle = \langle \alpha, \beta, \gamma \rangle$$

for all $\gamma \in H^*_{CR,c}(\mathcal{G})$ of the appropriate degree, where now $\cup$ denotes not the Chen-Ruan cup product but the ordinary cup product. In order for the above to be nonzero, we must have

$\gamma \in H^*(\mathcal{G}^1(1))$. And for $\gamma \in H^*(\mathcal{G}^1_{(1)})$, the left-hand side of the above is

$$\int_{\mathcal{G}} (\alpha \cup \beta) \wedge I^*(\gamma) = \int_{\mathcal{G}} \alpha \wedge \beta \wedge \gamma.$$

Observe that

(1) If $e_i : \mathcal{G}^3_{(1,1,1)} \to \mathcal{G}^1_{(1)}$ is any of the three evaluation maps, then $e_i = \mathrm{id}_{\mathcal{G}}$.
(2) The obstruction bundle $E_{(1,1,1)}$ on $\mathcal{G}^3_{(1,1,1)}$ is trivial, since

$$\mathrm{rank}(E_{(1,1,1)}) = \dim(\mathcal{G}^3_{(1,1,1)}) - \dim(\mathcal{G}) + 2\iota_{(1)} + 2\iota_{(1)} + 2\iota_{(1)} = 0.$$

Therefore, we have

$$\langle \alpha \cup \beta, \gamma \rangle = \int_{\mathcal{G}} \alpha \wedge \beta \wedge \gamma = \int_{\mathcal{G}^3_{(1,1,1)}} e_1^*\alpha \wedge e_2^*\beta \wedge e_3^*\gamma \wedge e(E_{(1,1,1,)}) = \langle \alpha, \beta, \gamma \rangle,$$

as required.

We collect these and a few other facts that we won't prove in the following list:

<u>Properties of the Chen-Ruan Cup Product</u>

Let $\mathcal{G}$ be an almost complex orbifold groupoid with almost complex structure J and $\dim_{\mathbb{C}} \mathcal{G} = n$. Then we have

$$\cup : H^p_{CR}(\mathcal{G}; \mathbb{C}) \otimes H^q_{CR}(\mathcal{G}; \mathbb{C}) \to H^{p+q}_{CR}(\mathcal{G}; \mathbb{C}),$$

and it satisfies:

- $H^*_{CR}(\mathcal{G}; \mathbb{C})$ is a ring with unit $e^0_{\mathcal{G}} \in H^0(\mathcal{G}; \mathbb{C})$, the constant function 1 on $\mathcal{G}$.
- $\cup$ is invariant under deformations of J and under Morita equivalence.
- If all the degree-shifting numbers are integers, then the Chen-Ruan cohomology is integrally graded and we have $\alpha_1 \cup \alpha_2 = (-1)^{\deg(\alpha_1)\deg(\alpha_2)} \alpha_2 \cup \alpha_1$.
- On the nontwisted sector, $\cup$ agrees with the ordinary cup product on $\mathcal{G}$.

We would like is a decomposition of $\alpha \cup \beta$ in accordance with the direct sum decomposition of $H^*_{CR}(\mathcal{G})$. This is given by the following lemma:

Decomposition Lemma. Let $\alpha \in H^*(\mathcal{G}^1_{(g_1)})$ and $\beta \in H^*(\mathcal{G}^1_{(g_2)})$. Then

$$\alpha \cup \beta = \sum_{(h_1,h_2) \in T_2, h_i \in (g_i)} (\alpha \cup \beta)_{(h_1,h_2)},$$

where $(\alpha \cup \beta)_{(h_1,h_2)} \in H^*(\mathcal{G}_{(h_1h_2)})$ is defined by the relation

$$\langle (\alpha \cup \beta)_{(h_1,h_2)}, \gamma \rangle = \int_{\mathcal{G}_{(h_1,h_2,(h_1h_2)^{-1})}} e_1^*\alpha \wedge e_2^*\beta \wedge e_3^*\gamma \wedge e(E_{(\mathbf{g})})$$

for all $\gamma \in H^*_c(\mathcal{G}_{((h_1h_2)^{-1})})$.

