A Model Reference Adaptive Control Approach for Uncertain Dynamical Systems with Strict Component-wise Performance Guarantees

Mario Luca Fravolini
University of Perugia, Perugia, Italy

Ehsan Arabi and Tansel Yucelen
University of South Florida, Tampa, FL, USA

A critical problem for adaptive control systems is the characterization of the system response during transients. In fact a major issue in adaptive system design is the inability to achieve, a-priori, non-conservative user-defined performance guarantees. At present most of the available analysis tools provide performance bounds depending on the norm of uncertain quantities. Since it is extremely difficult to quantify these quantities, conservative upper bounds are used in their place; these, in turn, produce conservative performance bounds of limited practical utility.

To face these problems some of the authors have recently introduced a set-theoretic adaptive controller based on generalized restricted potential functions. The key feature of this approach is that it allows the norm of the tracking error to be less than a-priori user-defined worst-case performance bound, and hence, it has the capability to enforce strict performance guarantees. Since this performance is expressed as function of the norm of the error vector it is not possible to have the direct control on the amplitude of the single error components.

In this paper the method is improved by allowing the control of the shape of the performance (ellipsoidal) set that is guaranteed to contain the tracking error trajectories. The design problem is formalized as a linear optimization with LMI constraints that allows specifying independent componentwise requirements for the error components.

Different linear optimization cost functions have been evaluated with the purpose of computing the largest ellipsoidal domain contained in an a-priori specified tracking error polyhedral domain and the smallest ellipsoidal domain containing an a-priori specified ellipsoidal initial condition set. A detailed simulation study in the aeronautic context has been used to highlight the efficacy of the method and the role of the different design parameters.

I. Introduction
Adaptive control systems are well-known for their capability in compensating for large uncertainties, partial failures, exogenous disturbances and sudden changes in system operating point\(^1\). Despite these attractive features, a major issue is the inability to achieve a-priori, user-defined performance guarantees with adaptive control algorithms. Considering Model Reference Adaptive Controllers (MRAC), although a large number of adaptation laws and robust modifications have been proposed for years to guarantee the asymptotic convergence of the system to the model

---

\(^1\) M.L. Fravolini is an Associate Professor of the Engineering Department at the University of Perugia, Perugia, Italy (Email: mario.fravolini@unipg.it).

\(^2\) E. Arabi is a Graduate Research Assistant of the Department of Mechanical Engineering, University of South Florida, Tampa, FL, USA (Email: ehsanarabi@mail.usf.edu).

\(^3\) T. Yucelen is an Assistant Professor of the Department of Mechanical Engineering, University of South Florida, Tampa, FL, USA (Email: yucelen@lacis.team).

\(^*\) This research was supported in part by the National Aeronautics and Space Administration under Grant NNX15AM51A.
states, the problem of predicting the tracking error during transients and the problem of relating adaptive controller parameters to time domain specifications is still, to a large extent, an open area of research.\(^{3,4}\)

To face these challenges, the current practice to perform validation of adaptive control algorithms relies either on systematic closed loop testing or on the development of specific analytical tools to perform worst-case analysis. Systematic system testing, typically, is a time consuming task, furthermore this strategy provides performance guarantees only for what has been tested and the generalization to a different dynamic context may result unreliable due to the nonlinear and involved relation between the dynamics of the uncertain system and the dynamics of the adaptation algorithm.

As for existing analytical tools for designing and validating adaptive control algorithms, over the years a substantial research effort has been devoted toward the development of robust adaptation strategies featuring improved transitory performance.\(^{5,6,7,8,9,10,11,12}\) In these research efforts the adaptation schemes were developed with the purpose of minimizing performance metrics such as the \(L_1\), \(L_2\) and \(L_\infty\) norm of the error with the effect of improving, in general terms, the response of the system; however, there was never a clear relationships with the a-priori specified boundedness region for the error.

Actually, a limited number of design approaches is today available to guarantee specific strict time domain requirements on the transitory response of the adaptive systems. Some interesting results are reported in Refs. 13,14,15,17,18,19

Recently, some of the authors have proposed in Ref. 20 a Model Reference Adaptive Control architecture with strict performance guarantees under the assumption that system uncertainties do not depend on time. The approach is based on restricted potential functions (also known as barrier Lyapunov functions). The extension to the time-varying case to handle exogenous disturbances and changes in system dynamics was proposed in Ref. 21.

The key feature of the MRAC schemes proposed in Refs. 20,21 is that these guarantee the norm of the tracking error vector be less than a-priori, user-defined worst-case performance bound that does not depend on any unknown system parameter. Therefore, it is possible to establish a guaranteed worst-case performance using the proposed architecture at the pre-design stage. This is a significant difference from the existing, standard MRAC approaches, because in this case the error bounds, typically, depend on unknown parameters. Although in standard MRAC upper bounds on the norm of the transitory error can be theoretically computed as function of the initial error and of the uncertainty size, these bounds are always extremely conservative and, therefore, of scarce utility from an engineering point of view.

