A Generalization to Set-Theoretic Model Reference Adaptive Control Architecture for Enforcing User-Defined Time-Varying Performance Bounds

Ehsan Arabi and Tansel Yucelen

Abstract—It is a challenge to achieve user-defined performance guarantees while utilizing model reference adaptive control laws in the feedback loop. To this end, we recently introduced a set-theoretic model reference adaptive control framework. The key feature of this approach allows the distance between the state of an uncertain dynamical system and the state of a reference model (i.e., the system error) to be less than a user-defined constant performance bound.

In this paper, the set-theoretic model reference adaptive control framework is generalized to enforce user-defined time-varying performance bounds on the system error, which gives user the flexibility to control the closed-loop system performance as desired on different time intervals (e.g., transient time interval and steady-state time interval). For this purpose, an architecture is proposed for adaptive command following, where two numerical examples are provided to illustrate the efficacy of the proposed contribution.

I. INTRODUCTION

A. Literature Review

A reference model, an update law, and a controller are the main ingredients of model reference adaptive control architectures. Briefly, a desired closed-loop system performance can be captured by the reference model, where its output or state is compared with the output or state of an uncertain dynamical system to form a system error. This system error drives the update law and then the controller adapts feedback gains to suppress the system error using the information received from the update law. It is a challenge to achieve user-defined performance guarantees while utilizing model reference adaptive control laws in the feedback loop, although these controllers have the capability to cope with adverse effects resulting from exogenous disturbances and system uncertainties. To this end, the authors of this paper recently introduced a set-theoretic model reference adaptive control framework [1] and [2].

It should be noted that the key feature of the set-theoretic model reference adaptive control framework allows the system error to be less than a user-defined performance bound. As compared with other notable contributions in the literature enforcing similar performance constraints [3]–[11], the set-theoretic framework proposed in [1] and [2] does not assume that the control signals can access every element of the state vector as in [3], does not assume that a desired trajectory along with its derivatives are all available as in [4], enforces user-defined performance bounds not only on measurable output signals as in [5] but also on measurable state signals, and does not use a backstepping procedure for adaptive control as in [6]–[11]. In addition, it should be noted that the results in [2] go beyond the ones in [1] in that both the exogenous disturbances and the system uncertainties can depend on time in [2] to capture dynamic environment conditions and changes in system dynamics. However, a common denominator of [1] and [2] is that both papers assume the user-defined performance bound to be constant.

B. Contribution

In this paper, we generalize the set-theoretic model reference adaptive control framework to enforce user-defined time-varying performance bounds on the system error. This generalization gives user a flexibility to control the closed-loop system performance as desired on different time intervals (for example, transient time interval and steady-state time interval). Specifically, an architectures is proposed for adaptive command following in the presence of exogenous disturbances and system uncertainties. This architecture presents a direct approach through designing a new control architecture to enforce user-defined time-varying performance bounds. The proposed architecture’s efficacy is further illustrated via two detailed numerical examples.

II. NECESSARY PRELIMINARIES

A. Notation

We start with the notation. In particular, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^n$ denotes the set of $n \times 1$ real column vectors, $\mathbb{R}^{n \times m}$ denotes the set of $n \times m$ real matrices, $\mathbb{R}_+$ (respectively, $\mathbb{R}_+$) denotes the set of positive (respectively, non-
negative-definite) real numbers, $\mathbb{R}^{n \times n}_+$ (respectively, $\mathbb{R}^{n \times n}_+$) denotes the set of $n \times n$ positive-definite (respectively, non-negative-definite) real matrices, $\mathbb{D}^{n \times n}$ denotes the set of $n \times n$ real matrices with diagonal scalar entries, $0_{n \times n}$ denotes the $n \times n$ zero matrix, and “$\triangleq$” denotes equality by definition. 