The proof of this lemma is actually quite obvious: it essentially says that when we are pairing $\alpha \cup \beta$ with γ in order to define the Chen-Ruan cup product, we only need to bother pairing it with γ for which the three-point function has a chance of being nonzero.

One indication that Chen-Ruan cohomology is the "right" notion of orbifold cohomology is that, conjecturally, it agrees with the ordinary cohomology of certain crepant resolutions. Specifically:

Conjecture. If $\pi : Y \to X$ is a crepant resolution with Y hyperkähler or symplectic, then

$$H^*_{CR}(X;\mathbb{C}) \cong H^*(Y;\mathbb{C}).$$

If we do not assume that Y satisfies either of the above two conditions, then

$$H^*_{CR}(X;\mathbb{C}) \cong H^*_{\pi}(Y;\mathbb{C}),$$

where the latter is the "Ruan cohomology" of Y, defined as ordinary cohomology with a correction factor in the cup product coming from Gromov-Witten invariants.

Global Quotient Case

In case $\mathcal{X} = Y/G$ is a global quotient, we have an isomorphism on the level of vector spaces:

$$H^*_{CR}(Y/G) = H^*(\wedge(Y/G)) = H^*\left(\left(\bigsqcup_{g\in G} Y^g\right)/G\right) = \left(\bigoplus_{g\in G} H^*(Y^g)\right)^G,$$

where the last space is the G-invariant part under the induced action on cohomology coming from the natural action of G by conjugation. There is actually a way to put a product structure on this direct sum in such a way that, as rings in addition to vector spaces, $H^*_{CR}(Y/G)$ is the G-invariant part.

The main point is that

$$\wedge\mathcal{X} = \bigsqcup_{(g)} Y^g/C(g),$$

so we need to lift everything we did on the level of $Y^g/C(g)$ to the level of Y^g. Throughout, let $Y^{g_1,\dots,g_k} = Y^{g_1} \cap \dots \cap Y^{g_k} \times \{(g_1,\dots,g_k)\}$.

First, define evaluation maps:

$$e_{i_1,\dots,i_\ell} : Y^{g_1,\dots,g_k} \to Y^{g_{i_1},\dots,g_{i_\ell}}$$

as the inclusions. These are indeed lifts of the previously-defined evaluation maps, since under the equivalence

$$\mathcal{X}^k \sim \bigsqcup_{(g_1,\dots,g_k)_G} Y^{g_1,\dots,g_k}/C(g_1)\cap\dots\cap C(g_k),$$

the evaluation map on multisectors is the quotient of the inclusion:

$$Y^{g_1,\dots,g_k}/C(g_1)\cap\dots C(g_k) \to Y^{g_{i_1},\dots,g_{i_\ell}}/C(g_{i_1})\cap\dots\cap C(g_{i_\ell}).$$

Similarly, the involution I lifts to the identity map

$$Y^{g_1,\dots,g_k} \to Y^{g_1^{-1},\dots,g_k^{-1}}.$$

To lift the Poincaré pairing, we can define the degree-shifting number of Y^g as the degree-shifting number of the conjugacy class (g) just as before, and then define the pairing $\langle\ ,\ \rangle$ on $\bigoplus H^*(Y^g)$ as the direct sum of the pairings

$$\langle\ ,\ \rangle_{(g)} : H^{d-2\iota_{(g)}}(Y^g) \times H^{2n-d-2\iota_{g^{-1}}}(Y^{(g^{-1})})$$

given by

$$\langle\alpha,\beta\rangle_{(g)} = \frac{1}{|G|}\int_{Y^g} \alpha\wedge I^*\beta.$$

Here, the factor of $1/|G|$ is necessary in order to ensure that our new construction lifts our old one. For we have

$$\mathcal{X}^1_{(g)} = \left(\bigsqcup_{h\in(g)} Y^h\right)/G,$$

and so integration satisfies

$$\int_{\mathcal{X}^1_{(g)}} \alpha\wedge I^*\beta = \frac{1}{|G|}\int_{Y^g} \alpha\wedge I^*\beta,$$

identifying the cohomology classes with their lifts.