A specific feature of the guaranteed performance bounds in Refs. 20,21 is that these are expressed as functions of the norm of the error vector. Specifically, the adopted barrier Lyapunov functions force the error trajectories to evolve within an ellipsoidal domain whose shape matrix is derived, indirectly, from the solution of a Lyapunov equation. As a result it in not immediate to manage the shape of this ellipsoidal domain along the different directions of the error. This aspect is relevant especially for large dimensional systems where the knowledge of the norm of the error vector provides only a rough upper bound of the actual value of the error components.

In the present study the design approaches in Refs 20,21 are extended by allowing the designer to define, strict component-wise tracking error requirements so that he has the direct control on the shape of the performance set. This was achieved by fixing, at design stage, a desired polyhedron that defines performance requirements along any direction in the error space. Then, Linear Matrix Inequality (LMI) conditions are derived to constraint the ellipsoidal set within the polyhedral domain and, at the same time, to be external to a specified ellipsoidal initial condition set. The computation of a suitable ellipsoidal set satisfying these design requirements was cast as a convex optimization problem with LMI constraints. Specifically, we evaluated two optimization problems. The first problem deals with the maximization of the "size" (volume) of the ellipsoid contained in the performance polyhedron and the second deals with the minimization of the "size" of the ellipsoid containing a pre specified initial condition set. The above constrained convex optimization problems can be efficiently solved using standard optimization software.\(^23\)

The optimized design of the performance domain is illustrated through a detailed study taken from the aeronautic context where a strict performance MRAC controller is designed starting from predefined strict componentwise tracking error requirements.

### A. Notation

The notation used in this paper is standard. Specifically, \(\mathbb{N}\) and \(\mathbb{R}\) denotes the set of natural and real numbers respectively, \(\mathbb{R}^m\) denotes the set of \(n\times m\) real column vectors, \(\mathbb{R}^{m\times n}\) denotes the set of \(n\times m\) real matrices; \(\mathbb{R}_+\) and \(\mathbb{R}_-\) denotes, respectively, the set of positive and non-negative-definite real numbers; \(\mathbb{R}_+^{m\times n}\) and \(\mathbb{R}_-^{m\times n}\) denotes, respectively, the set of \(n\times n\) positive-definite and non-negative-definite real matrices; \(\mathbb{D}^{m\times n}\)denotes the set of \(n\times n\) real diagonal matrices,
and "\(\triangleq\)" denotes equality by definition. In addition, we write \(\|x\|_2\) for the Euclidean norm. Furthermore, we write \(\|A\|_2 \triangleq \sqrt{\lambda_{\text{max}}(A^T A)}\) for the weighted Euclidean norm of \(x \in \mathbb{R}^n\) with the matrix \(A \in \mathbb{R}^{n \times n}, \|A\|_2 \triangleq \sqrt{\lambda_{\text{max}}(A^T A)}\) for the induced 2-norm of the matrix \(A \in \mathbb{R}^{n \times n}\), \(\lambda_{\text{min}}(A)\) and \(\lambda_{\text{max}}(A)\) for the minimum and (respectively) for the maximum eigenvalue of the matrix \(A \in \mathbb{R}^{n \times n}\), \(\text{tr}(\cdot)\) for the trace operator, and \(\bar{x}\) and \(\overline{x}\) for the lower bound (respectively) upper bound of a bounded signal \(x(t) \in \mathbb{R}^n\), that is, \(\bar{x} \leq \|x(t)\|_2\), and \(\|x(t)\|_2 \leq \overline{x}\).

II. Standard Model Reference Adaptive Control Overview

In this section, we present an overview on the standard model reference adaptive control problem. Consider the uncertain dynamical system given by

\[
\dot{x}_p(t) = A_p x(t) + B_p \Lambda(u(t) + B_p \delta_p(t, x_p(t))), \quad x_p(0) = x_{p0},
\]

where \(x_p(t) \in \mathbb{R}^n\), is the (measurable) state vector, \(u(t) \in \mathbb{R}^m\), is the control input, \(A_p \in \mathbb{R}^{n \times n}\) and \(B_p \in \mathbb{R}^{n \times m}\) are known system and input matrices respectively, \(\delta_p: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^m\) is a system uncertainty, \(\Lambda \in \mathbb{R}^{m \times m}\) is an unknown control effectiveness matrix and the pair \((A_p, B_p)\) is assumed to be controllable.

Assumption-1

We now introduce a quite standard assumption, see for instance\(^3,4\), to characterize the uncertainty in adaptive control systems, that is \(\delta_p\) is parameterized as follows

\[
\delta_p(t, x_p) = W_p(t) \sigma_p(x_p),
\]

where \(W_p(t) \in \mathbb{R}^{m \times m}\), is an unknown but bounded weight matrix (i.e., \(W_p(t)\|_2 \leq w_p\)) with a bounded time rate of change (\(\dot{W}_p(t)\|_2 \leq \dot{w}_p\)) and \(\sigma_p: \mathbb{R}^m \rightarrow \mathbb{R}^m\) is a known vector of basis functions of the form \(\sigma_p(x_p) = [\sigma_{p1}(x_p), \sigma_{p2}(x_p), \ldots, \sigma_{pn}(x_p)]\).

