Adaptive Set-Theoretic Emulator Reference Architecture (ASTERA): Control of Uncertain Dynamical Systems with Performance Guarantees and Smooth Transients

Tansel Yucelen

University of South Florida

Sivasubramanya Balakrishnan†

Missouri University of Science and Technology

Ehsan Arabi‡

University of South Florida

In control of uncertain dynamical systems with model reference adaptive control methods, simultaneously achieving user-defined performance guarantees and smooth transients is a challenging problem. Motivated from this point, an adaptive set-theoretic emulator reference architecture (ASTERA) is proposed in this paper. Specifically, the ASTERA has two major components — a state emulator and a set-theoretic adaptive controller. Based on a given reference model capturing a desired closed-loop system performance, the state emulator alters the trajectories of this model with the system error (the difference between the uncertain dynamical system state and the state emulator state) for the purpose of achieving smooth transients — transients that do not necessarily exhibit high-frequency oscillations (a property that does not necessarily exist in standard set-theoretic model reference adaptive control methods). In addition, the set-theoretic adaptive controller assures the weighted Euclidean norm of this system error to be less than a priori, user-defined scalar worst-case bound, which ensures the state emulator trajectories stay close to the given reference model trajectories as desired (a property that does not exist in standard state emulator-based standard adaptive control methods), for the purpose of achieving performance guarantees. An illustrative numerical example is also presented to complement the proposed theoretical contribution — the ASTERA.

I Introduction

A Literature Review

In control of uncertain dynamical systems with model reference adaptive control methods, simultaneously achieving user-defined performance guarantees and smooth transients is a challenging problem. Based on
a given reference model capturing a desired closed-loop system performance, these adaptive control methods minimize the difference between the uncertain dynamical system state and this model [1–3]. From a practical standpoint, however, this difference can be large during transient response. A classical solution to this problem is to judiciously increase the learning rate of the adaptive controller for achieving stringent performance guarantees. Yet, adaptive controllers with high learning rates may yield to signals with high-frequency oscillations — a property not desired in practice.

The contributions in [4–17] (see also references therein) present notable methods to address this challenge. In particular, the authors of [4] use a low-pass filter to reduce high-frequency oscillations due to high learning rates while achieving a desirable performance. In addition, the authors of [5, 6] present model reference adaptive control methods that allow high learning rates in achieving a desirable performance without significantly exciting high-frequency oscillations. The contributions in [7–9] respectively propose command governor, artificial basis functions, direct uncertainty minimization methods for achieving stringent performance without necessarily resorting to high learning rates in the adaptive controller.

The authors of [10–17] propose state emulator-based or state emulator-like methods to adaptive control. Specifically, based on a given reference model, the state emulator alters the trajectories of this model with the system error (the difference between the uncertain dynamical system state and the state emulator state) for the purpose of achieving smooth transients — transients that do not necessarily exhibit high-frequency oscillations. While these approaches are promising and inspiring, they do not have a-priori given, user-defined worst-case bounds on the system error. Thus, since one cannot strictly ensure if the state emulator trajectories stay close to the given reference model trajectories a-priori at the pre-design stage, there can be time instants where these approaches can exhibit smooth-yet-poor performance.

In fact, the common denominator of the contributions in [4–17] is that their respective worst-case system error bounds are conservative and not user-defined, since they are generally calculated using classical Lyapunov theory arguments that gives sufficient stability conditions only. To address this problem from a standard model reference adaptive control point, set-theoretic model reference adaptive control methods, which are predicated on a set-theoretic controller construction using generalized restricted potential functions (generalized barrier Lyapunov functions), are proposed in [18–24] (we refer to, for example, [Section 1, 19] for details on other adaptive control methods with strict performance guarantees such as [25–32]). In particular, the key feature of [18–24] is to guarantee the weighted Euclidean norm between the uncertain dynamical system state and the given reference model state to be less than a-priori, user-defined scalar worst-case bound, where this implies that they achieve strict performance guarantees. However, unlike the contributions in [4–17], these approaches can suffer from high-frequency oscillations since their system error dependent learning rate can be high for specific time instants to enforce the aforementioned user-defined scalar worst-case bound. Thus, there is a practical need to develop a new adaptive control method to simultaneously achieve user-defined performance guarantees and smooth transients.