In addition, we write $(\cdot)^T$ for the transpose operator, $(\cdot)^{-1}$ for the inverse operator, $\text{tr}(\cdot)$ for the trace operator, and $\| \cdot \|_2$ for the Euclidean norm. Furthermore, we write

$$
\|x\|_A \triangleq \sqrt{x^T A x},
$$

(1)

for the weighted Euclidean norm of $x \in \mathbb{R}^n$ with the matrix $A \in \mathbb{R}^{n \times n}$, $\|A\|_2 \triangleq \sqrt{\lambda_{\text{max}}(A^T A)}$ for the induced 2-norm of the matrix $A \in \mathbb{R}^{n \times n}$, $\lambda_{\text{min}}(A)$ (resp., $\lambda_{\text{max}}(A)$) for the minimum (resp., maximum) eigenvalue of the matrix $A \in \mathbb{R}^{n \times n}$, and $\varpi$ (resp., $\varpi$) for the lower bound (resp., upper bound) of a bounded signal $x(t) \in \mathbb{R}^n$, $t \geq 0$, that is, $\varpi \leq \|x(t)\|_2$, $t \geq 0$ (resp., $\|x(t)\|_2 \leq \varpi$, $t \geq 0$).

### B. Projection Operator

We now introduce a definition of the projection operator, which is adopted in this paper as well as in our early result documented in [2]. For this purpose, let

$$
\Omega = \{ \theta \in \mathbb{R}^n : (\theta_i^{\text{min}} \leq \theta_i \leq \theta_i^{\text{max}})_{i=1,2,\ldots,n} \},
$$

(2)

be a convex hypercube in $\mathbb{R}^n$, where $(\theta_i^{\text{min}}, \theta_i^{\text{max}})$ represent the minimum and maximum bounds for the $i$th component of the $n$-dimensional parameter vector $\theta$. Additionally, for a sufficiently small positive constant $\nu$, a second hypercube is defined by

$$
\Omega_\nu = \{ \theta \in \mathbb{R}^n : (\theta_i^{\text{min}} + \nu \leq \theta_i \leq \theta_i^{\text{max}} - \nu)_{i=1,2,\ldots,n} \},
$$

(3)

where $\Omega_\nu \subset \Omega$. Then, the projection operator $\text{Proj} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is defined component-wise by

$$
\text{Proj}(\theta, y) \triangleq \left\{ \begin{array}{ll}
(\frac{\theta_i - \theta_i^{\text{min}}}{\nu}, y_i), & \text{if } \theta_i > \theta_i^{\text{max}} - \nu \\
(\theta_i, y_i), & \text{if } \theta_i < \theta_i^{\text{min}} + \nu \\
y_i, & \text{otherwise}
\end{array} \right.
$$

(4)

where $y \in \mathbb{R}^n$ [12]. Based on this definition, note [12], [13]

$$
\text{Proj} (\theta - \theta^*)^T (\text{Proj}(\theta, y) - y) \leq 0, \quad \theta^* \in \mathbb{R}^n.
$$

(5)

This definition can be further generalized to matrices as $\text{Proj}_{\text{m}}(\Theta, Y) = (\text{Proj}(\text{col}_1(\Theta), \text{col}_1(Y)), \ldots, \text{Proj}(\text{col}_m(\Theta), \text{col}_m(Y)))$, where $\Theta \in \mathbb{R}^{n \times m}$, $Y \in \mathbb{R}^{n \times m}$ and $\text{col}_i(\cdot)$ denotes the $i$th column operator. In this case, for a given $\Theta^* \in \mathbb{R}^{n \times m}$, it follows from (5) that

$$
\text{tr} \left[ (\Theta - \Theta^*)^T (\text{Proj}_{\text{m}}(\Theta, Y) - Y) \right] = \sum_{i=1}^m \left[ \text{col}_i((\Theta - \Theta^*)^T (\text{Proj}(\text{col}_i(\Theta), \text{col}_i(Y)) - \text{col}_i(Y)) \right] \leq 0.
$$

### C. Generalized Restricted Potential Functions

A key ingredient for the set-theoretic model reference adaptive control framework proposed in this paper as well as in our early approaches documented in [1] and [2] is generalized restricted potential functions. To define such functions, let

$$
\|z\|_H = \sqrt{z^T H z},
$$

(6)

be a weighted Euclidean norm, where $z \in \mathbb{R}^p$ is a real column vector and $H \in \mathbb{R}^p \times p$. We define $\phi(\|z\|_H)$, $\phi : \mathbb{R}^p \rightarrow \mathbb{R}$, to be a generalized restricted potential function (generalized barrier Lyapunov function) on the set

$$
\mathcal{D}_\epsilon \triangleq \{ \|z\|_H : \|z\|_H \in [0, \epsilon] \},
$$

(7)

with $\epsilon \in \mathbb{R}_+$ being a user-defined constant, if the following statements hold:

i) If $\|z\|_H = 0$, then $\phi(\|z\|_H) = 0$.