Remark-1: Uncertainty parameterization (2) can be used to capture time-varying changes in system dynamics due to system reconfiguration, deployment of a payload, docking, or structural damage. By letting the first element of the basis function to be a constant (i.e., \(\sigma_{p1}(x_p) = 1\)), then this parameterization is also sufficient to capture exogenous disturbances depending on time such as winds or turbulent flows.

We now consider the control law given by

\[
u(t) = u_{n1}(t) + u_{n2}(t) + u_a(t),
\]

where \(u_{n1}(t) \in \mathbb{R}^m\) is the nominal (linear) feedback contributions, \(u_{n2}(t) \in \mathbb{R}^m\) is feedforward reference contribution and \(u_a(t) \in \mathbb{R}^m\) is the adaptive control contribution. In more details the \(u_{n1}(t)\)contribution is defined as

\[
u_{n1}(t) = -K_1 x(t)
\]

such that \(A_p \triangleq A - BK\), and \(K \in \mathbb{R}^{m \times m}\) is a Hurwitz matrix. To guarantee, zero steady state tracking error (in nominal conditions) of the reference signals \(c(t) \in \mathbb{R}^m\), the \(u_{n2}(t)\) contribution is selected as

\[
u_{n2}(t) = K_2 c(t)
\]

where \(K_2 \triangleq -(CA^{-1}B)^{-1}\) and \(C\) is proper state selection matrix and \(B_2 \triangleq BK_2\). Using (2,3,4) and (5) in (1) yields

\[
\dot{x}(t) = A_p x(t) + B_p c(t) + B A [u_a(t) + W(t) \sigma(x(t))], \quad x(0) = x_0.
\]
where $W(t) \triangleq \begin{bmatrix} W^T_{p}(t) \frac{1}{(\Lambda^{-1} - I_{n+m})}K_1 - (\Lambda^{-1} - I_{n+m})K_2 \end{bmatrix}$, is an unknown (aggregated) weight matrix and $\sigma(x(t)) \triangleq \begin{bmatrix} \sigma_p(x_p(t))^T, x(t)^T, c(t)^T \end{bmatrix}^T \in \mathbb{R}^{s+n+m}$, is a known (aggregated) basis function vector. The adaptive contribution is defined as

$$u_a(t) = -\hat{W}(t)\sigma(x(t))$$

(7)

where $\hat{W}(t) \in \mathbb{R}^{(s+n+m)\times m}$, is the estimate of the unknown $W(t)$ vector, satisfying the update law

$$\dot{\hat{W}}(t) = \gamma \text{Proj}_m(\hat{W}(t), \sigma(x(t))e(t)^T PB)$$

(8)

where $\gamma \in R_+$ is the (constant) learning rate (i.e., adaptation gain) and $\text{Proj}_m(\cdot)$ is the standard matrix weights projection operator defined, for instance in Refs 4,20,21 that is used to guarantees the boundedness of the norm $\|\hat{W}(t)\| \leq \hat{W}_{max}$ (where $\hat{W}_{max}$ is a user defined upper bound) despite the presence of unstructured uncertainties and persistent disturbances acting on the system. The matrix $P \in \mathbb{R}^{n×n}$ in (8) is derived from the solution of the Lyapunov equation given by

$$A_rP + PA_r = -R$$

(9)

with $R \in \mathbb{R}^{n×n}$, and $e(t) \triangleq x(t) - x_r(t)$, is the system tracking error where $x_r(t) \in \mathbb{R}^n$ is the reference state vector, which is produced by the linear reference model given by

$$\dot{x}_r(t) = A_r x_r(t) + B_r c(t), \quad x_r(0) = x_{r0},$$

(10)

Using (7), and considering the dynamic equations (6) and (10), the system error dynamics is given by

$$\dot{e}(t) = A_e e(t) - B e \hat{W}(t)\sigma(x(t)), \quad e(0) = e_0,$$

(11)

where $\hat{W}(t) \triangleq \hat{W}(t) - W(t) \in \mathbb{R}^{(s+n+m)}$, is the weight estimation error and $e_0 \triangleq x_0 - x_{r0}$. Note that the assumption that $W(t)\| \leq \kappa$ and $\hat{W}(t)\| \leq \hat{\kappa}$, automatically holds as a direct consequence of the properties of the uncertainty given by Assumption-1.

The closed loop stability of the error dynamics (11) with the parameter adaptation law (8) was proved in Ref. 21 using the Lyapunov stability theory for nonlinear systems using the quadratic Lyapunov function candidate

$$V(e, \hat{W}) = e^P e + \gamma^{-1} tr(\hat{W} \Lambda^{1/2})^T (\hat{W} \Lambda^{1/2})$$

In particular in Ref. 21 it was shown that the tracking error is Ultimate Bounded (UB) and satisfies the worst-case upper bound.

$$\|e(t)\|_2 \leq \sqrt{\frac{2\eta^2 + \gamma^{-1}\hat{\kappa}^2 \| \Lambda \|_2}{\lambda_{max}(P)}}$$

(12)

where $\eta \triangleq \frac{2\gamma^{-1}\hat{\kappa} \| \Lambda \|_2}{\lambda_{min}(R)}$ and $\hat{\kappa} = \hat{W}_{max} + \kappa$.

