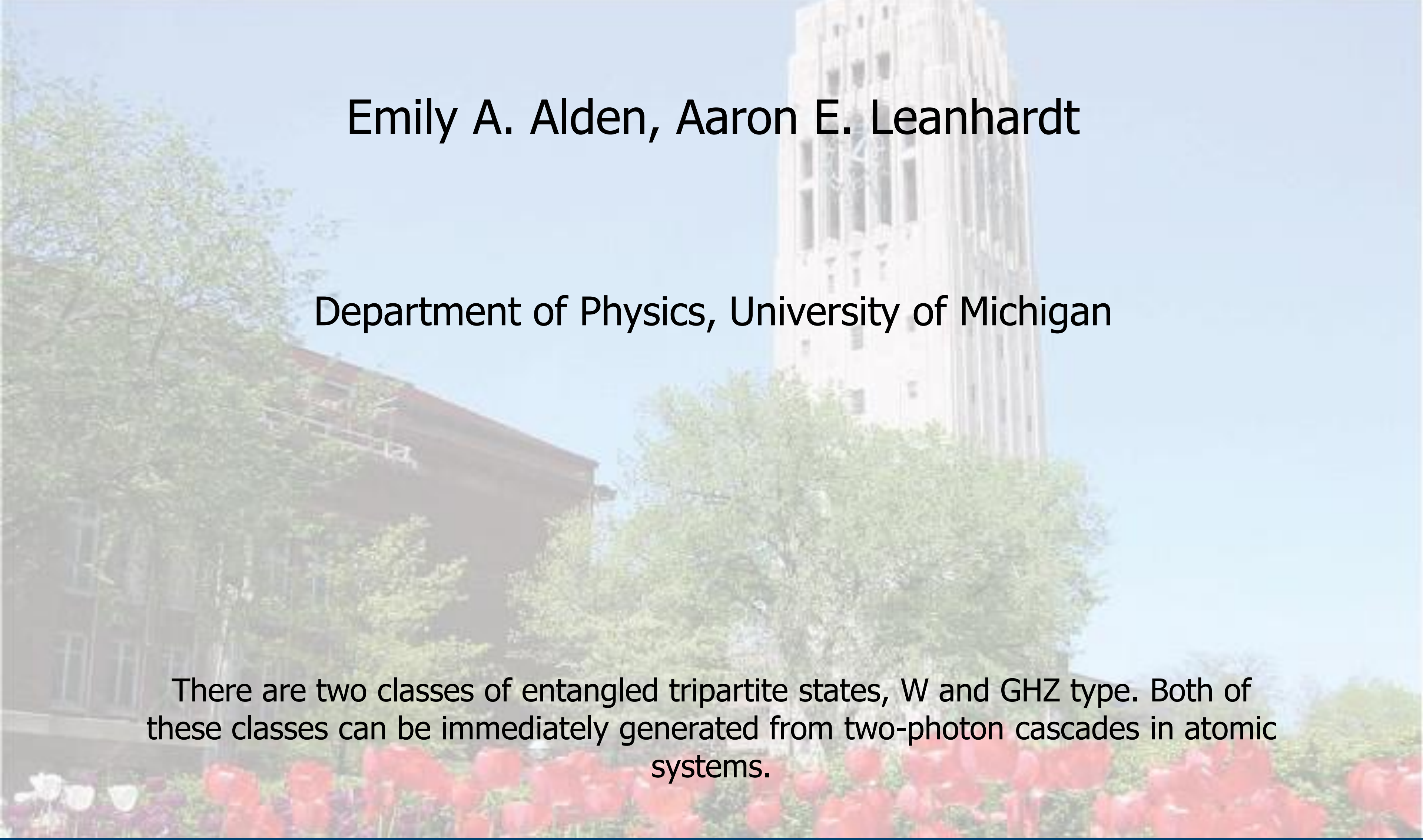
Tripartite Entanglement from Two-Photon Cascades

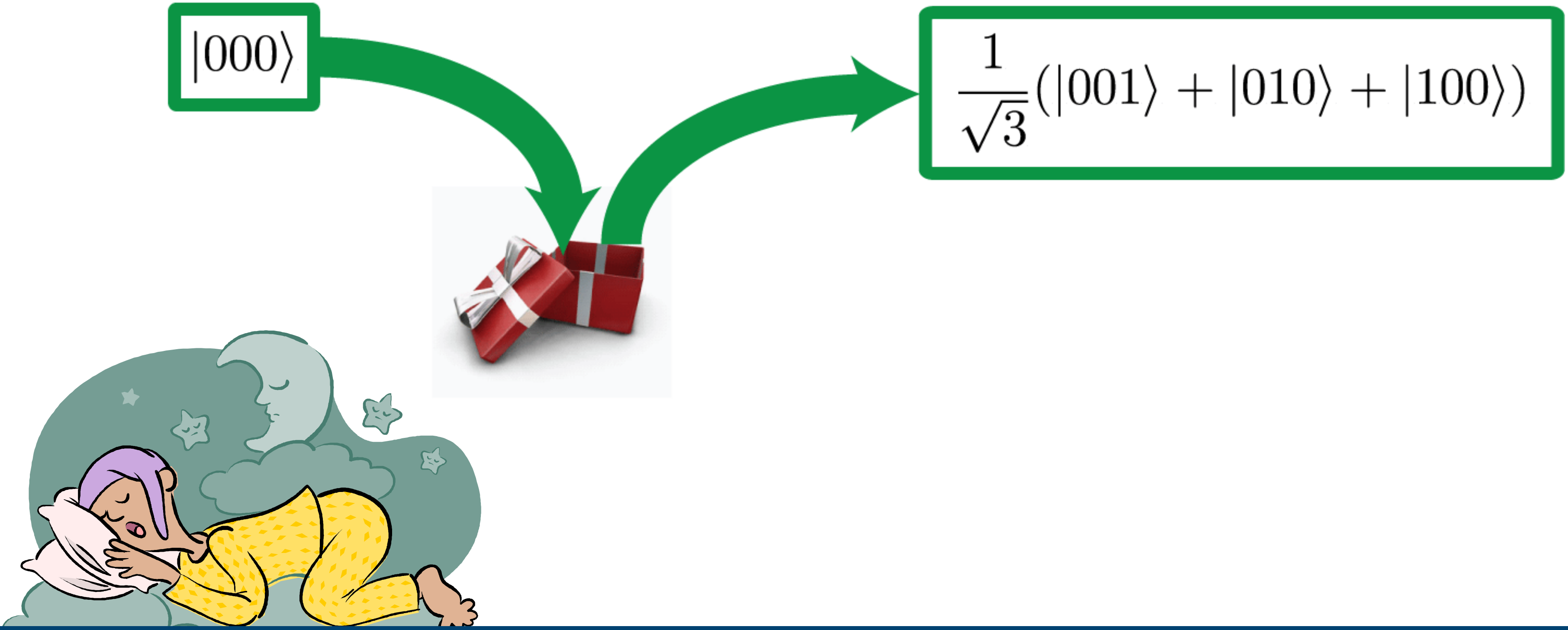


Emily A. Alden, Aaron E. Leanhardt

Department of Physics, University of Michigan

There are two classes of entangled tripartite states, W and GHZ type. Both of these classes can be immediately generated from two-photon cascades in atomic systems.