ii) If $\|z\|_H \in \mathcal{D}_\epsilon$ and $\|z\|_H \neq 0$, then $\phi(\|z\|_H) > 0$.

iii) If $\|z\|_H \rightarrow \infty$, then $\phi(\|z\|_H) \rightarrow \infty$.

iv) $\phi(\|z\|_H)$ is continuously differentiable on $\mathcal{D}_\epsilon$.

v) If $\|z\|_H \in \mathcal{D}_\epsilon$, then $\phi_d(\|z\|_H) > 0$, where

$$
\phi_d(\|z\|_H) \triangleq \frac{d \phi(\|z\|_H)}{d \|z\|_H}.
$$

(8)

vi) If $\|z\|_H \in \mathcal{D}_\epsilon$, then

$$
2\phi_d(\|z\|_H) \|z\|_H^2 - \phi(\|z\|_H) > 0.
$$

(9)

As noted in [2], this definition generalizes the definition of the restricted potential functions (barrier Lyapunov functions) used by the authors of [1], [6]–[11].

### III. PROBLEM FORMULATION

We first introduce a formulation for the adaptive command following problem in this section in the presence of exogenous disturbances and systems uncertainties. We then briefly overview the set-theoretic model reference adaptive control architecture proposed in [2].

#### A. A Formulation on Adaptive Command Following

Here, we consider the uncertain dynamical system

$$
\dot{x}_p(t) = A_p x_p(t) + B_p u(t) + B_p \delta_p(t, x_p(t)),
$$

(10)

where $x_p(t) \in \mathbb{R}^{n_p}$, $t \geq 0$, is the measurable state vector, $u(t) \in \mathbb{R}^m$, $t \geq 0$, is the control input, $A_p \in \mathbb{R}^{n_p \times n_p}$ is a known system matrix, $B_p \in \mathbb{R}^{n_p \times m}$ is a known input matrix, $\delta_p : \mathbb{R}_+ \times \mathbb{R}^{n_p} \rightarrow \mathbb{R}^m$ is a system uncertainty, $\Lambda \in \mathbb{R}^{m \times m}$ is an unknown control effectiveness matrix, and the pair $(A_p, B_p)$ is controllable.
We now introduce a standard system uncertainty parameterization. Specifically, the system uncertainty given by (10) is parameterized as
\[ \delta_p(t, x_p) = W_p^T(t)\sigma_p(x_p), \]  
(11)
where \( W_p(t) \in \mathbb{R}^{s \times m}, t \geq 0 \), is a bounded unknown weight matrix (i.e., \( \|W_p(t)\|_2 \leq w_p, t \geq 0 \)) with a bounded time rate of change (i.e., \( \|W_p(t)\|_2 \leq w_p, t \geq 0 \)) and \( \sigma_p : \mathbb{R}^{np} \to \mathbb{R}^s \) is a known basis function of the form
\[ \sigma_p(x_p) = [\sigma_{p1}(x_p), \sigma_{p2}(x_p), \ldots, \sigma_{ps}(x_p)]^T. \]  
(12)
Note that by letting the first element of the basis function be a constant; that is,
\[ \sigma_{p1}(x_p) = b, \]  
(13)
the parameterization given by (11) also captures exogenous.

To address command following, let \( c(t) \in \mathbb{R}^{n_c}, t \geq 0 \), be a given bounded piecewise continuous command and \( x_c(t) \in \mathbb{R}^{n_c}, t \geq 0 \), be the integrator state satisfying
\[ \dot{x}_c(t) = E_{x_p}x_p(t) - c(t), \quad x_c(0) = x_{c0}, \quad t \geq 0, \]  
(14)
where \( E_{x_p} \in \mathbb{R}^{n_c \times np} \) allows the selection of a subset of \( x_p(t) \), \( t \geq 0 \), to follow \( c(t) \), \( t \geq 0 \). Using (10) and (14), we write
\[ \dot{x}(t) = Ax(t) + B\Lambda u(t) + BW_p^T(t)\sigma_p(x_p(t)) + B_c c(t), \]  
(15)
where
\[ x(t) = [x_p^T(t), x_c^T(t)]^T \in \mathbb{R}^n, \quad t \geq 0, \quad n = n_p + n_c, \]  
(16)
is the augmented state vector, \( x_0 = [x_{p0}^T, x_{c0}^T]^T \).