Considering the upper bound on the norm of the tracking error in (12), the following observations can be drawn. The bound (12) does not solely depend on user-defined design parameters but also on the upper bounds of the norm of unknown uncertain matrices such as $W$ and $\Lambda$. It is extremely difficulty to compute tight and reliable upper bounds for the norm of $W$ and $\Lambda$, it follows that extremely conservative values are often used in the evaluation of (12). The immediate consequence is that the resulting tracking error bound (12) for the $\|e(t)\|_2$ is overly conservative and, often, of scarce utility from a validation and verification standpoint. This last consideration motivated the development of the set-Theoretic MRAC approach proposed in Ref. 21 and summarized in the following section.
III. Set-Theoretic Model Reference Adaptive Control

In this section, the set-theoretic model reference adaptive control architecture is summarized that allows the system error bound between the state of the uncertain dynamical system (1) and the state of a linear reference model (10) to be less than a-priori, user-defined worst-case performance bound in the presence of time-varying exogenous disturbances and system uncertainties\textsuperscript{20,21} (see also, Refs. 24, 25, 26, 27, 28). For this purpose, we introduce the following definitions.

**Definition-1.**
Let \( \| z \|_H = \sqrt{z^T H z} \) be a weighted Euclidean norm, where \( z \in \mathbb{R}^p \) is a real column vector and \( H \in \mathbb{R}^{p \times p} \). We define \( \phi(\| z \|_H) : \mathbb{R}^p \to \mathbb{R} \), to be a generalized restricted potential function (generalized barrier Lyapunov function) on the set \( D_c \triangleq \{ \| z \|_H : \| z \|_H \leq \varepsilon \} \), with \( \varepsilon \in \mathbb{R}_+ \) being an a-priori, user-defined constant, if the following statements hold:

- If \( \| z \|_H = 0 \), then \( \phi(\| z \|_H) = 0 \).
- If \( \| z \|_H \in D_c \) and \( \| z \|_H \neq 0 \), then \( \phi(\| z \|_H) > 0 \).
- If \( \| z \|_H \to \varepsilon \), then \( \phi(\| z \|_H) \to \infty \).
- \( \phi(\| z \|_H) \) is continuously differentiable on \( D_c \).
- If \( \| z \|_H \in D_c \), then \( 2\phi(\| z \|_H) - \phi(\| z \|_H^2) \geq \frac{d\phi(\| z \|_H)}{d\| z \|_H} \).

(13)

**Remark-2:** Note that the above definition generalizes the definition of the restricted potential functions (barrier Lyapunov functions) used by the authors of Refs. 15, 16, 17. A candidate generalized restricted potential function satisfying the conditions given in Definition-1 has the form

\[
\phi(\| z \|_H) = \frac{\| z \|_H^2}{\varepsilon - \| z \|_H^2}, \quad \| z \|_H \in D_c.
\]

(14)

Next, consider the augmented uncertain dynamical system given by (6) with the feedback control law in (7) and define the system error as

\[
e(t) \triangleq x(t) - x_r(t)
\]

(15)

and the weight estimation error as

\[
\bar{W}(t) \triangleq \hat{W}(t) - W(t)
\]

(16)

In Ref. 21 it has been proposed the (set-theoretic) update law based on generalized restricted potential functions given by

\[
\dot{\hat{W}}(t) = \gamma \text{Proj}_m \left( \hat{W}(t), \phi_e(\| e(t) \|_p) \sigma(x(t))e^T(t)PB \right), \quad \hat{W}(0) = \hat{W}_0.
\]

(17)

where \( \text{Proj}_m(\cdot) \) is the matrix projection operator defined in Refs. 4, 21, \( \gamma \in \mathbb{R}_+ \) is the learning rate, \( P \in \mathbb{R}^{n \times n} \) is the solution of the Lyapunov equation given by (9) with \( R \in \mathbb{R}^{n \times n} \), with \( x_r(t) \in \mathbb{R}^n \) being the reference state vector, which satisfies the linear reference model given by (10). Note that \( \phi_e(\| e(t) \|_p) \) in (17) can be viewed as an error dependent learning rate. Similarly to the standard MRAC control the overall system and the weight estimation error dynamics is

\[
\dot{e}(t) = A_e e(t) - B \hat{W}(t)^T \sigma(x(t)), \quad e(0) = e_0, \quad (18)
\]

\[
\dot{\hat{W}}(t) = \gamma \text{Proj}_m \left( \hat{W}(t), \phi_e(\| e(t) \|_p) \sigma(x(t))e(t)^T PB \right) - \hat{W}(t), \quad \hat{W}(0) = \hat{W}_0.
\]

(19)

American Institute of Aeronautics and Astronautics
The closed loop set-theoretic properties of the uncertain adaptive system can be derived by exploiting the theorem-1 originally derived in Ref. 21 and reported below.

**Theorem 1** [21] - Consider the uncertain dynamical system given by (6), the modelling uncertain subject to Assumption 1, the linear reference model given by (10), and the control law given by (3). If \( \| e(0) \|_\rho < \epsilon \), then the closed-loop trajectories of the dynamical system given by (18) and (19) are bounded, and the bound on the system error strictly satisfies an a-priori given, user-defined worst-case performance

\[
\| e(t) \|_\rho < \epsilon
\]

(20)

If, in addition, the unknown weight matrix in (2) is constant, then \( \lim_{t \to \infty} e(t) = 0 \).