The lifted obstruction bundle is

$$E_{h_1,h_2,(h_1h_2)^{-1}} = (e^*_{(h_1,h_2,(h_1h_2)^{-1})}TY \otimes H^{0,1}(\Sigma))^{\langle h_1,h_2\rangle},$$

which indeed lifts the old obstruction bundle

$$E_{h_1,h_2,(h_1h_2)^{-1}} = (e^*_{(h_1,h_2,(h_1h_2)^{-1})}T(Y/G) \otimes H^{0,1}(\Sigma))^{\langle h_1,h_2\rangle},$$

the latter being the quotient by G of the former. So we obtain a lifted three-point function

$$\langle \alpha, \beta, \gamma \rangle = \frac{1}{|G|} \sum_{h_1,h_2} \int_{Y^{h_1,h_2,(h_1h_2)^{-1}}} e_1^*\alpha \wedge e_2^*\beta \wedge e_3^*\gamma \wedge e(E_{h_1,h_2,(h_1h_2)^{-1}})$$

for $\alpha, \beta, \gamma \in \bigoplus H^*(Y_g)$, where the factor of $1/|G|$ arises for the same reason as mentioned above.

Finally, we may define the cup product on $\bigoplus H^*(Y_g)$ by the relation

$$\langle \alpha \cup \beta, \gamma \rangle = \langle \alpha, \beta, \gamma \rangle.$$

Example: $\mathcal{G} = \bullet^G$

The decomposition of $\wedge\mathcal{G}$ into twisted sectors is

$$\wedge\mathcal{G} = \bigsqcup_{(g)\in G_*} \bullet^{C(g)},$$

so

$$\mathcal{G}_{(g)} = \bullet^{C(g)}.$$

Its degree-shifting number is 0 (since the action on the 0-dimensional tangent space to $\bullet$ is obviously trivial), and its de Rham cohomology is

$$H^*(\mathcal{G}_{(g)}) = \{C(g) - \text{invariant forms on } \bullet\} = H^0(\bullet) = \mathbb{C}.$$

Let $x_{(g)}$ denote the constant function 1 on $\mathcal{G}_{(g)}$, which generates the cohomology of that twisted sector. Then the Poincaré pairing is given by

$$\langle x_{(g)}, x_{(g^{-1})}\rangle = \int_{\mathcal{G}_{(g)}} x_{(g)} \wedge I^* x_{(g^{-1})} = \frac{1}{|C(g)|}\int_\bullet 1 = \frac{1}{|C(g)|}.$$

For the cup product, use the Decomposition Lemma to write

$$x_{(g_1)} \cup x_{(g_2)} = \sum_{(h_1,h_2), h_i\in(g_i)} (x_{(g_1)} \cup x_{(g_2)})_{(h_1,h_2)},$$

where $(x_{(g_1)} \cup x_{(g_2)})_{(h_1,h_2)} \in H^*(\mathcal{G}_{(h_1,h_2)})$ and hence

$$(x_{(g_1)} \cup x_{(g_2)})_{(h_1,h_2)} = d_{(h_1,h_2)} x_{(h_1h_2)}$$

for some number $d_{(h_1,h_2)}$. This number is determined by the relation

$$\langle (x_{(g_1)} \cup x_{(g_2)})_{(h_1,h_2)}, x_{((h_1h_2)^{-1})}\rangle = d_{(h_1,h_2)}\langle x_{(h_1h_2)}, x_{((h_1h_2)^{-1})}\rangle = \int_{\mathcal{G}_{(h_1,h_2)}} 1 = \frac{1}{|C(h_1)\cap C(h_2)|},$$

where we have used that the obstruction bundle is trivial, as an easy computation of its rank verifies. We conclude that

$$d_{(h_1,h_2)} = \frac{|C(h_1h_2)|}{|C(h_1)\cap C(h_2)|},$$

justifying the first half of the following:

> The Chen-Ruan cohomology of $\bullet^G$ is generated by $x_{(g)} \in H^0(\mathcal{G}_{(g)}) \subset H^0_{CR}(\mathcal{G})$ for each conjugacy class $(g) \in G_*$. The multiplication is given by
>
> $$x_{(g_1)} \cup x_{(g_2)} = \sum_{(h_1,h_2), h_i\in(g_i)} \frac{|C(h_1h_2)|}{|C(h_1)\cap C(h_2)|} \cdot x_{(h_1h_2)}.$$
>
> This is isomorphic as a ring to $Z(\mathbb{C}G)$, the center of the group algebra. The isomorphism is given on the generators by
>
> $$x_{(g)} \mapsto \tau_{(g)} = \sum_{h\in(g)} h.$$

We omit the proof of the second statement.