Dream Entangler



Boring & Unentangled

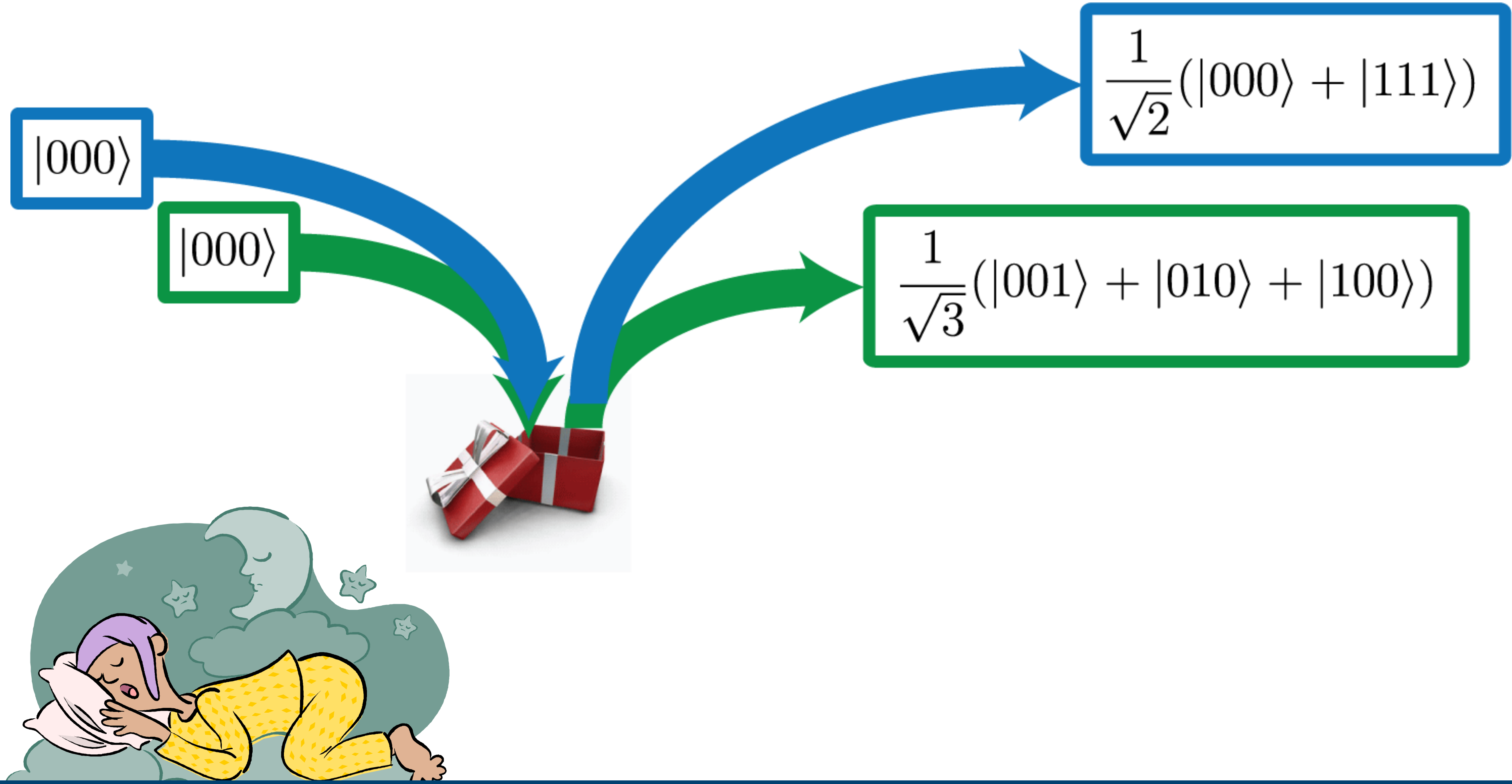
Useful & Entangled

$$|000\rangle$$

$$\frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$$

Dreamier Entangler

both types of tripartite entangled states



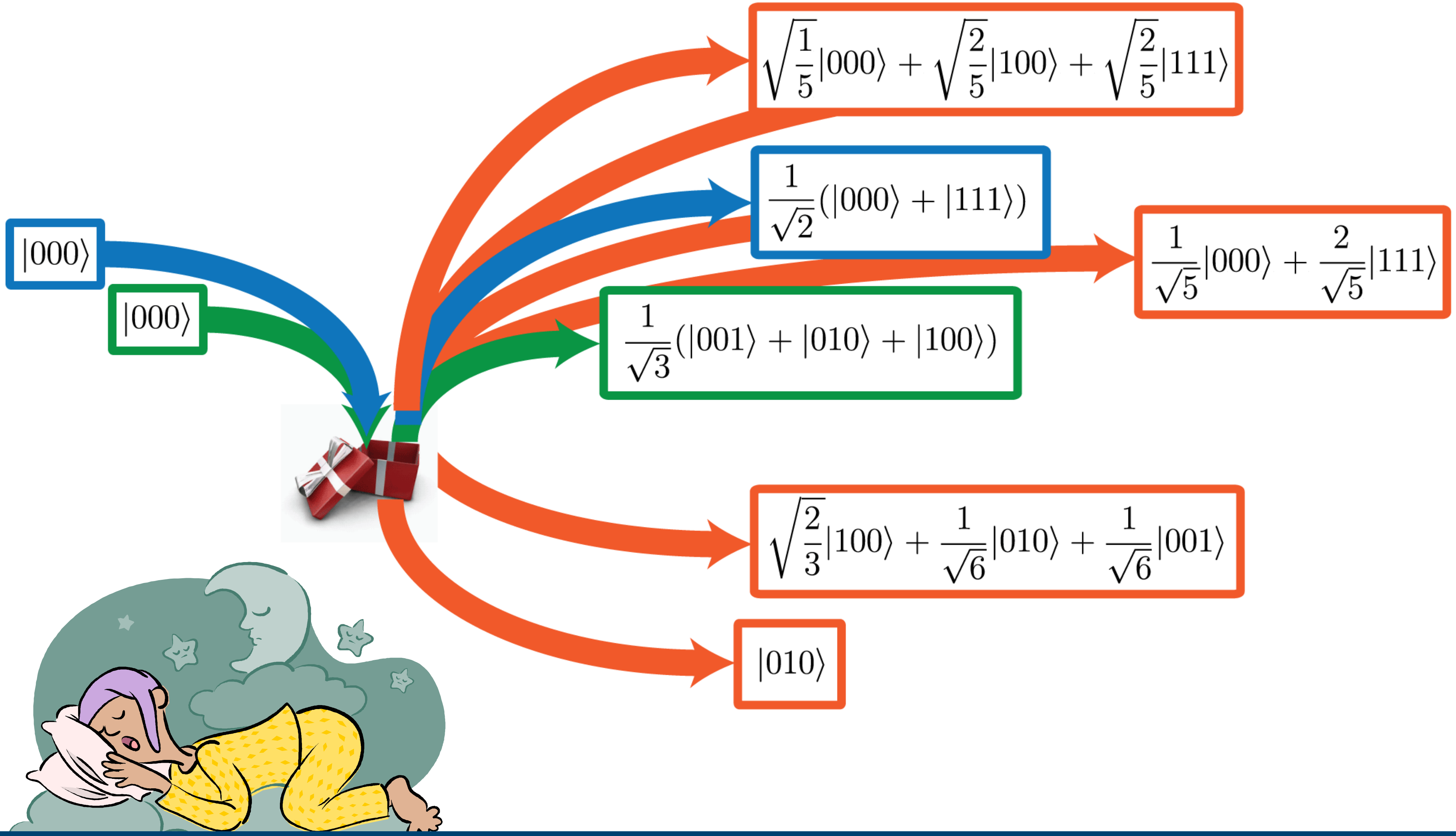
$|000\rangle$

$|000\rangle$

$$\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

$$\frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$$

Dreamiest Entangler



$|000\rangle$

$|000\rangle$

$\sqrt{\frac{1}{5}}|000\rangle + \sqrt{\frac{2}{5}}|100\rangle + \sqrt{\frac{2}{5}}|111\rangle$