In Theorem 1 the update law given by (17) for the set-theoretic model reference adaptive control architecture was derived considering the following energy function

\[
V(e, \bar{W}) = \phi(\| e(t) \|_\rho) + \gamma^{-1} \text{tr}[(\bar{W} \Lambda^{1/2})^T (\bar{W} \Lambda^{1/2})]
\]

where

\[
D_\epsilon \triangleq \{ \| e(t) \|_\rho: \| e(t) \|_\rho < \epsilon \},
\]

(22)

and \( P \in \mathbb{R}^{n \times n}_+ \) is a solution of the Lyapunov equation (9) with \( R \in \mathbb{R}^{n \times n}_+ \). Note that \( V(0,0) = 0 \) and that \( V(e, \bar{W}) > 0 \) for all \( (e, \bar{W}) \neq (0,0) \). In Ref. 21 it was shown that

\[
\dot{V}(e(t), \bar{W}(t)) \leq -\frac{1}{2} \alpha V(e, \bar{W}) + \mu,
\]

(23)

where \( \alpha \triangleq \frac{\lambda_{\min}(R)}{\lambda_{\max}(P)} \), \( d \triangleq 2\gamma^{-1} \bar{W} \hat{w} \| \Lambda \|_2 \), and \( \mu \triangleq \frac{1}{2} \alpha \gamma^{-1} \bar{w}^2 \| \Lambda \|_2 + d \).

By applying Lemma 1 of Refs 16 and 17, one can now conclude the boundedness of the closed-loop dynamical system given by (18) and (19) as well as the strict performance bound on the system error given by (20).

**Remark-3:** It is important to note that compared with the upper bounds on the norm of the system error given by (12) for the (standard) MRAC problem, the worst-case performance bound given by (20) not only is strict but also solely depends on a-priori given, user-defined design parameter \( \epsilon \). That is, since (20) does not depend on the unknown matrices \( \bar{W} \) and \( \Lambda \), unlike (20), this performance bound is a-priori computable at the pre-design stage, which yields to a guaranteed system performance.

**IV. Design constraints for the performance set \( D_\epsilon \)**

In the previous sections it has been shown that the proposed adaptive control guarantees the evolution of the tracking error \( e(t) \) within the domain \( D_\epsilon \) defined in (22). Typically the \( D_\epsilon \) shape matrix \( P \) is derived from the solution of the Lyapunov equation (9) for a fixed \( R \in \mathbb{R}^{n \times n}_+ \) (often it is selected \( R=I \)), therefore it is not immediate to have the control of the size of the set \( D_\epsilon \) along the different components of the tracking error vector. In this section, we show how the set \( D_\epsilon \) can be shaped at the design stage to meet specific componentwise performance requirements on the tracking error components. Specifically, the design problem is cast as an optimization problem with LMI constraints. To simplify the following analysis it is somewhat convenient to work with the ellipsoidal set \( \Omega_\epsilon \) that descends immediately from \( D_\epsilon \) and is defined as follows

\[
\Omega_\epsilon \triangleq \{ e(t): \| e(t) \|_\rho^2 = e^T(t) Pe(t) \leq \epsilon^2 \}
\]

(24)
At this point we introduce a set of LMI conditions for the fulfillment of componentwise error requirements for the set $\Omega_e$. In the following discussion the LMI conditions will be expressed as function of the matrix $Q$ such that $Q = P^{-1}$.

### A. Requirements on the Lyapunov inequality

The first condition descends from the Lyapunov condition (9). Here, instead of fixing the matrix $R$ and deriving $Q$ from the solution of the Lyapunov equation, the matrix $P = Q^{-1} \in \mathbb{R}^{n\times n}$ is assumed to be a free design matrix that has to satisfy the Lyapunov inequality $A_r P + PA_r^T < 0$. This implies that the matrix $P$ can be derived to satisfy specific design objectives. Expressing these above inequality requirements as function of the $Q$ matrix, the following LMI conditions results

$$Q > 0$$

$$A_r Q + QA_r^T < 0$$

### B. Requirements on the tracking error components

In this study the *componentwise* tracking error requirements for the error components are formalized as follows:

$$|e_i(t)| \leq e_{Mi}, \quad i = 1,...,n$$

where $e_{Mi}$ is the desired maximum error for the $i$-th error component. The requirements in (27) define the following tracking error performance (box) domain $\Pi_e$

$$\Pi_e \triangleq \{e : [-e_{M1};+e_{M1}] \times [-e_{M2};+e_{M2}] \times \ldots [-e_{Mn};+e_{Mn}]\}$$