Example: $\mathcal{G} = G \ltimes \mathbb{C}^n$, G a finite subgroup of $SL(n, \mathbb{C})$

The sectors in $\wedge\mathcal{G}$ are $\mathcal{G}_{(g)} = (\mathbb{C}^n)^g/C(g)$. Since the fixed-point sets are linear subspaces of $\mathbb{C}^n$ and hence are contractible, we find

$$H^{p,q}(\mathcal{G}_{(g)}) = \begin{cases} 0 & p \text{ or } q > 0 \\ \mathbb{C} & p = q = 0. \end{cases}$$

It follows that

$$H^*_{CR}(\mathcal{G}) = \bigoplus_{(g)} H^{p-\iota_{(g)}, q-\iota_{(g)}}(\mathcal{G}^1_{(g)}) = \bigoplus_{p} \bigoplus_{(g)\,|\,\iota_{(g)}=p} \mathbb{C}.$$

To compute the product structure, let $x_{(g)}$ be the constant function 1 on $\mathcal{G}_{(g)}$, which generates the copy of $\mathbb{C}$ corresponding to (g). Then, as in the previous example, we have

$$x_{(g_1)} \cup x_{(g_2)} = \sum_{(h_1,h_2), h_i \in (g_i)} d_{(h_1,h_2)} x_{(h_1h_2)}.$$

Now, $x_{(g_1)} \cup x_{(g_2)} \in H^{2\iota_{(g_1)}+2\iota_{(g_2)}}(\mathcal{G})$. Since any h_1, h_2 in the above sum have $h_i \in (g_i)$, the degree-shifting numbers are the same, and we have $x_{(g_1)} \cup x_{(g_2)} \in H^{2\iota_{(h_1)}+2\iota_{(h_2)}}(\mathcal{G})$. On the other hand, $x_{(h_1h_2)} \in H^{2\iota_{(h_1h_2)}}(\mathcal{G})$. Thus, we clearly have $d_{(h_1,h_2)} = 0$ unless $\iota_{(h_1)} + \iota_{(h_2)} = \iota_{(h_1h_2)}$.

Furthermore, it suffices to assume that $\gamma = \mathrm{vol}_c((\mathbb{C}^n)^{h_1h_2})$ is a compactly-supported, $C(h_1h_2)$-invariant, top-dimensional form with volume 1 on $(\mathbb{C}^n)^{h_1h_2}$, since such a form generates the top compactly-generated cohomology of $\mathcal{G}_{((h_1h_2)^{-1})}$. So we are computing

$$\int_{(\mathbb{C}^n)^{h_1} \cap (\mathbb{C}^n)^{h_2}/C(h_1)\cap C(h_2)} e_3^*(\mathrm{vol}_c((\mathbb{C}^n)^{h_1h_2})) \wedge e(E).$$

Since $\mathrm{vol}_c((\mathbb{C}^n)^{h_1h_2})$ is a top form on $(\mathbb{C}^n)^{h_1h_2}$, though, and we're integrating it over the subspace

$$(\mathbb{C}^n)^{h_1} \cap (\mathbb{C}^n)^{h_2} \subset (\mathbb{C}^n)^{h_1h_2},$$

it will be of the wrong dimension unless the above inclusion is actually an equality. Thus, if

$$I_{g_1,g_2} = \{(h_1, h_2) \mid h_i \in (g_i), \iota_{(h_1)} + \iota_{(h_2)} = \iota_{(h_1h_2)}, (\mathbb{C}^n)^{h_1} \cap (\mathbb{C}^n)^{h_2} = (\mathbb{C}^n)^{h_1h_2}\},$$

then

$$x_{(g_1)} \cup x_{(g_2)} = \sum_{(h_1,h_2) \in I_{g_1,g_2}} d_{(h_1,h_2)} x_{(h_1h_2)}.$$