$\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$

$\frac{1}{\sqrt{5}}|000\rangle + \frac{2}{\sqrt{5}}|111\rangle$

$\frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$

$\sqrt{\frac{2}{3}}|100\rangle + \frac{1}{\sqrt{6}}|010\rangle + \frac{1}{\sqrt{6}}|001\rangle$

$|010\rangle$

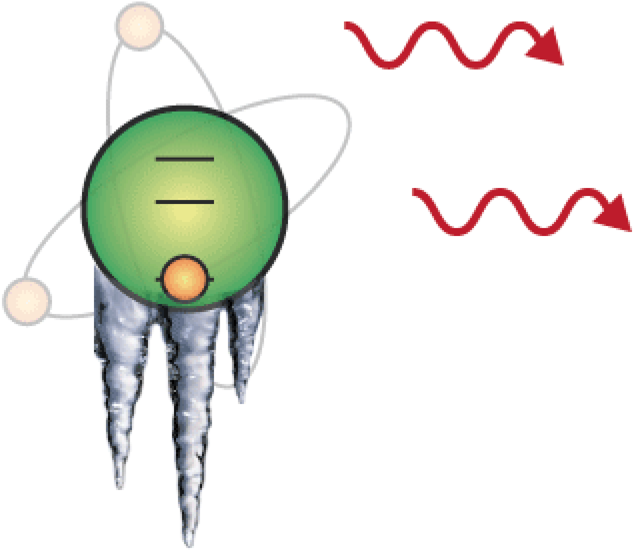
Proposed Entangler: three-level atom

Tripartite entanglement from Spontaneous Two-Photon Cascade

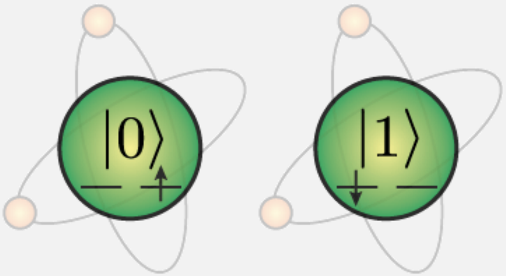
magic box

≈

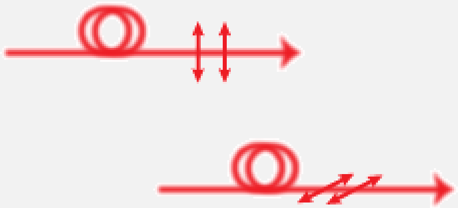
1 cold atom +
2 photons



Meet the Qubits



Atomic Spin Qubit
ground state F= ½



Photon Polarization Qubits

Chosen Basis

ATOM

$$|\Psi_A\rangle = c_{i\downarrow}|\tfrac{1}{2},-\tfrac{1}{2}\rangle + c_{i\uparrow}|\tfrac{1}{2},\tfrac{1}{2}\rangle$$

$$|0\rangle_A \equiv |\tfrac{1}{2},\tfrac{1}{2}\rangle$$

$$|1\rangle_A \equiv |\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

PHOTON

$$|\Psi_B\rangle = c_{j\uparrow}|\widehat{\theta_1}\rangle + c_{j\downarrow}|\widehat{\phi_1}\rangle$$

$$|0\rangle_B \equiv |\widehat{\theta_1}\rangle$$

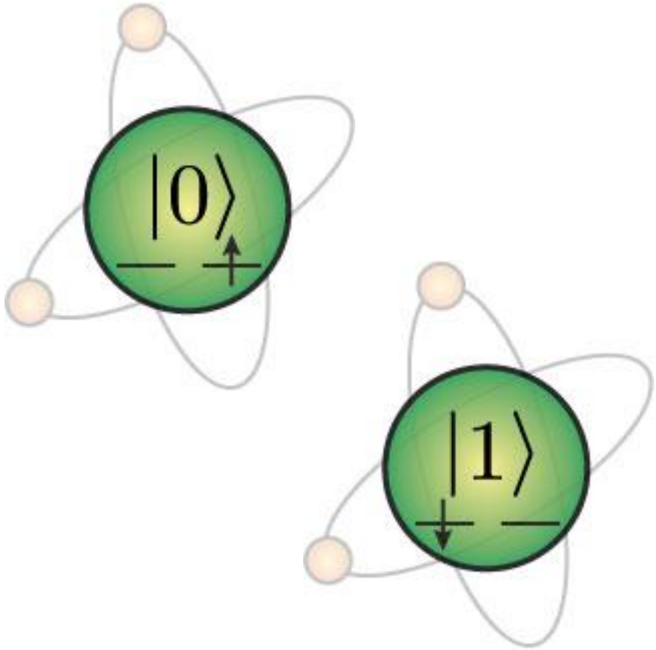
$$|1\rangle_B \equiv |\widehat{\phi_1}\rangle$$

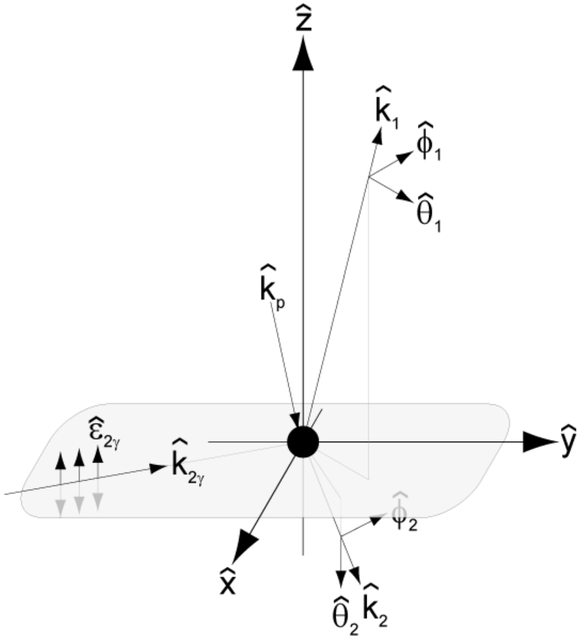
PHOTON

$$|\Psi_C\rangle = c_{k\uparrow}|\widehat{\theta_2}\rangle + c_{k\downarrow}|\widehat{\phi_2}\rangle$$