To satisfy the tracking error requirements (27), we require that the ellipsoid $\Omega_e$ is contained within the polyhedron $\Pi_e$. Exploiting geometric considerations it can be shown that the condition $\Omega_e \subset \Pi_e$ holds if the following set of LMIs are satisfied

$$\begin{bmatrix} Q / e^2 & Qg_i \\ (Qg_i)^T & e_{Mi}^2 \end{bmatrix} > 0 \quad i = 1,...,n$$

where $g_i = [0,...,1,...,0]^T$ is the $i$th base vector of $\mathbb{R}^n$. Sometimes, the purpose of the design is to reduce, as much as possible, the size of the original performance set $\Pi_e$, in this case it is also useful to introduce a scaled performance domain $\Pi_{e\rho}$ defined as

$$\Pi_{e\rho} \triangleq \{e : \rho_1[-e_{M1};+e_{M1}] \times \rho_2[-e_{M2};+e_{M2}] \times \ldots \rho_n[-e_{Mn};+e_{Mn}]\}$$

where the $\rho_i$ are scalar (scaling) parameters that will be determined in the optimized design. Similarly to LMI (29) the condition $\Omega_{e\rho} \subset \Pi_{e\rho}$ is given by the following set of LMIs

$$\begin{bmatrix} Q / e^2 & Qg_i \\ (Qg_i)^T & \rho_i^2 e_{Mi}^2 \end{bmatrix} > 0 \quad i = 1,...,n$$

### C. Requirements on the initial condition for the tracking error $e(0)$

To take into account possible initialization errors on the tracking errors at $t=0$, (for instance due to measurement noise) we defined an initial condition domain $\Omega_{e0}$ such that $e(0) \in \Omega_{e0}$. In the design we require the set $\Omega_e$ to be “large enough” to contain the a-priori specified initial condition domain $\Omega_{e0}$. In this study we considered an ellipsoidal shape for the domain $\Omega_{e0}$ defined as

$$\Omega_{e0} \triangleq \{e(0) : e^T(0)P_0e(0) \leq 1\}$$
where $P_0$ is the ellipsoid shape matrix defined at design stage. We require that $\Omega_{e0} \subseteq \Omega_e$. Exploiting geometric considerations it can be shown\(^\text{22}\) that the containment of the ellipsoid $\Omega_{e0}$ in another ellipsoid $\Omega_e$ is guaranteed if the following LMI condition is satisfied

$$
\begin{bmatrix}
-P_0 \tau & 0 & I \\
0 & -1 + \tau & 0 \\
I & 0 & -\epsilon^2 Q
\end{bmatrix} < 0
$$

(33)

where $\tau$ is an arbitrary positive scalar. In this study the matrix $P_0$ is defined as

$$
R_0 = \begin{bmatrix}
\epsilon_{01}^2 & 0 & 0 \\
0 & \ldots & 0 \\
0 & 0 & \epsilon_{0n}^2
\end{bmatrix}^{-1}
$$

(34)

where the $\epsilon_{0i}$, define the length of the semi axis of the ellipsoid $\Omega_{e0}$ along the $i$th error component.

V. Optimized design of the set $\Omega_e$

Thanks to the fact that the design requirements formulated in section IV are all expressed as LMIs, we are in the position to formulate the design of the set $\Omega_e$ as a convex optimization problem whose optimization variables are the free parameters and matrices involved in the LMIs. Two optimized design problems were considered in the present study.

A. Maximum size ellipsoid contained in the performance domain $\Pi_e$

In this case the optimization objective is the maximization of the "size" of the set $\Omega_e$ that has to be contained within the performance domain $\Pi_e$; further the set $\Omega_e$ has to be external to the initial conditions set $\Omega_{e0}$. To measure the size of the set $\Omega_e$, we exploit the property that the trace of the matrix $Q$ is equal to the sum of the squares of the semi axis lengths of the ellipsoid $\Omega_e$\(^\text{29}\), therefore the maximization of the size of the set $\Omega_e$ is formulated as the following optimization problem:

$$
\begin{align*}
\text{Maximize} & \quad J = \text{sum}(\text{trace}(Q)) \\
\text{subject to} & \quad (25, 26, 29, 33)
\end{align*}
$$

(35)

Problem (35) is a standard linear optimization problem with LMI constraints and can be solved efficiently with many existing optimization software. In this study we have used the LMI toolbox of Matlab\(^\text{29}\).

B. Minimum size ellipsoid containing the initial condition set $\Omega_{e0}$

In this case the optimization objective is the minimization of the "size" of the set $\Omega_e$ that is contained in the scaled performance set $\Pi_{e0}$; further, the set $\Omega_e$ has to be external to the initial conditions set $\Omega_{e0}$. The problem is formalized as the following optimization problem

$$
\begin{align*}
\text{Minimize} & \quad J = \sum_{i=1}^{n} \rho_i \\
\text{subject to} & \quad (25, 26, 31, 33)
\end{align*}
$$

(36)

Also problem (36) is a standard linear optimization problem with LMI constraints and can be solved using the same optimization software used for problem (35).