We next consider the feedback control law given by
\[ u(t) = u_n(t) + u_a(t), \quad t \geq 0, \]  
(20)
where \( u_n(t) \in \mathbb{R}^m, t \geq 0 \), and \( u_a(t) \in \mathbb{R}^m, t \geq 0 \), are the nominal and adaptive control laws, respectively. Furthermore, let the nominal control law be
\[ u_n(t) = -Kx(t), \quad t \geq 0, \]  
(21)
such that
\[ A_r \equiv A - BK, \quad K \in \mathbb{R}^{m \times n}, \]  
(22)
is Hurwitz. Using (20) and (21) in (15) yields
\[ \dot{x}(t) = A_r x(t) + B_r c(t) + B\Lambda [u_n(t) + W_p^T(t)\sigma_p(x(t))], \]  
(23)
where
\[ W(t) \triangleq [\Lambda^{-1}W_p^T(t), (\Lambda^{-1} - I_{m \times m})K]^T \in \mathbb{R}^{(s + n) \times m}, \quad t \geq 0, \]  
(24)
is an unknown (aggregated) weight matrix and
\[ \sigma(x(t)) \triangleq [\sigma_p^T(x_p(t)), x_t^T(t)]^T \in \mathbb{R}^{s + n}, \quad t \geq 0, \]  
(25)
is a known (aggregated) basis function. Considering (23), let the adaptive control law be
\[ u_a(t) = -\dot{W}(t)\sigma(x(t)), \quad t \geq 0, \]  
(26)
where \( \dot{W}(t) \in \mathbb{R}^{(s + n) \times m} \) is an estimate of \( W(t) \).

Note that the theoretical development of an update law to construct the estimate \( \hat{W}(t), t \geq 0 \), is crucial in any model reference adaptive control design for achieving desired command following characteristics captured by the reference model given by
\[ \dot{x}_r(t) = A_r x_r(t) + B_r c(t), \quad x_r(0) = x_{r0}, \quad t \geq 0, \]  
(27)
with \( x_r(t) \in \mathbb{R}^n, t \geq 0 \), being the reference state vector. Although there are many update law candidates for this purpose, they do not achieve user-defined performance guarantees for the adaptive command following problem formulated in this section. As discussed, a notable exception is entitled as set-theoretic model reference adaptive control architecture [2], where we next briefly overview this result.

### B. Set-Theoretic Model Reference Adaptive Control: A Concise Overview

Following the architecture presented in [2], consider the update law for (26) given by
\[ \dot{\hat{W}}(t) = \gamma_1 \text{Proj}_m\left(\hat{W}(t), \phi_d(\|e(t)\|_p)\sigma(x(t))e^T(t)PB\right), \]  
(28)
with \( \hat{W}_{\text{max}} \) being the projection norm bound. In (28), additionally, \( \gamma_1 \in R_+ \) is the learning rate (i.e., adaptation gain), \( P \in R_+^{n \times n} \) is a solution of the Lyapunov equation given by
\[ 0 = AP + PA_r + R, \]  
(29)
where \( R \in R_+^{n \times n} \), and
\[ e(t) \triangleq x(t) - x_r(t), \quad t \geq 0, \]  
(30)
is the system error.

Next, one can write the system error dynamics and the weight estimation error dynamics respectively as
\[ \dot{e}(t) = A_r e(t) - B\Lambda W_p^T(t)\sigma(x(t)), \quad e(0) = e_0, \quad t \geq 0, \]  
(31)
\[ \dot{\hat{W}}(t) = \gamma_1 \text{Proj}_m\left(\hat{W}(t), \phi_d(\|e(t)\|_p)\sigma(x(t))e^T(t)PB\right), \]  
(32)
where
\[ \dot{W}(t) \triangleq \dot{W}(t) - W(t), \quad t \geq 0, \] (33)
is the weight estimation error. Note that \( \phi_d(\|e(t)\|_P) \) in (28) can be viewed as an error dependent learning rate and \( \|W(t)\|_2 \leq w, t \geq 0 \), and \( \|\dot{W}(t)\|_2 \leq \dot{w}, t \geq 0 \), automatically holds.