$$|0\rangle_C \equiv |\widehat{\theta_2}\rangle$$

$$|1\rangle_C \equiv |\widehat{\phi_2}\rangle$$

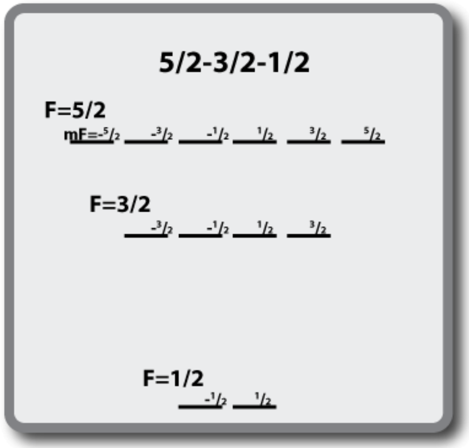




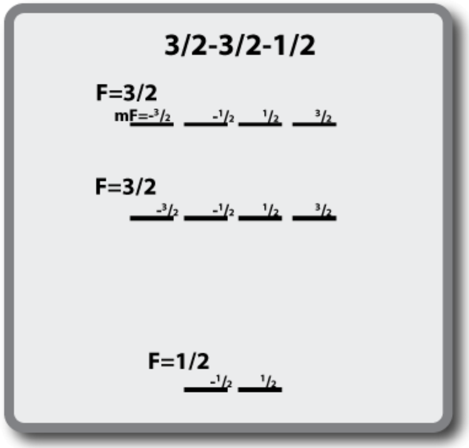
Cold Atom

with F=1/2 ground state

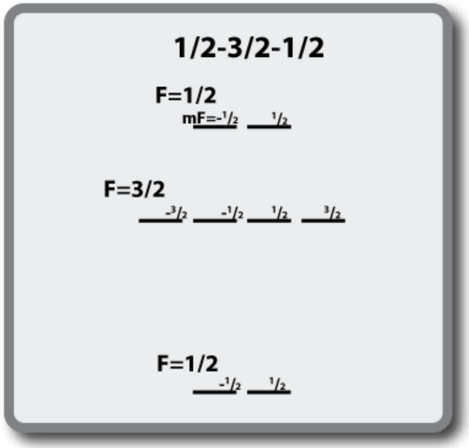
Dipole selection rules only permit 5 systems