Remark-4: Minimization problem (36) can have also a different interpretation. In fact considering $\Omega_{e0}$ as an a-priori defined (minimum size) set that has to be contained within the domain $\Omega_e$, then the solution of the optimization problem (36) provides a minimum size $\Omega_e$ such that $\Omega_{e0} \subseteq \Omega_e$.
VI. Design Example

The proposed design methodology was applied to the short period longitudinal dynamics of an F16 aircraft model under MRAC control in the presence of matched and control efficiency uncertainty. It was considered the dynamical model reported in Ref. 30 that is

\[
\begin{bmatrix}
\dot{x}_1(t) \\
\dot{x}_2(t)
\end{bmatrix} =
\begin{bmatrix}
-1.01 & 0.90 \\
0.82 & -1.07
\end{bmatrix}
\begin{bmatrix}
x_1(t) \\
x_2(t)
\end{bmatrix} +
\begin{bmatrix}
-0.0022 \\
-0.1756
\end{bmatrix}
A\left(\delta_p(x,t) + u_{a1}(t) + u_{a2}(t) + u_\delta(t)\right)
\]

where \(x_1(t)\) is the angle of attack \(\alpha(t)\) (rad), \(x_2(t)\) is the pitch rate \(\dot{q}(t)\) (rad/sec), \(u_{a1}(t)\), \(u_{a2}(t)\) and \(u_\delta(t)\) are adaptive and linear control contributions respectively. The matched uncertainty \(\delta(x,t)\) is linearly parameterized as \(\delta(x,t) = \sin(0.25t) + x_1(t) - 1.5x_2(t)\) and the control efficiency is \(\Lambda = 0.3\). A baseline tracking controller for \(\alpha(t)\) was designed (in nominal conditions) using the pole placement technique by placing the poles of the matrix \(A_r = A - BK_r\) in \(s=[-0.7\, -1]\). This design produced a linear feedback controller \(K_1 = [4.7432\,\ -2.3163]\) and the feedforward gain \(K_2 = -([1,0]A_r^{-1}B)^{-1} = -4.3396\) was selected to guarantee a reference/output gain equal to one; finally \(B_r = BK_2\). The range for the reference command is \(|c(t)| \leq r_c = 5\) deg. As for the weight adaptation algorithm, the adaptation rate in (17) was fixed at \(\gamma = 1\) and the maximum projection weights norm was fixed at \(W_{\text{max}} = 2\).

The component-wise requirements for the tracking error are \(|e_1| \leq 0.01\) rad and \(|e_2| \leq 0.015\) rad/sec; the resulting performance box is \(\Pi_e \triangleq \{e: \ [-0.01; +0.01]\times[-0.015; +0.015]\}\). The initial condition set \(Po\) was specified using (34) where \(e_{01} = 0.2e_{M1}\) and \(e_{02} = 0.8e_{M2}\).

A. Design of the optimized ellipsoid \(\Omega_e\)

In the first study it was computed the maximum size ellipsoid solving the optimization problem (35). In this study the free parameter \(\epsilon\) that defines the set \(\Omega_e\) was selected equal to one \((\epsilon = 1)\). Fig. 1a shows the relevant sets. It can be observed that all the design requirements are satisfied, that is the set \(\Omega_e\) is external to \(\Omega_{e0}\) and it is also tangent to the \(\Pi_e\) domain, that is the \(\Omega_e\) set is the largest ellipsoidal set contained in \(\Pi_e\). Then it was solved the optimization problem (36) that computes the minimum size ellipsoid containing the initial condition set \(\Omega_{e0}\) whose results are shown in Fig. 1b. In this case it is observed that the scaled performance domain \(\Pi_{e0}\) reduces as much as possible the size of the original set \(\Pi_e\), in fact \(\Pi_{e0}\) is tangent to \(\Omega_e\), that in turn, is tangent to \(\Omega_{e0}\). The scaling coefficients that characterize the set \(\Pi_{e0}\) are \(\rho_1 = 0.64\) and \(\rho_2 = 0.80\) showing that it is possible to reduce significantly the size of the original performance domain \(\Omega_e\) along both ror components.
achieved cases achieved for the same input reference command.

Finally, case bounds resulting high gain, for the case ϵ=0.1 they are also feasible for any ϵ>0. On the other hand ϵ has a direct influence on the norm of the matrix P. In particular, the larger is ϵ the larger is ||P||₂.

Since P and ϵ enter directly in the adaptation law (17), these have a direct impact on the closed loop performance. To emphasize this aspect we show in Figs.2, 3 and 4 the results archived for ϵ=0.01, ϵ=0.1 and ϵ=1 respectively (derived from the solution of problem (35). Analyzing these figures it is evident that the best tracking performance is achieved for ϵ=1. As for the amplitude of the control signal u(t); this results comparable in the three cases. In more details for ϵ=0.01 we observed high peaks in the time varying adaptation rate φ(t(⋅)) that, in turns, causes sudden variations in the control signal u(t); this negative behavior is less evident for ϵ=0.1 and ϵ=1.

For ϵ=0.1 and ϵ=1 small amplitude and high frequency oscillations appear in the control signal that are typical of high gain controls. In fact for ϵ=0.01, ϵ=0.1 and ϵ=1 there is a remarkable difference in the norm of the matrix P, resulting ||P||₂=1.005, ||P||₂=100.5 and ||P||₂=1.005e+04 respectively. This fact highlights that the value of ϵ has to be selected with care.

Figs. 5, 6 and 7 show the tracking error components for e₁(t) and e₂(t) as long as the performance ||e(t)||ₚ for ϵ=0.01, ϵ=0.1 and ϵ=1 respectively. Analyzing these figures it is evident that in all cases the componentwise tracking error bounds are satisfied for e₁(t) and e₂(t). In the case ϵ=0.01, the tracking error amplitude are significantly larger that in case ϵ=0 1 and ϵ=1.