B Contribution

In this paper, we propose an adaptive set-theoretic emulator reference architecture (ASTERA). The proposed architecture is a generalization of both set-theoretic model reference adaptive control methods [18–24] and state emulator-based or state emulator-like standard adaptive control methods [10–17] to simultaneously achieve user-defined performance guarantees and smooth transients. Specifically, the ASTERA has two major components — a state emulator and a set-theoretic adaptive controller. The state emulator alters the trajectories of a given reference model for the purpose of achieving smooth transients (a property that does not necessarily exist in standard set-theoretic model reference adaptive control methods). In addition, the set-theoretic adaptive controller assures the weighted Euclidean norm of the difference between the uncertain dynamical system state and the state emulator state to be less than a-priori, user-defined scalar worst-case bound, which ensures the state emulator trajectories stay close to the given reference model trajectories as desired (a property that does not exist in standard state emulator-based or state emulator-like standard adaptive control methods), for the purpose of achieving performance guarantees. An illustrative numerical example is also presented to complement the proposed theoretical contribution — the ASTERA.
Throughout this paper, we use a fairly standard notation. In particular, \( \mathbb{R} \) denotes the set of real numbers, \( \mathbb{R}^n \) denotes the set of \( n \times 1 \) real column vectors, \( \mathbb{R}^{n \times m} \) denotes the set of \( n \times m \) real matrices, \( \mathbb{R}_+ \) (respectively, \( \mathbb{R}_+^n \)) denotes the set of positive (respectively, nonnegative-definite) real numbers, \( \mathbb{R}^{n \times n}_+ \) (respectively, \( \mathbb{R}_+^{n \times n} \)) denotes the set of \( n \times n \) positive-definite (respectively, nonnegative-definite) real matrices, \( \mathbb{S}^{n \times n}_+ \) denotes the set of \( n \times n \) symmetric real matrices, \( \mathbb{D}^{n \times n} \) denotes the set of \( n \times n \) real matrices with diagonal scalar entries, and “\( \triangleq \)” denotes equality by definition. Furthermore, we write \( (\cdot)^{-1} \) for the inverse operator, \( \text{det}(\cdot) \) for the determinant operator, \( \text{tr}(\cdot) \) for the trace operator, and \( \| \cdot \|_2 \) for the Euclidean norm. We also write \( \lambda_{\min}(A) \) (respectively, \( \lambda_{\max}(A) \)) for the minimum (respectively, maximum) eigenvalue of the Hermitian matrix \( A \). \( \bar{x} \) (respectively, \( \underline{x} \)) for the lower bound (respectively, upper bound) of a bounded signal \( x(t) \in \mathbb{R}^n \), that is, \( \underline{x} \leq \| x(t) \|_2 \) (respectively, \( \| x(t) \|_2 \leq \bar{x} \)).

II Mathematical Preliminaries

A Uncertain Dynamical System

In this paper, we focus on a class of uncertain dynamical systems given by

\[
\dot{x}(t) = Ax(t) + B(Au(t) + W_0^T x(t)), \quad x(0) = x_0. \tag{1}
\]

In (1), \( x(t) \in \mathbb{R}^n \) is the measurable state vector, \( u(t) \in \mathbb{R}^m \) is the control input, \( A \in \mathbb{R}^{n \times n} \) is a known system matrix, \( B \in \mathbb{R}^{n \times m} \) is a known input matrix, \( W_0 \in \mathbb{R}^{n \times m} \) is an unknown weight matrix, and \( \Lambda \in \mathbb{R}^{m \times m}_+ \cap \mathbb{D}^{m \times m} \) is an unknown control effectiveness matrix. For the well-posedness of the problem considered in this paper, we implicitly assume that the pair \((A, B)\) is controllable.

Note that this paper’s results can be readily extended to a broader classes of uncertain dynamical systems. For example, the term “\( W_0^T x(t) \)” in (1) can be replaced with “\( W_0^T \omega(x(t)) \)” based on the works in, for example, [3, 8, 19, 33–36], where \( \omega(x(t)) \) is a known nonlinear basis function in this case. Furthermore, one can also relax the known assumption on \( \omega(x(t)) \) by resorting to universal function approximation tools such as neural networks and fuzzy logic algorithms as in [20, 37–41]. As another example, the term “\( \Lambda \)” in (1) can be relaxed with its nonlinear version under certain well-posedness assumptions based on the works in, for example, [42–44]. Thus, we focus on (1) without loss in generality and for brevity of the presentation.

B Augmenting Feedback Control

In this paper, we consider the feedback controller given by

\[
u(t) = u_n(t) + u_a(t). \tag{2}\]

In (2), the nominal controller \( u_n(t) \in \mathbb{R}^m \) augments the adaptive controller \( u_a(t) \in \mathbb{R}^m \). As standard, purpose of the nominal controller is to guarantee a desired closed-loop dynamical system performance in the absence of system uncertainties (i.e., \((W_0, \Lambda) \equiv (0, I))\). In particular, let the nominal controller be given by

\[
u_n(t) = -K_1 x(t) + K_2 c(t), \tag{3}\]

where \( K_1 \in \mathbb{R}^{m \times n} \) and \( K_2 \in \mathbb{R}^{m \times n_c} \) are design matrices and \( c(t) \) is a uniformly continuous bounded reference command. That is, when \((W_0, \Lambda) \equiv (0, I))\) (thus, \( u_n(t) \equiv 0 \) in the absence of system uncertainties), this selection of the nominal controller leads to desired closed-loop dynamical system trajectories given by

\[
\dot{x}(t) = A_r x(t) + B_r c(t), \quad \text{where} \ A_r \triangleq A - BK_1 \quad \text{and} \ B_r \triangleq BK_2.
\]