The update law given by (28) for the set-theoretic model reference adaptive control architecture can be derived by considering the following energy function
\[ V(e, \dot{W}) = \phi(\|e\|_P) + \gamma_1^{-1} \text{tr} \left[ (\dot{W} A_{1/2})^T (\dot{W} A_{1/2}) \right], \] (34)
where
\[ D_\epsilon \triangleq \{ \|e\|_P : \|e\|_P < \epsilon \}, \] (35)
and \( P \in \mathbb{R}_{++}^{n \times n} \) is a solution of the Lyapunov equation in (29) with \( R \in \mathbb{R}_{++}^{n \times n} \). Note that \( V(0, 0) = 0 \), \( V(e, \dot{W}) > 0 \) for \( (e, \dot{W}) \neq (0, 0) \), and
\[ \dot{V}(e(t), \dot{W}(t)) \leq -\frac{1}{2} \alpha V(e, \dot{W}) + \mu, \] (36)
where \( \alpha \triangleq \frac{\lambda_{\min}(R)}{\lambda_{\max}(P)} \), \( d \triangleq 2\gamma_1^{-1} \dot{w} \|A\|_2 \), and \( \mu \triangleq \frac{1}{2} \alpha \gamma_1^{-1} \dot{w}^2 \|A\|_2 + d \). By applying Lemma 1 of [8], [9], one can now conclude the boundedness of the closed-loop dynamical system given by (31) and (32) as well as the strict performance bound on the system error given by
\[ \|e(t)\|_P < \epsilon, \quad t \geq 0. \] (37)

IV. A GENERALIZATION TO ENFORCE TIME-VARYING PERFORMANCE BOUNDS IN SET-THEORETIC MODEL REFERENCE ADAPTIVE CONTROL

For generalizing the results in [2], which utilize constant performance bounds, we now present an architecture for adaptive command following problem overviewed in the previous section. In particular, this architecture presents a direct approach in that a new control architecture is designed to enforce user-defined time-varying performance bounds.

Consider the generalized restricted potential function
\[ \phi(\|z\|_H) = \frac{\|z\|_H^2}{\varepsilon^2(t) - \|z\|_H^2}, \quad \|z\|_H \in D_\epsilon, \] (38)
which has the partial derivative
\[ \phi_d(\|z\|_H) = \frac{\varepsilon^2(t)}{(\varepsilon^2(t) - \|z\|_H^2)^2} > 0, \quad \|z\|_H \in D_\epsilon, \] (39)
with respect to \( \|z\|_H^2 \) and
\[ 2\phi_d(\|z\|_H)\|z\|_H^2 - \phi(\|z\|_H) = \frac{\varepsilon^2(t)\|z\|_H^2 + \|z\|_H^4}{(\varepsilon^2(t) - \|z\|_H^2)^2} > 0, \]
\[ \|z\|_H \in D_\epsilon. \] (40)
It is clear that (38) satisfies all the generalized restricted potential function properties stated in Section 2 with \( \epsilon(t) \) being positive and bounded-away-from-zero.

Next, let the adaptive control law be given by
\[ u_n(t) = -\dot{W}^T(t) \sigma(x(t)) - v(t), \quad t \geq 0, \] (41)
where \( v(t) \in \mathbb{R}^m, t \geq 0 \), is an additive signal, and \( \dot{W}(t) \in \mathbb{R}^{(s+n) \times m}, t \geq 0 \), is the estimate of \( W(t), t \geq 0 \), satisfying the update law (28). Specifically, we let the additive signal \( v(t), t \geq 0 \), be
\[ v(t) = B^T P e(t) \tilde{q}(t) \frac{\|\tilde{e}(t)\|_P}{\epsilon(t)} \lambda_{\max}(P), \quad t \geq 0, \] (42)
where \( \tilde{q}(t) \in \mathbb{R}_+, t \geq 0 \), is the estimate of the unknown parameter
\[ q \triangleq \lambda_{\min}^{-1}(P B A B^T P), \] (43)
satisfying the adaptive parameter update law
\[ \dot{q}(t) = \gamma_2 \text{Proj} \left( \tilde{q}(t), \phi_d(\|e(t)\|_P)\|e(t)\|_P^2 \|\tilde{e}(t)\|_P^2 \right), \]
\[ \dot{q}(0) = \tilde{q}_0 \in \mathbb{R}_+, \quad t \geq 0, \] (44)
with \( \dot{q} \) being the minimum projection bound and the maximum projection bound respectively and \( \gamma_2 \in \mathbb{R}_+ \) being the learning rate.