We can compute $d_{(h_1,h_2)}$ exactly as in the previous example. In summary:

In summary:

> The Chen-Ruan cohomology of $G \ltimes \mathbb{C}^n$, where G is a finite subgroup of $SL(n, \mathbb{C})$ is generated by $x_{(g)} \in H^0(\mathcal{G}_{(g)}) \subset H^{2\iota_{(g)}}_{CR}(\mathcal{G})$ for each conjugacy class $(g) \in G_*$. The multiplication is given by
>
> $$x_{(g_1)} \cup x_{(g_2)} = \sum_{(h_1,h_2) \in I_{g_1,g_2}} \frac{|C(h_1h_2)|}{|C(h_1) \cap C(h_2)|} \cdot x_{(h_1h_2)},$$
>
> where
>
> $$d_{(h_1,h_2)} = \tfrac{|C(h_1h_2)|}{|C(h_1)\cap C(h_2)|}.$$

Example: $\mathcal{G} = W\mathbb{P}(w_1, w_2)$

Recall that the weighted projective space $W\mathbb{P}(w_1, w_2)$ is an orbifold $\mathbb{S}^2$ with singular points $x = [1:0]$ and $y = [0:1]$ of orders w_1 and w_2, respectively. To compute the twisted sectors, let us consider the equivalence relation on $(g)_{G_p}$. Any class $(1)_{G_p}$ is equivalent to any other, giving the nontwisted sector. The only $(g)_{G_p}$ with $g \neq 1$ are $(g)_{G_x}$ and $(h)_{G_y}$. However, we cannot have $(g)_{G_x} \approx (g)_{G_y}$ for $g, h \neq 1$, because any path of linear charts joining x to y contains a chart with trivial isotropy. Furthermore, the same reasoning shows that $(g)_{G_x} \approx (g')_{G_x}$ if and only if $(g)_{G_x} = (g')_{G_x}$, and this is true if and only if $g = g'$, since G_x is abelian; the same holds for $(g)_{G_y}$.

Thus, there is a sector $\mathcal{G}_{(g)}$ for each $g \in \mathbb{Z}_{w_1} \cup \mathbb{Z}_{w_2}$, for a total of $(w_1 - 1) + (w_2 - 1)$ twisted sectors and one nontwisted sector. A twisted sector $\mathcal{G}_{(g)}$ with $g \in \mathbb{Z}_{w_1}$ looks like

$$\mathcal{G}_{(g)} = \bullet^{\mathbb{Z}_{w_1}}.$$

To compute the degree-shifting number, notice that if $g = e^{2\pi i a/w_1}$ for $1 \leq a < w_1$, then g acts on a chart $U_x \cong \mathbb{C}$ by multiplication, and hence its derivative is equal to itself. It follows that $\rho_p(g) = e^{2\pi i a/w_1}$, so that $\iota_{(g)} = a/w_1$.

So the Chen-Ruan cohomology, as a vector space, has

$$H^*_{CR}(\mathcal{G}) = H^*(\mathcal{G}) \oplus \bigoplus_{(g) \neq 1 \in \mathbb{Z}_{w_1} \cup \mathbb{Z}_{w_2}} \mathbb{C}_{(2\iota_{(g)})},$$

where our notation indicates that the copy of $\mathbb{C}$ indexed by (g) lies in degree $2\iota_{(g)}$. In particular, the nontrivial cohomology lies in degrees 0 and 2 (from $H^*(\mathcal{G}) = H^*(S^2)$) as well as degrees $2i/w_1$ and $2j/w_2$ for $1 \leq i < w_1$ and $1 \leq j < w_2$.

Let $\alpha^k \in H^{2k/w_1}_{CR}(\mathcal{G}) = H^0(\mathcal{G}_{e^{2\pi i k/w_1}})$ and $\beta^\ell \in H^{2\ell/w_2}_{CR}(\mathcal{G}) = H^0(\mathcal{G}_{e^{2\pi i \ell/w_2}})$ denote the constant 1 functions on the various twisted sectors. These clearly generate the Chen-Ruan cohomology. What are the relations among them?