To suppress the effect of system uncertainties, we now design the adaptive controller. For this purpose,
using (2) in (1) gives
\[ \dot{x}(t) = Ax(t) + BA(u_n(t) + u_a(t) + \Lambda^{-1}W_0^T x(t)) \]
\[ = A_r x(t) + B_r c(t) + BA(u_n(t) + W^T \sigma(x(t), u_n(t))), \tag{4} \]
where \( W^T \triangleq [A^{-1}W_0^T, I - \Lambda^{-1}] \in \mathbb{R}^{(n+m) \times m} \) and \( \sigma(x(t), u_n(t)) \triangleq [x^T(t), u_n^T(t)]^T \in \mathbb{R}^{n+m} \). Motivated by the structure \( "u_a(t) + W^T \sigma(x(t), u_n(t))" \) in (4), let the adaptive controller be given by
\[ u_n(t) = -\hat{W}(t) \sigma(x(t), u_n(t)), \tag{5} \]
where \( \hat{W}(t) \in \mathbb{R}^{(n+m) \times m} \) is an estimate of the unknown weight matrix \( W \) satisfying an update law to be defined later. Specifically, using (5) in (4) gives
\[ \dot{x}(t) = A_r x(t) + B_r c(t) - B A \hat{W}(t) \sigma(x(t), u_n(t)), \tag{6} \]
where \( \hat{W}(t) \triangleq \hat{W}(t) - W \), and hence, the selection of the update law plays a crucial role to suppress the effect of system uncertainties.

**C State Emulator**

We now introduce the state emulator dynamics, which is also called as the modified reference model by the authors of [10,11,16], the observer-like reference model by the author of [12], the modified state observer by the authors of [10,11,16], the observer-like reference model by the author of [12], the modified state observer by the authors of [13,14], and the closed-loop reference model by the authors of [15]. In particular, consider the state emulator given by
\[ \dot{x}_r(t) = A_r x_r(t) + B_r c(t) + \kappa (x(t) - x_r(t)), \quad x_r(0) = x_{r0}, \tag{7} \]
where \( \kappa \in \mathbb{R}_+ \) is the state emulator design parameter.

The following observations are now immediate: i) As shown in [12,15] (also see other related references cited above), the selection of the state emulator design parameter \( \kappa \) is important for the purpose of achieving smooth transients with adaptive controller (5) — transients that do not necessarily exhibit high-frequency oscillations. ii) When the state emulator design parameter is zero (i.e., \( \kappa = 0 \)), (7) reduces to a given (i.e., ideal) reference model. However, this reference model can result in an adaptive closed-loop dynamical system with high-frequency oscillations. iii) For any positive state emulator design parameter (i.e., \( \kappa \in \mathbb{R}_+ \)), (7) converges to the given reference model at steady-state if the adaptive controller (5) with its update law guarantees \( \lim_{t \to \infty} (x(t) - x_r(t)) = 0 \). However, since the term \( "\kappa (x(t) - x_r(t))" \) in (7) modifies the given reference model, the state emulator trajectories can be far different than the reference model trajectories during the transient response — for time instants, for example, when the system error \( c(t) \triangleq x(t) - x_r(t) \) is large. In addition, if the adaptive controller (5) employs a standard update law, it cannot enforce a-priori, user-defined worst-case bound to the system error. Thus, since one cannot strictly upper bound the last term on the right hand side of (7) for guaranteeing the state emulator trajectories staying close to the given reference model trajectories, there can be time instants where the state emulator-based adaptive controllers can exhibit smooth-yet-poor performance.

To elucidate this point, consider a standard and widely-adopted update law given by
\[ \dot{\hat{W}}(t) = \gamma \text{Proj} \left( \hat{W}(t), \sigma(x(t), u_n(t)) (x(t) - x_r(t))^T PB \right), \quad \hat{W}(0) = \hat{W}_0, \tag{8} \]
where \( \gamma \in \mathbb{R}_+ \) is the learning rate, \( P \in \mathbb{R}_+^{n \times n} \) is the solution to the Lyapunov equation given by
\[ 0 = A_r^T P + PA_r + R, \tag{9} \]
for \( R \in \mathbb{R}_+^{n \times n} \), and Proj is the projection operator defined next [3,45].
Definition 1 (Projection Operator). Let $\Omega = \{ \theta \in \mathbb{R}^n : (\theta_i^{\min} \leq \theta_i \leq \theta_i^{\max})_{i=1,2,\ldots,n} \}$ be a convex hypercube in $\mathbb{R}^n$, where $(\theta_i^{\min}, \theta_i^{\max})$ represent the minimum and maximum bounds for the $i$th component of the $n$-dimensional parameter vector $\theta$. Additionally, for a sufficiently small positive constant $\nu$, define the second hypercube as $\Omega_\nu = \{ \theta \in \mathbb{R}^n : (\theta_i^{\min} + \nu \leq \theta_i \leq \theta_i^{\max} - \nu)_{i=1,2,\ldots,n} \}$, where $\Omega_\nu \subset \Omega$. With $y \in \mathbb{R}^n$, the projection operator $\text{Proj} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is then defined component-wise by $\text{Proj}(\theta, y) = \left( \frac{\theta - \theta^{\min} - \nu}{\nu} \right)y_i$ if $\theta_i > \theta_i^{\max} - \nu$ and $y_i > 0$, $\text{Proj}(\theta, y) = \left( \frac{\theta - \theta^{\min} + \nu}{\nu} \right)y_i$ if $\theta_i < \theta_i^{\min} + \nu$ and $y_i < 0$, and $\text{Proj}(\theta, y) \triangleq y_i$ otherwise. Based on the above formulation, note that $(\theta - \theta^*)^T(\text{Proj}(\theta, y) - y) \leq 0$ holds, where this inequality can be also readily generalized to matrices using $\text{Proj}(\Theta, Y) = \{ \text{Proj}(\text{col}_1(\Theta), \text{col}_1(Y)), \ldots, \text{Proj}(\text{col}_m(\Theta), \text{col}_m(Y)) \}$, where $\Theta \in \mathbb{R}^{n \times m}$, $Y \in \mathbb{R}^{n \times m}$, and $\text{col}_i(\cdot)$ denotes $i$th column operator.