Note that one can write the system error dynamics, the weight estimation error dynamics, and the adaptive parameter estimation error dynamics respectively as
\[ \dot{e}(t) = A e(t) - B A W^T(t) \sigma(x(t)) - B A v(t), \]
\[ e(0) = e_0, \quad t \geq 0, \] (45)
\[ \dot{\tilde{W}}(t) = \gamma_1 \text{Proj}_{\Omega}(\tilde{W}(t), \phi_d(\|e(t)\|_P)\sigma(x(t))e^T(t) PB) \]
\[ - \tilde{W}(t), \quad \tilde{W}(0) = \tilde{W}_0, \quad t \geq 0, \] (46)
\[ \dot{\tilde{q}}(t) = \gamma_2 \text{Proj} \left( \tilde{q}(t), \phi_d(\|e(t)\|_P)\|e(t)\|_P^2 \|\tilde{e}(t)\|_P^2 \right), \]
\[ \dot{\tilde{q}}(0) = \tilde{q}_0, \quad t \geq 0, \] (47)
where
\[ \tilde{q}(t) \triangleq \tilde{q}(t) - q, \quad t \geq 0, \] (48)
is the adaptive parameter estimation error. Here, we inherently assume \( \epsilon(t), t \geq 0 \), and \( \dot{\epsilon}(t), t \geq 0 \), are smooth and bounded user-defined functions.

**Theorem 1**: Consider the uncertain dynamical system given by (10) the reference model given by (27), and the feedback control law given by (20) along with (21), (41), (42), (28) and (44). If \( \|e_0\|_P < \epsilon(0) \), then the closed-loop dynamical system given by (45), (46), and (47) are bounded, where the bound on the system error satisfies the a-priori given, time-varying user-defined worst-case performance bound
\[ \|e(t)\|_P < \epsilon(t), \quad t \geq 0. \] (49)
Proof. Due to page limitations, a detailed proof showing all the key steps is omitted from this paper. Concisely, the result follows by considering the energy function \( V : \mathcal{D}_e \times \mathbb{R}^{(n+1) \times m} \times \mathbb{R} \rightarrow \mathbb{R}_+ \) given by
\[
V(e, \tilde{W}, \tilde{q}) = \phi(\|e\|_P) + \gamma_1^{-1} \text{tr}[\tilde{W}A^{1/2}T(WA^{1/2})] \\
\quad + \gamma_2^{-1} \tilde{q}^2 \lambda_{\text{max}}(P)\lambda_{\text{min}}(PBAB^TP),
\]
with
\[
\mathcal{D}_e \triangleq \{\|e(t)\|_P : \|e(t)\|_P < \epsilon(t)\},
\]
and \( P \in \mathbb{R}_+^{n \times n} \) being a solution of the Lyapunov equation in (29) subject to \( R \in \mathbb{R}_+^{n \times n} \), and calculating its time derivative along the closed-loop system trajectories (45), (46), and (47).

\[\square\]

V. NUMERICAL EXAMPLES

We now present two numerical examples to illustrate the approach presented in Theorem 1.

**Example 1:** We start simple. Consider a scalar uncertain dynamical system given by
\[
\dot{x}_p(t) = x_p(t) + \Lambda u(t) + W_p(t), \quad x(0) = 0, \quad t \geq 0,
\]
where \( \Lambda = 0.75 \) and \( W_p(t) = 2 \sin(0.5t) \). For command following, we let \( E_p = 1 \) in (14) and choose a linear nominal controller gain matrix
\[
K = [4, 3.2],
\]
in (21), which yields to the reference model given by
\[
\dot{x}_r(t) = \begin{bmatrix} -3 & -3.16 \end{bmatrix} x_r(t) + \begin{bmatrix} 0 \end{bmatrix} c(t), \quad x_r(0) = 0,
\]
\[ t \geq 0. \quad (54) \]

For the proposed set-theoretic model reference adaptive control architecture in Theorem 1, we use the generalized restricted potential function given by (38). We choose the smooth time-varying performance bound \( \epsilon(t), t \geq 0 \), such that it is large at the beginning of each step input and then gradually gets smaller prior to the end of each step input. Finally, we set the projection norm bounds imposed on each element of the weight estimate and adaptive parameter estimate to \( \tilde{W}_{\text{max}} = 3 \) and \( \tilde{q}_{\text{max}} = 10 \) respectively, and use \( R = I \) to calculate \( P \) from (29) for the resulting \( A_r \) matrix.