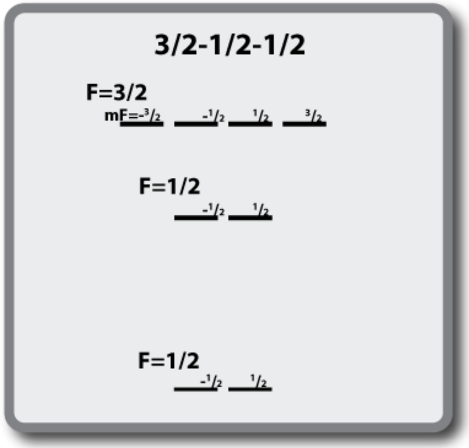
5/2-3/2-1/2



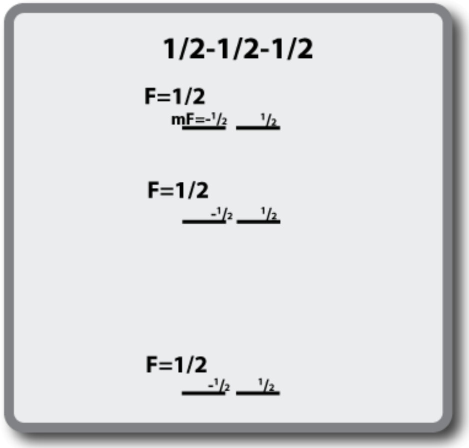
3/2-3/2-1/2



1/2-3/2-1/2



3/2-1/2-1/2



1/2-1/2-1/2

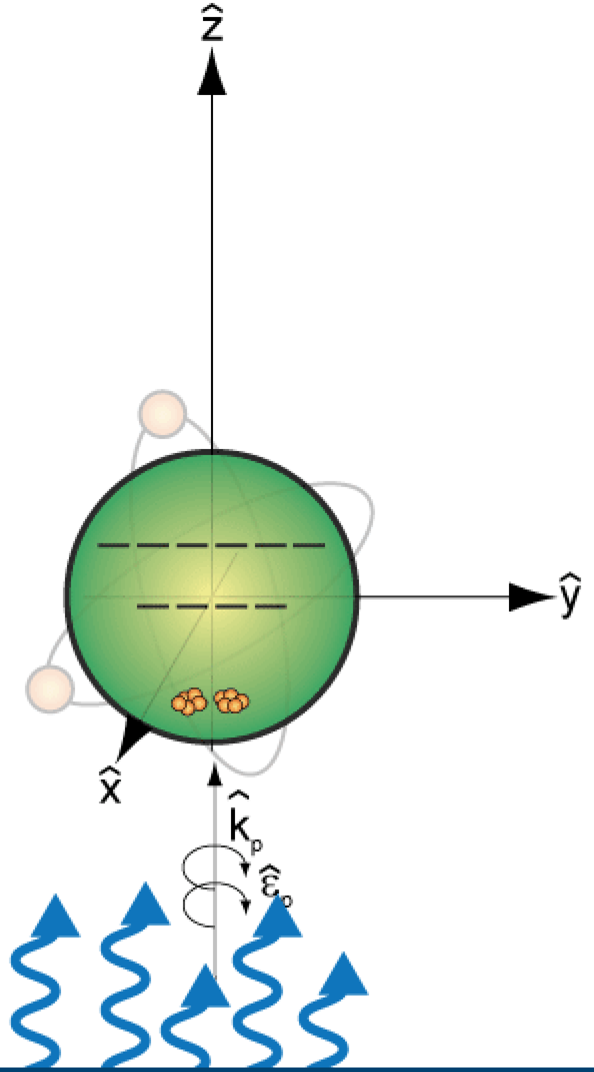
$\Delta F=\pm 1,0$

Sample Entanglement Cascade – ZXY

Prepare the Ground State

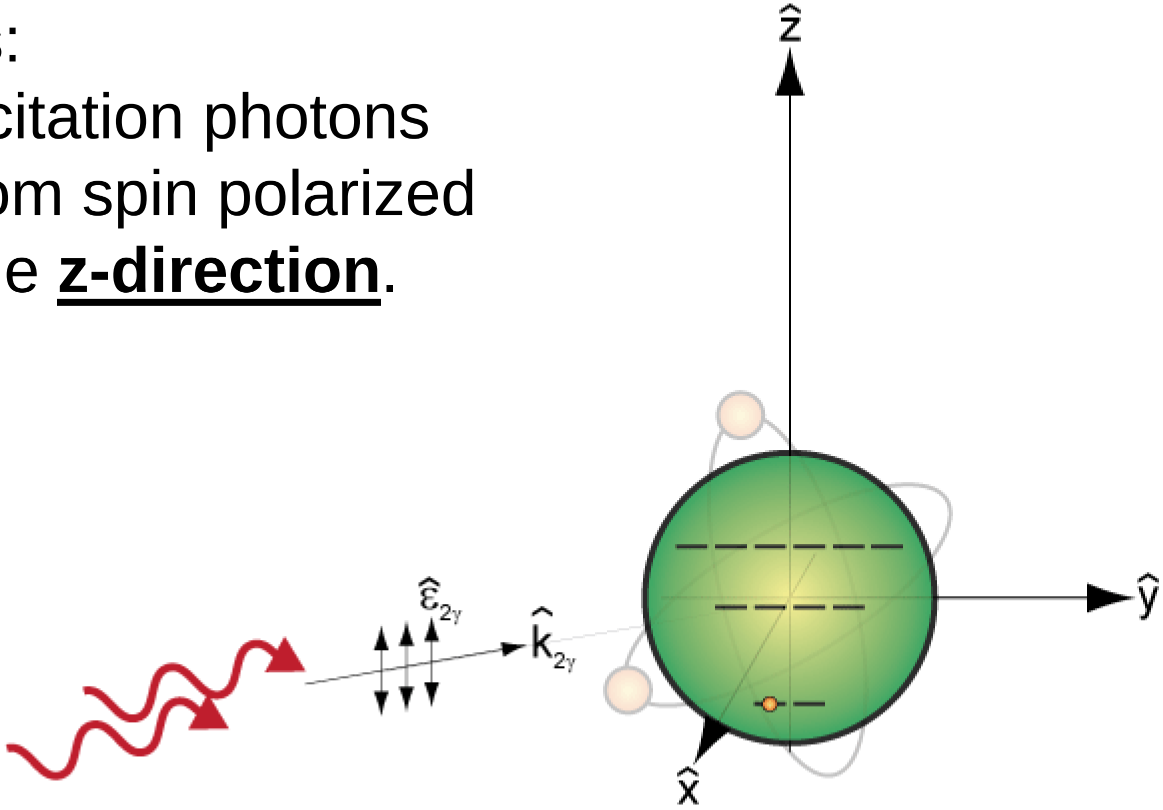
Optically pump the atom so the ground state population is all in $|\frac{1}{2},-\frac{1}{2}\rangle$.

The atom is spin polarized along the **<u>z-axis</u>**.