Now, using the system error $e(t) = x(t) - x_r(t)$ and the weight estimation error $\hat{W}(t) = \hat{W}(t) - W$ defined above, one can write
\[
\begin{align*}
\dot{e}(t) &= (A_r - \kappa I)e(t) - B\hat{W}^T(t)\sigma(x(t), u_n(t)), \quad e(0) = e_0, \quad (10) \\
\dot{\hat{W}}(t) &= \gamma \text{Proj}\left(\hat{W}(t), \sigma(x(t), u_n(t))(x(t) - x_r(t))^T PB\right), \quad \hat{W}(0) = \hat{W}_0, \quad (11)
\end{align*}
\]
where $e_0 \triangleq x_0 - x_{r0}$ and $\hat{W}_0 \triangleq W_0 - W$. From (10) and (11), it can be easily shown that the pair $(e(t), \hat{W}(t))$ is bounded for all initial conditions and $\lim_{t \rightarrow \infty} e(t) = 0$. To see this, consider the Lyapunov function candidate
\[
\mathcal{V}(e, \hat{W}) = e^TPe + \gamma^{-1} \text{tr}\left((\hat{W}\Lambda^1/2)^T(\hat{W}\Lambda^1/2)\right). \quad (12)
\]
where its derivative along the trajectories of (10) and (11) has the upper bound given by
\[
\dot{\mathcal{V}}(e(t), \hat{W}(t)) \leq -e^T(t)(R + 2\kappa P)e(t) \leq 0. \quad (13)
\]
Here, (13) yields the boundedness of the pair $(e(t), \hat{W}(t))$ for all initial conditions and $\lim_{t \rightarrow \infty} e(t) = 0$ follows from the Barbalat’s lemma. In addition, owing to the fact that $\dot{\mathcal{V}}(e(t), \hat{W}(t)) \leq 0$, one can write
\[
\begin{align*}
\mathcal{V}(e(t), \hat{W}(t)) &\leq \mathcal{V}(e(0), \hat{W}(0)) \\
&= e_0^TPe_0 + \gamma^{-1} \text{tr}\left((\hat{W}_0\Lambda^1/2)^T(\hat{W}_0\Lambda^1/2)\right). \quad (14)
\end{align*}
\]
Since $\lambda_{\min}(P)||e(t)||_2^2 \leq \mathcal{V}(e(t), \hat{W}(t))$, (14) gives
\[
||e(t)||_2 \leq e^*, \quad e^* \triangleq \sqrt{\lambda_{\min}^{-1}(P)[e_0^TPe_0 + \gamma^{-1} \text{tr}\left((\hat{W}_0\Lambda^1/2)^T(\hat{W}_0\Lambda^1/2)\right)]}, \quad t \geq 0. \quad (15)
\]
Note that the transient response upper bound (15) is conservative and not user-defined.

Finally, to show our claim stated in iii) on the paragraph above (8), define the (ideal) reference model as
\[
\begin{align*}
\dot{x}_{rI}(t) &= A_r x_{rI}(t) + B_r c(t), \quad x_{rI}(0) = x_{r0}, \quad (16)
\end{align*}
\]
and let $\tilde{x}_r(t) \triangleq x_r(t) - x_{rI}(t)$ be the state emulator error (the error between the state emulator (7) and the ideal reference model (16)), which satisfies
\[
\begin{align*}
\dot{\tilde{x}}_r(t) &= A_r \tilde{x}_r(t) + \kappa e(t), \quad \tilde{x}_r(0) = 0. \quad (17)
\end{align*}
\]
Since $||e^{A_r t}|| \leq \omega e^{-\beta t}$, $t \geq 0$, for the Hurwitz matrix $A_r$ [46], where $\omega > 0$, $0 < \beta < -\alpha(A_r)$, and $\alpha(A_r) \triangleq \max\{\text{Re}(\lambda) : \lambda \in \text{spec}(A_r)\}$, it follows that
\[
\begin{align*}
||\tilde{x}_r(t)||_2 &\leq \omega \beta^{-1} \kappa ||e(t)||_2 \\
&\leq \omega \beta^{-1} \kappa e^*, \quad t \geq 0, \quad (18)
\end{align*}
\]
and hence, since the transient response upper bound (15) is conservative and not user-defined, the error between the state emulator and the ideal reference model given by (18) is conservative and not user-defined as well. Thus, if the adaptive controller (5) employs the standard update law given by (8), one cannot strictly guarantee that the state emulator trajectories are staying close to the given reference model trajectories during the transient response, which justifies our claim that the state emulator-based adaptive controllers can exhibit smooth-yet-poor performance. In the next section, the ASTERA addresses the problem of simultaneously achieving user-defined performance guarantees and smooth transients.