Note that the closed-loop dynamical system performance with the nominal controller does not give a satisfactory performance satisfying the given performance limits and the corresponding figures are omitted due to, once again, page restrictions. Next, we apply the proposed set-theoretic adaptive controller with \( \gamma_1 = 0.5 \) and \( \gamma_2 = 1 \) in Figure 1, where Figure 2 clearly illustrates its efficacy as well as validates the results of Theorem 1.

**Example 2:** We next consider an uncertain controlled wing rock dynamics model given by
\[
\dot{x}_p(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} x_p(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \left( \Lambda u(t) + \delta_p(t, x_p(t)) \right), \quad x_p(0) = 0 \quad t \geq 0,
\]
where \( x_p(t) = [x_{p1}(t) \ x_{p2}(t)]^T \) with \( x_{p1}(t) \) representing the roll angle (in rad) and \( x_{p2}(t) \) representing the roll rate (in rad/sec). In (55), \( \delta_p(t, x_p(t)) \) represents an uncertainty of the form
\[
\delta_p(t, x_p(t)) = \alpha_1 \sin(t) + \alpha_2 x_{p1} + \alpha_3 x_{p2} + \alpha_4 |x_{p1}| x_{p2} + \alpha_5 |x_{p2}| x_{p2} + \alpha_6 x_{p1}^3,
\]
with \( \alpha_1 = 0.25, \alpha_2 = 0.5, \alpha_3 = 1.0, \alpha_4 = -1.0, \alpha_5 = 1.0, \) and \( \alpha_6 = 1.0, \) and \( \Lambda = 0.75 \) represents an uncertain control effectiveness matrix. For command following, we let \( E_p = [1, 0] \) in (14) and choose a linear nominal controller gain matrix
\[
K = [5.49, 3.78, 2.89],
\]
in (21), which yields to the reference model given by
\[
\dot{x}_r(t) = \begin{bmatrix} 0 & 1 & 0 \\ -5.5 & -3.8 & -2.9 \end{bmatrix} x_r(t) + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} c(t), \quad x_r(0) = 0, \quad t \geq 0. \quad (58)
\]
In addition, we choose the basis function as
\[
\sigma(x) = [1, x_{p1}, x_{p2}, |x_{p1}|x_{p2}, |x_{p2}|x_{p2}, x_{p1}^3, x^T].
\]
(59)

For the proposed set-theoretic model reference adaptive control architecture in Theorem 1, we use the generalized restricted potential function given in (38). We choose the smooth time-varying performance bound \( \epsilon(t), t \geq 0 \), similar to that of Example 1. Finally, we set the projection norm bounds imposed on each element of the weight estimate and adaptive parameter estimate to \( \tilde{W}_{\text{max}} = 15 \) and \( \tilde{q}_{\text{max}} = 10 \) respectively, and use \( R = I \) to calculate \( P \) from (29) for the resulting \( A_r \) matrix.

Figures regarding the closed-loop dynamical system performance are not presented since the system response with the nominal controller is unstable. In Figure 3, we apply the proposed set-theoretic adaptive controller with \( \gamma_1 = 1 \) and \( \gamma_2 = 1 \), where Figure 4, once again, clearly illustrates its efficacy as well as the results of Theorem 1. \( \triangle \)

VI. CONCLUSION

For contributing to the previous studies on adaptive control systems, we generalized the set-theoretic model reference adaptive control framework to enforce user-defined time-varying performance bounds on the distance between
the state of an uncertain dynamical system and the state of a reference model (i.e., the system error). This generalization gives the user the flexibility to control the closed-loop system performance as desired on different time intervals (e.g., transient time interval and steady-state time interval).

REFERENCES