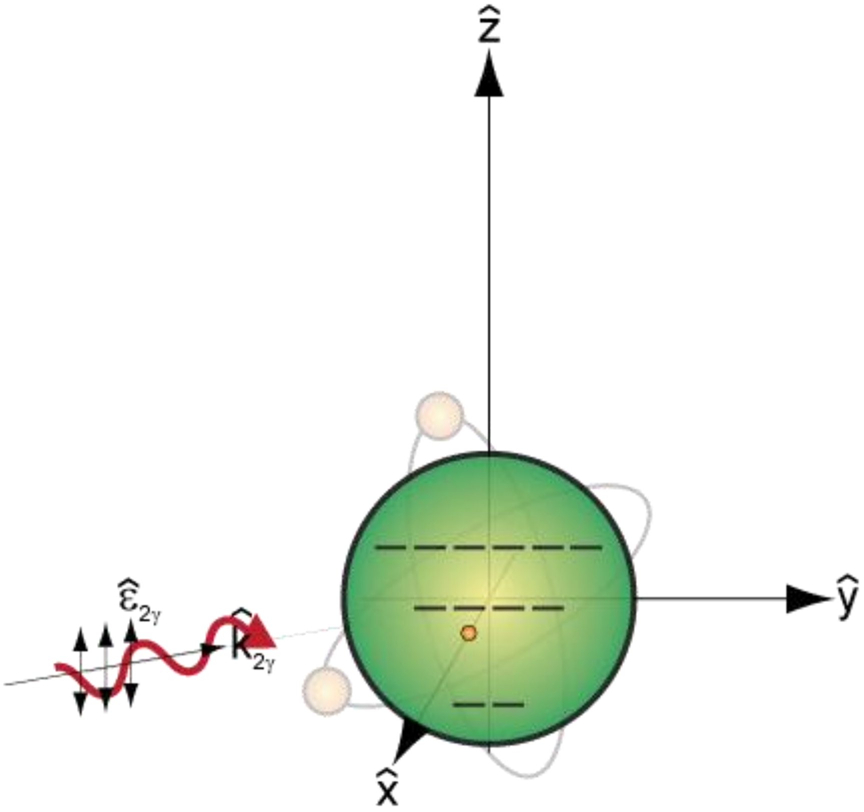


Sample Entanglement Cascade - ZXY

3 Inputs:
-two excitation photons
-one atom spin polarized
along the **z-direction**.

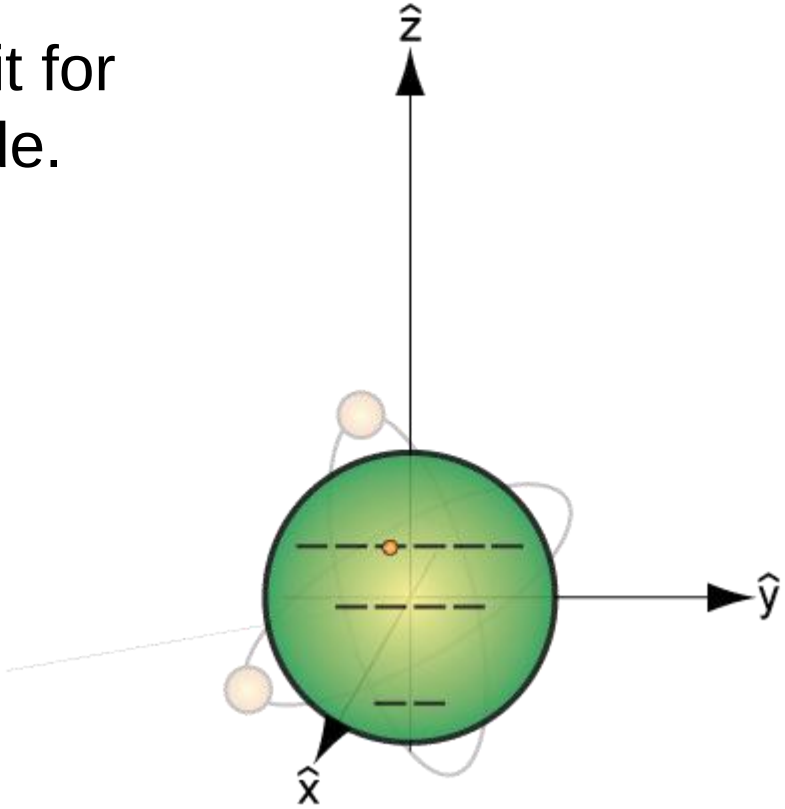


Sample Entanglement Cascade - ZXY



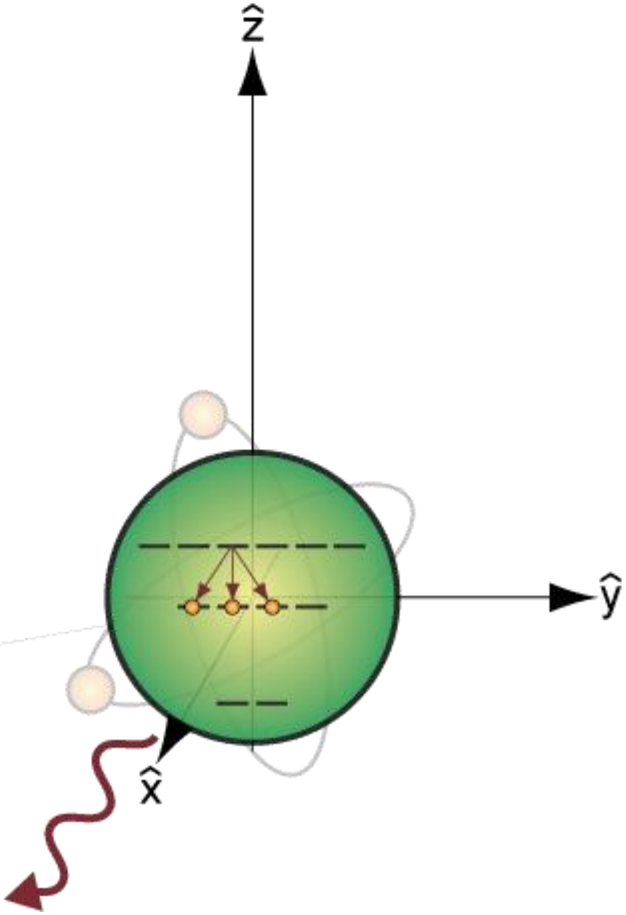
Sample Entanglement Cascade - ZXY

Excited atom, we wait for an entangling cascade.



Sample Entanglement Cascade - ZXY

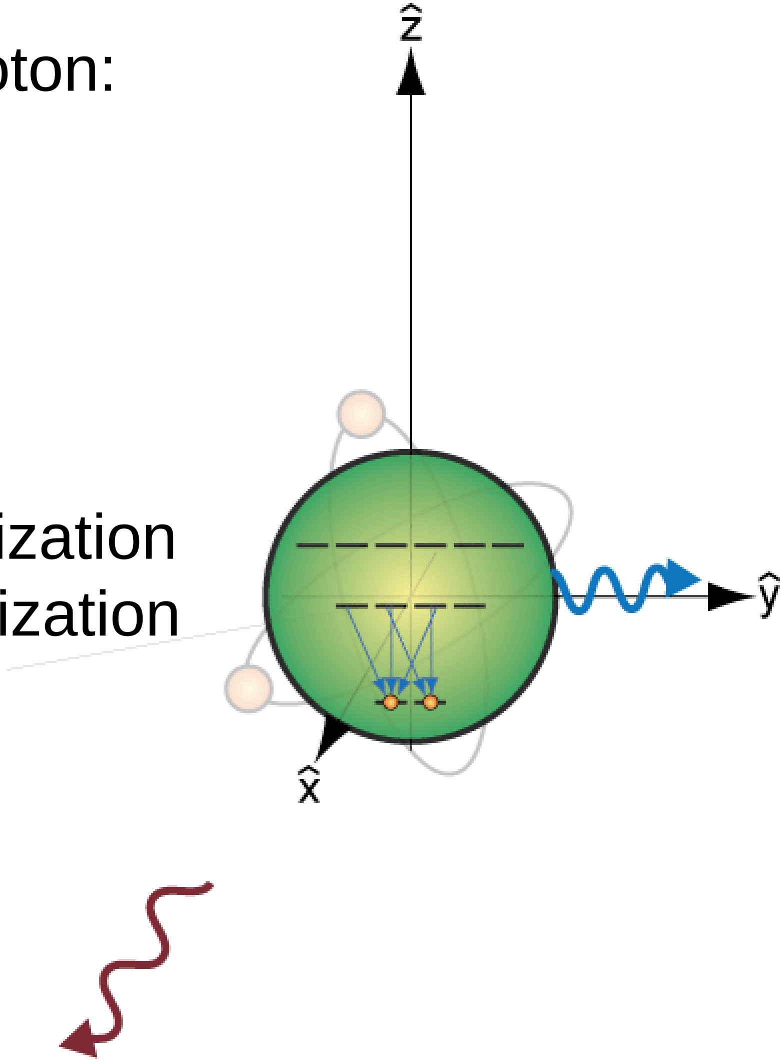
First cascade photon:
x-direction



Sample Entanglement Cascade - ZXY

Second cascade photon:
y-direction

Output:
A-Ground state atom
B-Lower photon polarization
C-Upper photon polarization



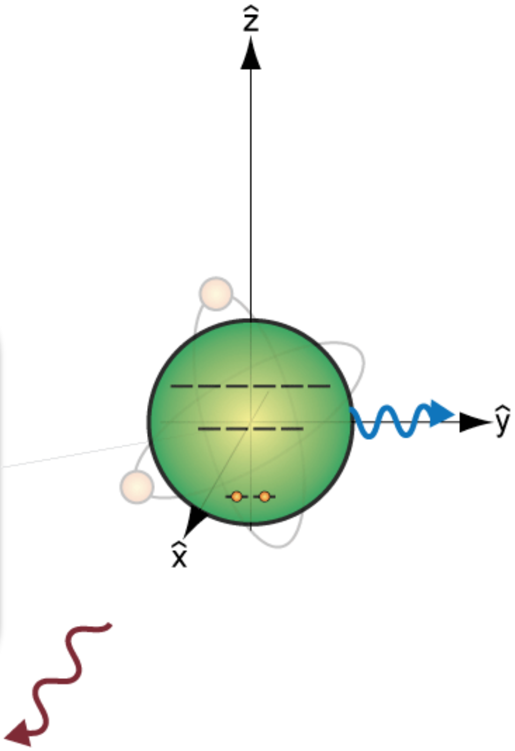
ZXY-cascade

$$|\Psi\rangle = \sqrt{\frac{2}{3}}|000\rangle - \frac{1}{\sqrt{6}}|101\rangle + \frac{i}{\sqrt{6}}|110\rangle$$

Not obviously entangled in our chosen basis.

Recall: Chosen Basis

ATOM	PHOTON	PHOTON
$\lvert\Psi_A\rangle = c_{i\downarrow}\lvert\frac{1}{2},-\frac{1}{2}\rangle + c_{i\uparrow}\lvert\frac{1}{2},\frac{1}{2}\rangle$	$\lvert\Psi_B\rangle = c_{j\uparrow}\lvert\widehat{\theta_1}\rangle + c_{j\downarrow}\lvert\widehat{\phi_1}\rangle$	$\lvert\Psi_C\rangle = c_{k\uparrow}\lvert\widehat{\theta_2}\rangle + c_{k\downarrow}\lvert\widehat{\phi_2}\rangle$
$\lvert 0\rangle_A \equiv \lvert\frac{1}{2},\frac{1}{2}\rangle$	$\lvert 0\rangle_B \equiv \lvert\widehat{\theta_1}\rangle$	$\lvert 0\rangle_C \equiv \lvert\widehat{\theta_2}\rangle$
$\lvert 1\rangle_A \equiv \lvert\frac{1}{2},-\frac{1}{2}\rangle$	$\lvert 1\rangle_B \equiv \lvert\widehat{\phi_1}\rangle$	$\lvert 1\rangle_C \equiv \lvert\widehat{\phi_2}\rangle$