III Adaptive Set-Theoretic Emulator Reference Architecture (ASTERA)

A Generalized Restricted Potential Function

Since the ASTERA is a generalization of both set-theoretic model reference adaptive control methods and state emulator-based or state emulator-like standard adaptive control methods to simultaneously achieve user-defined performance guarantees and smooth transients, we now need to adopt the following definition from, for example, [18,19].

**Definition 2 (Generalized Restricted Potential Function).** Let 
\[ \|z\|_H = (z^THz)^{1/2} \]
be a weighted Euclidean norm, where \( z \in \mathbb{R}^p \) is a real column vector and \( H \in \mathbb{R}^{p \times p} \). We define \( \phi(\|z\|_H), \phi : \mathbb{R}^p \rightarrow \mathbb{R} \), to be a generalized restricted potential function (generalized barrier Lyapunov function) on the set
\[ D_\epsilon \triangleq \{z : \|z\|_H \in [0, \epsilon)\}, \]
with \( \epsilon \in \mathbb{R}_+ \) being a-priori, user-defined constant, if the following statements hold:

1. If \( \|z\|_H = 0 \), then \( \phi(\|z\|_H) = 0 \).
2. If \( z \in D_\epsilon \) and \( \|z\|_H \neq 0 \), then \( \phi(\|z\|_H) > 0 \).
3. If \( \|z\|_H \rightarrow \epsilon \), then \( \phi(\|z\|_H) \rightarrow \infty \).
4. \( \phi(\|z\|_H) \) is continuously differentiable on \( D_\epsilon \).
5. If \( z \in D_\epsilon \), then \( \phi_d(\|z\|_H) > 0 \), where
\[ \phi_d(\|z\|_H) \triangleq \frac{d\phi(\|z\|_H)}{d\|z\|_H^2}. \]
6. If \( z \in D_\epsilon \), then
\[ 2\phi_d(\|z\|_H)\|z\|_H^2 - \phi(\|z\|_H) > 0. \]

As noted in [18,19], a candidate generalized restricted potential function satisfying the conditions given in Definition 2 (Generalized Restricted Potential Function) has the form \( \phi(\|z\|_H) = \|z\|_H^2/(\epsilon - \|z\|_H) \), \( z \in D_\epsilon \). This definition can be viewed as a generalized version of the restricted potential function (barrier Lyapunov function) definitions used by, for example, the authors of [28–32].

B Proposed Feedback Control Architecture

As discussed, the ASTERA proposed here has two major components — a state emulator and a set-theoretic adaptive controller. Specifically, for the uncertain dynamical system given by (1) subject to the augmenting feedback control given by (2), (3), and (5), it utilizes the set-theoretic update law given by
\[ \dot{\hat{W}}(t) = \gamma \text{Proj}\left(\hat{W}(t), \phi_d(\|x(t) - x_r(t)\|_P)\sigma(x(t), u_n(t))(x(t) - x_r(t))^TPB\right), \quad \hat{W}(0) = \hat{W}_0, \]
(22)
where $\gamma \in \mathbb{R}_+$ is the learning rate, $P \in \mathbb{R}^{n \times n}$ is the solution to the Lyapunov equation given by (9) for $R \in \mathbb{R}^{n \times n}$, Proj is the projection operator given in Definition 1 (Projection Operator), and $\phi_d(||x(t) - x_r(t)||_P)$ is the derivative of the generalized restricted potential function $\phi(||x(t) - x_r(t)||_P)$ with respect to $||x(t) - x_r(t)||_P^2$, with $\phi(||x(t) - x_r(t)||_P)$ satisfying the conditions stated in Definition 2 (Generalized Restricted Potential Function). Furthermore, $x_r(t)$ in (22) satisfies the state emulator dynamics given by (7).

Here, the purpose of the set-theoretic update law is to assure that the weighted Euclidean norm of the system error $||x(t) - x_r(t)||_P$ to stay less than a-priori, user-defined scalar worst-case bound $\epsilon$, and the purpose of the state emulator is to introduce the flexibility to achieve smooth transients — transients that do not necessarily exhibit high-frequency oscillations. For completeness, Table 1 summaries the elements of the proposed feedback control architecture.