First, note that $\alpha \cup \beta = 0$. Indeed, $\alpha \cup \beta$ is defined by the relation

$$\langle \alpha \cup \beta, \gamma \rangle = \langle \alpha, \beta, \gamma \rangle = \sum_{(g_1, g_2, g_3) \in T^0_3} \int_{\mathcal{G}_{(g_1, g_2 \cdot g_3)}} e_1^* \alpha \wedge e_2^* \beta \wedge e_3^* \gamma$$

for $\gamma \in H^*_{CR}(\mathcal{G}$. But e_i maps $\mathcal{G}_{(g_1, g_2, g_3)}$ to $\mathcal{G}_{(g_i)}$, so in order to get a nonzero integral above we must have $g_1 = e^{2\pi i/w_1}$ and $g_2 = e^{2\pi i/w_2}$, but of course, there is no such twisted sector $\mathcal{G}_{(g_1, g_2, g_3)}$.

Next, we claim that $\alpha^{k_1} \cup \alpha^{k_2} = \alpha^{k_1 + k_2}$ if $k_1 + k_2 < w_1$ (the same proof shows that the analogous property holds for the β classes). To prove this, we need to show that

$$\langle \alpha^{k_1 + k_2}, \gamma \rangle = \langle \alpha^{k_1}, \alpha^{k_2}, \gamma \rangle$$

for all $\gamma \in H^*_{CR}(\mathcal{G})$. Since

$$\alpha^{k_1 + k_2} \in H^2(k_1 + k_2)/w_1{}_{CR}(\mathcal{G}),$$

the left-hand side of the above is zero

$$\gamma \in H^{2-(k_1+k_2)/w_1}_{CR}(\mathcal{G}) = H^{2(w_1 - k_1 - k_2)/w_1}_{CR}(\mathcal{G}) = H^0(\mathcal{G}_{e^{2\pi i (w_1 - k_1 - k_2)/w_1}}).$$

One checks easily that the same constraint holds for the right-hand side. Thus, we may as well assume that $\gamma = \alpha^{w_1-k_1-k_2}$, since this generates the relevant cohomology group. In this case,

$$\langle \alpha^{k_1+k_2}, \gamma \rangle = \int_{\mathcal{G}_{(e^{2\pi i(k_1+k_2)/w_1})}} \alpha^{k_1+k_2} \wedge I^* \alpha^{w_1-k_1-k_2} = 1,$$

whereas

$$\langle \alpha^{k_1}, \alpha^{k_2}, \gamma \rangle = \int_{\mathcal{G}_{(e^{2\pi i k_1/w_1}, e^{2\pi i k_2/w_1}, e^{2\pi i(w_1-k_1-k_2)/w_1})}} e_1^* \alpha^{k_1} \wedge e_2^* \alpha^{k_2} \wedge e_3^* \alpha^{w_1-k_1-k-2} \wedge e(E_{(\mathbf{g})}) = 1,$$

since in each case, every factor of the integrand is equal to 1.

The same reasoning shows that $\alpha^{w_1} = \beta^{w_2}$, both being equal to the generator of $H^2_{CR}(\mathcal{G}; \mathbb{C}) = H^2(S^2) = \mathbb{C}$. We have thus proved the following:

The Chen-Ruan cohomology of $W\mathbb{P}(w_1, w_2)$, where w_1 and w_2 are coprime integers, is generated as a ring by

$$1 \in H^0_{CR}(\mathcal{G}; \mathbb{C})) = H^0(\mathcal{G}) = H^0(S^2),$$
$$\alpha \in H^{2/w_1}_{CR}(\mathcal{G}; \mathbb{C}) = H^0(\mathcal{G}_{e^{2\pi i/w_1}})$$

and

$$\beta \in H^{2/w_2}_{CR}(\mathcal{G}; \mathbb{C}) = H^0(\mathcal{G}_{e^{2\pi i/w_2}}),$$

with the relations

$$\alpha \cup \beta = 0,\ \alpha^{w_1} = \beta^{w_2},\ \alpha^{w_1+1} = \beta^{w_2+1} = 0.$$