Finally, Fig. 8 shows the optimized ellipsoids achieved for ϵ=0.01, ϵ=0.1 and ϵ=1 and closed loop error trajectories achieved for the same input reference command. It is observed that the shape of the 3 ellipsoids is the same in the 3 cases, while the tracking error is remarkably different. As explained this behavior is due to the difference of the ||P||₂ achieved in the 3 cases.

B. The role of parameter ϵ and time domain results

The set Ωₚ in (24) is uniquely defined by the matrix P and by the free parameter ϵ. The value of ϵ has not influence on the feasibility of problems (35) and (36) in the sense that if the problems are feasible for a certain ϵ>0 these are also feasible for any ϵ>0. On the other hand ϵ has a direct influence on the norm of the matrix P. In particular, the larger is ϵ the larger is ||P||₂.

Since P and ϵ enter directly in the adaptation law (17), these have a direct impact on the closed loop performance. To emphasize this aspect we show in Figs.2, 3 and 4 the results archived for ϵ=0.01, ϵ=0.1 and ϵ=1 respectively (derived from the solution of problem (35). Analyzing these figures it is evident that the best tracking performance is achieved for ϵ=1. As for the amplitude of the control signal u(t); this results comparable in the three cases. In more details for ϵ=0.01 we observed high peaks in the time varying adaptation rate φ(t(⋅)) that, in turns, causes sudden variations in the control signal u(t); this negative behavior is less evident for ϵ=0.1 and ϵ=1.

For ϵ=0.1 and ϵ=1 small amplitude and high frequency oscillations appear in the control signal that are typical of high gain controls. In fact for ϵ=0.01, ϵ=0.1 and ϵ=1 there is a remarkable difference in the norm of the matrix P, resulting ||P||₂=1.005, ||P||₂=100.5 and ||P||₂=1.005e+04 respectively. This fact highlights that the value of ϵ has to be selected with care.

Figs. 5, 6 and 7 show the tracking error components for e₁(t) and e₂(t) as long as the performance ||e(t)||ₚ for ϵ=0.01, ϵ=0.1 and ϵ=1 respectively. Analyzing these figures it is evident that in all cases the componentwise tracking error bounds are satisfied for e₁(t) and e₂(t). In the case ϵ=0.01, the tracking error amplitude are significantly larger that in case ϵ=0 1 and ϵ=1.

Finally, Fig. 8 shows the optimized ellipsoids achieved for ϵ=0.01, ϵ=0.1 and ϵ=1 and closed loop error trajectories achieved for the same input reference command. It is observed that the shape of the 3 ellipsoids is the same in the 3 cases, while the tracking error is remarkably different. As explained this behavior is due to the difference of the ||P||₂ achieved in the 3 cases.
VII. Conclusions

In this paper it has been show how the design of a previously introduced set-theoretic Model Reference Adaptive Controller for uncertain nonlinear systems can be improved by controlling, at design stage, the componentwise shape of the ellipsoidal set that is guaranteed to contain the closed loop tracking error trajectories. The design problem was formalized as a linear optimization problem with LMI constraints that allows specifying independent componentwise requirements for the tracking error components. Two linear optimization costs function have been proposed with the purpose of computing the largest invariant ellipsoid contained in a priori specified tracking error bounding box domain (first method); and the smallest invariant ellipsoid containing an a-priori specified ellipsoidal initial condition set (second method). The problem can be efficiently solved with on the shelf optimization software and can be used as effective design and analysis tool for performance oriented MRAC controllers. A detailed simulation study in the aeronautic context has clearly heighted the efficacy of the method and the role of the different design parameters.

Fig.2 - Tracking performance for $x_1(t)$ and $x_2(t)$ for $\epsilon=0.01$. It also shown the control signal $u(t)$ and the corresponding time varying adaptation rate $\phi(t)$. 
Fig. 3 - Tracking performance for $x_1(t)$ and $x_2(t)$ for $\epsilon=0.1$. It also shows the control signal $u(t)$ and the corresponding time varying adaptation rate $\phi_d(t)$.

Fig. 4 - Tracking performance for $x_1(t)$ and $x_2(t)$ for $\epsilon=1$. It also shows the control signal $u(t)$ and the corresponding time varying adaptation rate $\phi_d(t)$.
Fig. 5 - Tracking error components for $e_1(t)$ and $e_2(t)$; and $\|e(t)\|_p$ for $\epsilon = 0.01$.

Fig. 6 - Tracking error components for $e_1(t)$ and $e_2(t)$; and $\|e(t)\|_p$ for $\epsilon = 0.1$. 
Fig. 7 - Tracking error components for $e_1(t)$ and $e_2(t)$; and $\|e(t)\|_p$ for $\epsilon = 1$.

Fig. 8 - Optimized ellipsoid $\Omega_\epsilon$ and the tracking error trajectories (with $e(0)=0$) for $\epsilon=0.01$ (a), $\epsilon=0.1$ (b) and $\epsilon=1$ (c).
References