<table>
<thead>
<tr>
<th>Uncertain dynamical system</th>
<th>$\dot{x}(t) = Ax(t) + B(\Lambda u(t) + W_0^T x(t)), x(0) = x_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Augmenting feedback control</td>
<td>$u(t) = u_n(t) + u_a(t)$</td>
</tr>
<tr>
<td>Nominal control</td>
<td>$u_n(t) = -K_1 x(t) + K_2 c(t)$</td>
</tr>
<tr>
<td>Adaptive control</td>
<td>$u_a(t) = -W^T(t) \sigma(x(t), u_n(t)))$, $\sigma(x(t), u_n(t))) = [x^T(t), u_n^T(t)]^T$</td>
</tr>
<tr>
<td>Set-theoretic update law</td>
<td>$\dot{W}(t) = \gamma \text{Proj}(\dot{W}(t), \phi_d(</td>
</tr>
<tr>
<td>State emulator</td>
<td>$\dot{x}<em>r(t) = A_r x_r(t) + B_r c(t) + \kappa(x(t) - x_r(t))$, $x_r(0) = x</em>{r0}$</td>
</tr>
<tr>
<td>$A_r$ and $B_r$</td>
<td>$A_r = A - BK_1$ (Hurwitz) and $B_r = BK_2$</td>
</tr>
<tr>
<td>Lyapunov equation</td>
<td>$0 = A_r^T P + PA_r + R$, $R &gt; 0$ (i.e., $P &gt; 0$ since $A_r$ is Hurwitz)</td>
</tr>
</tbody>
</table>

Table 1. Uncertain dynamical system and the proposed feedback control architecture.

C Stability and Performance Analysis

For stability and performance analysis of the ASTERA, we write

$$ \begin{align*}
\dot{e}(t) &= (A_r - \kappa I)e(t) - BA W^T(t) \sigma(x(t), u_n(t)), \quad e(0) = e_0, \\
\dot{W}(t) &= \gamma \text{Proj}(\dot{W}(t), \phi_d(||e||_P)\sigma(\cdot)(x - x_r)PB), \quad \dot{W}(0) = \bar{W}_0,
\end{align*} 
$$

where $e(t) = x(t) - x_r(t)$ is the system error and $\dot{W}(t) = \dot{W}(t) - W$ is the weight estimation error defined above. The following theorem presents the first result of this paper.

**Theorem 1 (Overall Stability and Performance Guarantees on System Error).** Consider the uncertain dynamical system given by (1) subject to the augmenting feedback control architecture given by (2), (3), and (5). In addition, consider the set-theoretic update law given by (22) for (5) with the state emulator given by (7). If $||e_0||_P < \epsilon$, then the pair $(e(t), \dot{W}(t))$ is bounded for all time, where the transient performance bound on the weighted Euclidean norm of the system error satisfies $||e(t)||_P < \epsilon$ with $\epsilon \in \mathbb{R}_+$ being a user-defined scalar, and $\lim_{t \to \infty} e(t) = 0$.

**Proof.** Let $V : D_\epsilon \times \mathbb{R}^{(n+m)\times m}, \ D_\epsilon \triangleq \{e(t) : ||e(t)||_P < \epsilon\}$, be

$$ V(e, \tilde{W}) = \phi(||e||_P) + \gamma^{-1} \text{tr}((\tilde{W} A^{1/2})^T (\tilde{W} A^{1/2})). $$

In addition, for any $\psi \in \mathbb{R}_+$, let $\Psi \triangleq \{(e(t), \tilde{W}(t)) \in D_\epsilon \times \mathbb{R}^{(n+m)\times m} : V(e, \tilde{W}) \leq \psi\}$ denote the level sets of $V(e, \tilde{W})$. Note that $V(0,0) = 0$ and $V(e, \tilde{W}) > 0$ for all admissible $(e, \tilde{W}) \neq (0, 0)$. Now, it follows from (25) that one can write

$$ \dot{V}(e(t), \tilde{W}(t)) \leq -\phi_d(||e||_P) e^T(t)(R + 2\kappa P)e(t) \leq 0. $$

The rest of the proof follows similar to the proof of Theorem 5.3 in [18], which utilize the results from [47,48], and hence, is omitted.
Note that the ASTERA summarized in Table 1 can be also used identically when the unknown weight matrix is time-varying and bounded with a bounded time rate of change, i.e., when \( W^T(t)x(t) \) in (1) is replaced with \( W^T_0(t)x(t) \). In this case, the system error dynamics and the weight estimation error dynamics are respectively given by (23) and

\[
\dot{W}(t) = \gamma \text{Proj} \left( \dot{W}(t), \phi_d(\|e(t)\|_P)\sigma(x(t), u_0(t))e^T(t)PB \right) - \dot{W}(t), \quad \dot{W}(0) = \dot{W}_0,
\]

where \( W^T(t) = [\Lambda^{-1}W^T(t), I - \Lambda^{-1}] \). Following similar steps given in the proof of Theorem 1 (Overall Stability and Performance Guarantees on System Error) and the proof of Theorem 1 in [19], one can still show for this case if \( \|e_0\|_P < \epsilon \), then the pair \((e(t), \dot{W}(t))\) is bounded for all time, where the transient performance bound on the weighted Euclidean norm of the system error satisfies \( \|e(t)\|_P < \epsilon \) with \( \epsilon \in \mathbb{R}_+ \).

The above discussion and Theorem 1 (Overall Stability and Performance Guarantees on System Error) highlights that the transient performance bound on the weighted Euclidean norm of the system error is upper bounded by a user-defined scalar, i.e., \( \|e(t)\|_P < \epsilon \) with \( \epsilon \in \mathbb{R}_+ \) — unlike the case presented in (15) that results from a standard update law. From Remark 6 in [19], \( \|e(t)\|_P < \epsilon \) further implies that \( \|e(t)\|_2 < \lambda_{\min}^{-1}(P) \epsilon \) in terms of standard Euclidean norm. We next show that the ASTERA ensures the state emulator trajectories stay close to the given reference model trajectories as desired (a property that does not exist in standard state emulator-based or state emulator-like standard adaptive control methods, see Section II-C).