Change of Basis Example

Horizontal/Vertical Polarization Basis

$$|H\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |V\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$|R\rangle \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} \qquad |L\rangle \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$U_{HV \rightarrow RL} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$$

Right/Left-Circular Polarization Basis

$$|R\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |L\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$|H\rangle \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad |V\rangle \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ -i \end{pmatrix}$$

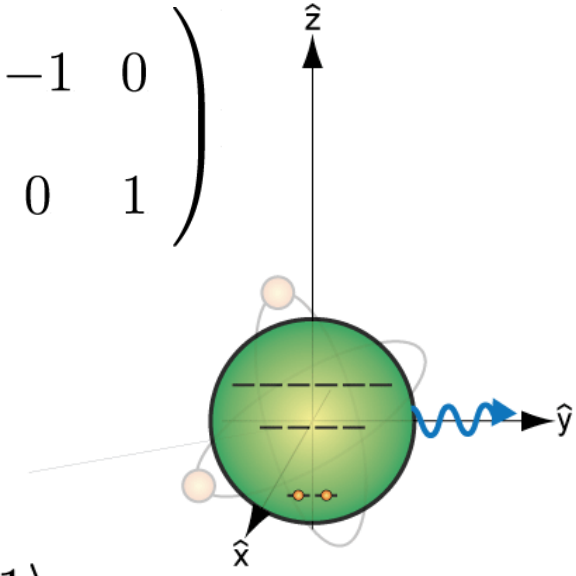
$$U_{RL \rightarrow HV} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$$

Find the Unique Basis

5/2-3/2-1/2 zxy

$$|\Psi\rangle = \sqrt{\frac{2}{3}}|000\rangle - \frac{1}{\sqrt{6}}|101\rangle + \frac{i}{\sqrt{6}}|110\rangle$$

$$U_A = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \quad U_B = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} \quad U_C = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$



Asymmetrical W-state:

$$|\Psi\rangle = \sqrt{\frac{2}{3}}|100\rangle + \frac{1}{\sqrt{6}}|010\rangle + \frac{1}{\sqrt{6}}|001\rangle$$

A. Acín, A. Andrianov, L. Costa, E. Jané, J. I. Latorre, and R. Tarrach, PRL. **85,** 1560 (2000)

Tripartite State Classification

There are three classes of tripartite states:

- Product states – No entanglement
- Bipartite states – Entanglement between two of the three qubits
- Tripartite entangled states – Entanglement between all three qubits

The ideal forms of the two classes of entangled tripartite states are:

$$|\Psi_W\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle)$$

$$|\Psi_{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

All tripartite entangled states are members of either the GHZ or the W class.

W. Dür, G. Vidal, and J. I. Cirac, Phys. Rev. A **62,** 062314 (2000)

Classification

$\hat{k}p,\hat{k}1,\hat{k}2$	$\frac{5}{2}-\frac{3}{2}-\frac{1}{2}$	$\frac{3}{2}-\frac{3}{2}-\frac{1}{2}$	$\frac{1}{2}-\frac{3}{2}-\frac{1}{2}$	$\frac{3}{2}-\frac{1}{2}-\frac{1}{2}$	$\frac{1}{2}-\frac{1}{2}-\frac{1}{2}$
$\hat{x},\hat{x},\hat{x}$	A–BC	A–BC	A–BC	A–B–C	A–B–C
$\hat{x},\hat{x},\hat{y}$	GHZ	GHZ	GHZ	GHZ	GHZ
$\hat{x},\hat{x},\hat{z}$	GHZ	GHZ	GHZ	GHZ	GHZ
$\hat{x},\hat{y},\hat{x}$	GHZ	GHZ	GHZ	C–AB	C–AB
$\hat{x},\hat{y},\hat{y}$	GHZ	GHZ	GHZ	GHZ	GHZ
$\hat{x},\hat{y},\hat{z}$	W	GHZ	GHZ	GHZ	GHZ
$\hat{x},\hat{z},\hat{x}$	GHZ	GHZ	GHZ	C–AB	C–AB
$\hat{x},\hat{z},\hat{y}$	W	GHZ	GHZ	GHZ	GHZ
$\hat{x},\hat{z},\hat{z}$	A–BC	GHZ	GHZ	GHZ	GHZ
$\hat{z},\hat{x},\hat{x}$	GHZ	GHZ	GHZ	GHZ	GHZ
$\hat{z},\hat{x},\hat{y}$	W	GHZ	GHZ	GHZ	GHZ
$\hat{z},\hat{x},\hat{z}$	GHZ	GHZ	GHZ	C–AB	C–AB
$\hat{z},\hat{z},\hat{x}$	GHZ	GHZ	GHZ	GHZ	GHZ
$\hat{z},\hat{z},\hat{z}$	A–BC	A–BC	A–BC	A–B–C	A–B–C

$$|\Psi_{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

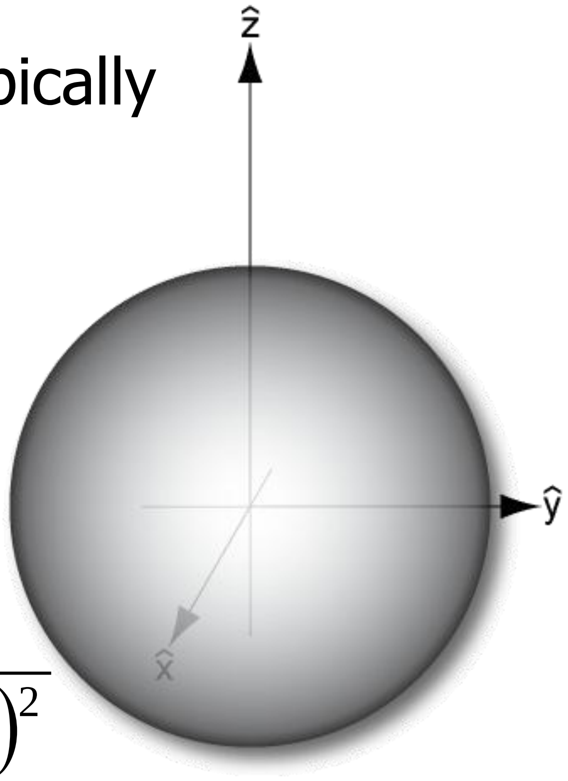
$$|\Psi_{W}\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle)$$

Experimental Concerns: I

1. Are entangling cascades frequent?

<u>Probability Distribution</u>

PD = 0	Event Never Happens
PD = 1	Event Happens Isotropically



GHZ: PD = 1
W : PD = 0.675

Scattering probability = $d\Omega 1 * d\Omega 2 * PD * \dfrac{1}{(4\pi)^2}$

Experimental Concerns: II

2. There is a balance between

- final state overlap with the perfect entangled state
- the probability of a scattering event.