**Corollary 1 (Performance Guarantees on the State Emulator Error).** Consider the uncertain dynamical system given by (1) subject to the augmenting feedback control architecture given by (2), (3), and (5). In addition, consider the set-theoretic update law given by (22) for (5) with the state emulator given by (7). If \( \|e_0\|_P < \epsilon \), then

\[
\|\tilde{x}_t(t) - x_{ri}(t)\|_2 < \beta^{-1} \lambda_{\min}^{-1/2}(P) \omega \kappa \epsilon,
\]

where \( \omega > 0 \), \( 0 < \beta < -\alpha(A_x) \), and \( \alpha(A_x) \triangleq \max \{\Re(\lambda) : \lambda \in \text{spec}(A_x)\} \) such that \( \|e^{A_xt}\| \leq \omega e^{-\beta t} \).

**Proof.** From Theorem 1 (Overall Stability and Performance Guarantees on System Error), we have \( \|e(t)\|_P < \epsilon \) that implies \( \|e(t)\|_2 < \lambda_{\min}^{-1/2}(P) \epsilon \). Furthermore, since \( \|e^{A_xt}\| \leq \omega e^{-\beta t}, t \geq 0 \), for the Hurwitz matrix \( A_x \) [46], where \( \omega > 0, 0 < \beta < -\alpha(A_x) \), and \( \alpha(A_x) \triangleq \max \{\Re(\lambda) : \lambda \in \text{spec}(A_x)\} \), it now follows from (17) that \( ||\tilde{x}_t(t)||_2 \leq \omega \beta^{-1} \lambda_{\min}^{-1/2}(P) \omega \kappa \epsilon \), which gives (28). \( \blacksquare \)

Note that all the terms appearing on the right hand side of (28) are known and user-defined, and hence, the ASTERA ensures the state emulator trajectories stay close to the given reference model trajectories as desired. Finally, the next result shows the transient upper bound between the uncertain dynamical system trajectories and the (ideal) reference model trajectories, i.e., the ideal system error \( e(t) \triangleq x(t) - x_{ri}(t) \).

**Corollary 2 (Performance Guarantees on the Ideal System Error).** Consider the uncertain dynamical system given by (1) subject to the augmenting feedback control architecture given by (2), (3), and (5). In addition, consider the set-theoretic update law given by (22) for (5) with the state emulator given by (7). If \( \|e_0\|_P < \epsilon \), then

\[
\|x(t) - x_{ri}(t)\|_2 < (1 + \beta^{-1} \omega \kappa) \lambda_{\min}^{-1/2}(P) \epsilon,
\]

where \( \omega > 0, 0 < \beta < -\alpha(A_x) \), and \( \alpha(A_x) \triangleq \max \{\Re(\lambda) : \lambda \in \text{spec}(A_x)\} \) such that \( \|e^{A_xt}\| \leq \omega e^{-\beta t} \).

**Proof.** From Theorem 1 (Overall Stability and Performance Guarantees on System Error), we have \( \|e(t)\|_P < \epsilon \) that implies \( \|e(t)\|_2 < \lambda_{\min}^{-1/2}(P) \epsilon \). Furthermore, from Corollary 1 (Performance Guarantees on the State Emulator Error), we have (28). Now, since \( \|x(t) - x_{ri}(t)\|_2 = ||e(t) + \tilde{x}_t(t)||_2 \leq ||e(t)||_2 + ||\tilde{x}_t(t)||_2 \), (29) readily follows. \( \blacksquare \)

Similar to the above discussion, all the terms appearing on the right hand side of (29) are known and user-defined. In addition, when \( \kappa = 0 \) in (29), it is not surprising that we recover the results presented in
Here, however, $\kappa > 0$ gives a flexibility to the ASTERA for achieving smooth transients — transients that do not necessarily exhibit high-frequency oscillations. This is illustrated in the next section.

**IV Illustrative Numerical Example**

Consider the uncertain dynamical system given by

$$
\dot{x}(t) = 
\begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
x(t) \\
x_{r}(t)
\end{bmatrix} + 
\begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
(0.75 u(t) + [0.25, 0.75] x(t)), \\
\gamma
\end{bmatrix},
\quad
x(0) = \begin{bmatrix} 0 \\ x_{0} \end{bmatrix}.
$$

(30)

To achieve a desired level of command following performance in the absence of system uncertainties (i.e., $(W_0, A) \equiv (0, I)$), we set $K_1 = [0.36, 0.96]$ and $K_2 = [0.36]$, which results in the state emulator dynamics

$$
\dot{x}_r(t) = 
\begin{bmatrix}
0 & 1 \\
-0.36 & -0.96
\end{bmatrix}
\begin{bmatrix}
x_r(t) \\
x_{r0}
\end{bmatrix} + 
\begin{bmatrix}
0 & 0.36 \\
-0.36 & 0
\end{bmatrix}
\begin{bmatrix}
c(t) + \kappa (x(t) - x_r(t)), \\
x_r(t)
\end{bmatrix},
\quad
x_r(0) = \begin{bmatrix} 0 \\ x_{r0} \end{bmatrix}.
$$

(31)

In (31), $c(t)$ is selected as a unity square wave signal filtered through a first-order low pass filter with a unity time constant. Command following performance in the absence and presence of system uncertainties and without the proposed adaptive controller (i.e., $u_a(t) \equiv 0$) are respectively shown in Figures 1 and 2. In particular, these figures show that the presence system uncertainties results in an unstable closed-loop system when no adaptation is utilized.

To implement the ASTERA summarized in Table 1, we use the generalized restricted potential function stated right after Definition 2 (Generalized Restricted Potential Function) for the set-theoretic update law (22), where we use $\gamma = 1$ as the learning rate and appropriately large projection operator bounds. Moreover, we

---

**Figure 1.** Command following performance in the absence of system uncertainties and without the proposed adaptive controller (i.e., $u_a(t) \equiv 0$).
Figure 2. Command following performance in the presence of system uncertainties and without the proposed adaptive controller (i.e., $u_a(t) \equiv 0$).

Figure 3. Command following performance in the presence of system uncertainties and the ASTERA. Note that we set $\epsilon = 1$ for (22) and $\kappa = 2$ for (31).
solve (9) for $P$ with $R = I$. For $\epsilon = 1$ and $\epsilon = 0.2$ in (22), Figures 3 and 4 respectively show the command following performances of the proposed adaptive control architecture with $\kappa = 2$ in (31). It is clear that both command following performances are acceptable and have smooth transients, but the latter performance is tighter as compared with the former one — since a smaller transient performance upper bound $\epsilon$ is used to constrain the time evolution of the weighted Euclidean norm of the system errors, i.e., $||e(t)||_P$. Specifically, notice that $||e(t)||_P$ in these figures always stay below $\epsilon$ as theoretically expected.

For Figures 3 and 4, one can also compute the resulting bound on the ideal system error using (29). To elucidate this point, we first find $\omega$ and $\beta$ for the selected $A_r$ matrix in (31) as $\omega = 2.15$ and $\beta = 0.35$ from Figure 5 such that $||e^{A_l t}|| \leq \omega e^{-\beta t}$ holds. It then follows from (29) that $||x(t) - x_{r_l}(t)||_2 < 16.9$ for Figure 3 and $||x(t) - x_{r_l}(t)||_2 < 3.4$ for Figure 4, where $\epsilon = 1$ is used for the former one and $\epsilon = 0.2$ is used for the latter one. We also compare the ASTERA with the standard set-theoretic model reference adaptive controller [18,19] by setting $\kappa$ to zero in (31). In particular, when $\kappa = 0$, Figures 6 and 7 show command following performances respectively for $\epsilon = 1$ and $\epsilon = 0.2$. Comparing these two figures with the ones in Figures 3 and 4, it is clear that the transient responses with the ASTERA are smoother. Thus, the flexibility to choose $\kappa$ through the introduction of the state emulator to the results in [18,19] gives a desired flexibility to the ASTERA in achieving transients that do not necessarily exhibit high-frequency oscillations. Finally, Figure 8 shows command following performances for $\epsilon = 0.2$ and $\kappa \in [0, 6]$, where it is interesting to note that the effective learning rate $\gamma_{\phi_d}(||x(t) - x_{r_l}(t)||_P)$ reduces as $\kappa$ increases.

V Conclusion

The contribution of this paper was an adaptive set-theoretic emulator reference architecture (ASTERA). The proposed architecture was a generalization of both set-theoretic model reference adaptive control methods [18–24] and state emulator-based or state emulator-like standard adaptive control methods [10–17] to simultaneously achieve user-defined performance guarantees and smooth transients. Specifically, the state
Figure 5. Plots of $|e^{At}|$ and $\omega e^{-\beta t}$ for $\omega = 2.15$ and $\beta = 0.35$, where $|e^{At}| \leq \omega e^{-\beta t}$ holds.

Figure 6. Command following performance when $\kappa$ is set to zero for the case in Figure 3.
Figure 7. Command following performance when $\kappa$ is set to zero for the case in Figure 4.

Figure 8. Command following performance in the presence of system uncertainties and the ASTERA. Note that we set $\epsilon = 0.2$ for (22) and $\kappa \in [0, 6]$ for (31) ($\kappa$ increases from “blue” to “red” colors).
emulator adopted in the ASTERA altered the trajectories of a given reference model for the purpose of achieving smooth transients — a property that does not necessarily exist in standard set-theoretic model reference adaptive control methods. Furthermore, the set-theoretic update law of the ASTERA assured the weighted Euclidean norm of the difference between the uncertain dynamical system state and the state emulator state to be less than a-priori, user-defined scalar worst-case bound, which ensures the state emulator trajectories stay close to the given reference model trajectories as desired — a property that does not exist in standard state emulator-based or state emulator-like standard adaptive control methods. The theoretical contributions were also illustrated on a numerical example to show the efficacy of the ASTERA.

References