↑ Acceptance angle ↓ Ideal entanglement

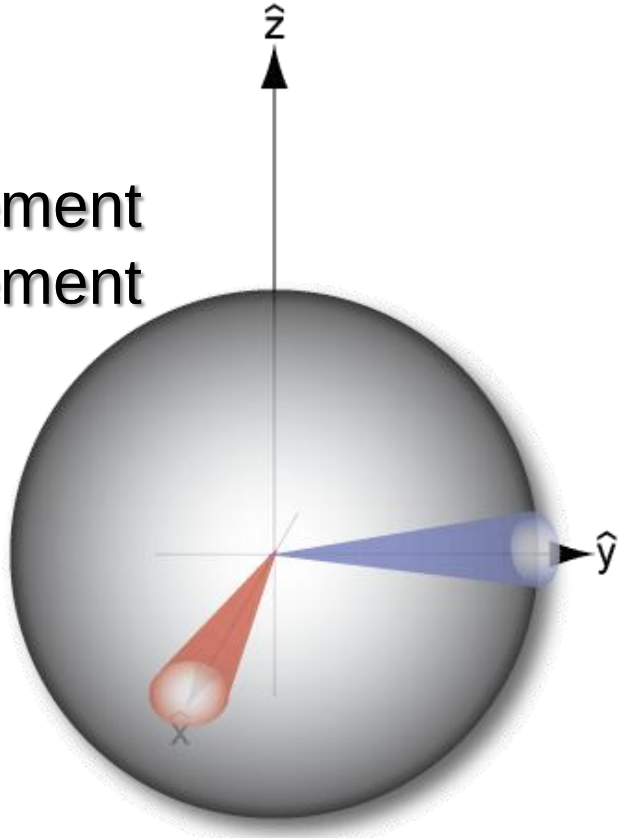
↓ Acceptance angle ↑ Ideal entanglement

W and GHZ type cascades with acceptance angles up to **12°**

•maintain 0.9 overlap

•scattering probabilities of 1E-4

For an atom with decay rate 1E6 Hz, we expect **100Hz** entangling cascade rates



Conclusions

We can produce prototypical W and GHZ states immediately in three level atomic systems.

A sample atomic system with the 5/2-3/2-1/2 level structure which can produce the prototypical W state is Yb^{171}.

An ideal W state can be used in a quantum router to employ a teleportation protocol with two potential recipients (Bob and Chris) from a single source (Alice).

Asymmetric W and GHZ states can be used in Quantum Information schemes.

P. Agrawal & A. Pati (2006). `Perfect Teleportation and Superdense Coding With W-States' .

AMO Group

Leanhardt Research Group

Aaron Leanhardt
(PI)

Jeongwon Lee
(grad)

Jinhai Chen
(post-doc)

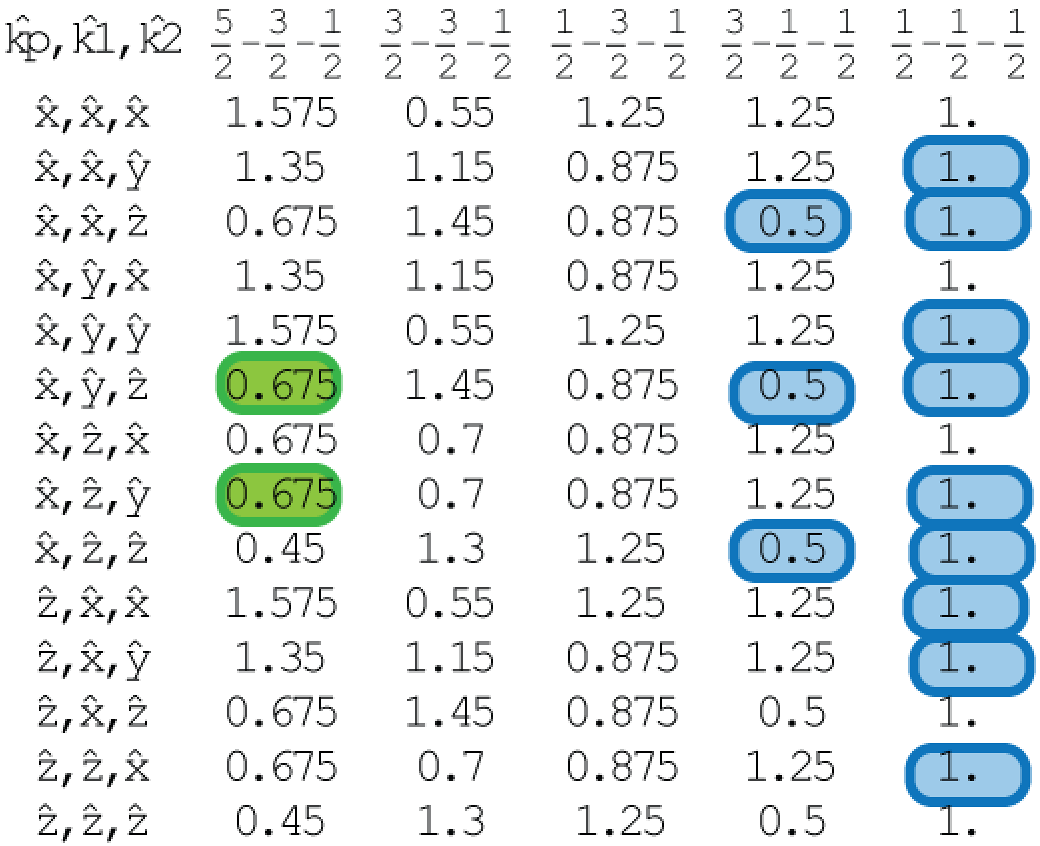
Emily Alden (+1)
(grad)

Yisa Rumala
(grad)

Kaitlin Moore
(post-bac)

Not Shown:
- Charlie Steiner (undergrad)

Entangling Cascade Frequency



$\hat{k}p,\hat{k}1,\hat{k}2$	$\frac{5}{2}-\frac{3}{2}-\frac{1}{2}$	$\frac{3}{2}-\frac{3}{2}-\frac{1}{2}$	$\frac{1}{2}-\frac{3}{2}-\frac{1}{2}$	$\frac{3}{2}-\frac{1}{2}-\frac{1}{2}$	$\frac{1}{2}-\frac{1}{2}-\frac{1}{2}$
$\hat{x},\hat{x},\hat{x}$	1.575	0.55	1.25	1.25	1.
$\hat{x},\hat{x},\hat{y}$	1.35	1.15	0.875	1.25	1.
$\hat{x},\hat{x},\hat{z}$	0.675	1.45	0.875	0.5	1.
$\hat{x},\hat{y},\hat{x}$	1.35	1.15	0.875	1.25	1.
$\hat{x},\hat{y},\hat{y}$	1.575	0.55	1.25	1.25	1.
$\hat{x},\hat{y},\hat{z}$	0.675	1.45	0.875	0.5	1.
$\hat{x},\hat{z},\hat{x}$	0.675	0.7	0.875	1.25	1.
$\hat{x},\hat{z},\hat{y}$	0.675	0.7	0.875	1.25	1.
$\hat{x},\hat{z},\hat{z}$	0.45	1.3	1.25	0.5	1.
$\hat{z},\hat{x},\hat{x}$	1.575	0.55	1.25	1.25	1.
$\hat{z},\hat{x},\hat{y}$	1.35	1.15	0.875	1.25	1.
$\hat{z},\hat{x},\hat{z}$	0.675	1.45	0.875	0.5	1.
$\hat{z},\hat{z},\hat{x}$	0.675	0.7	0.875	1.25	1.
$\hat{z},\hat{z},\hat{z}$	0.45	1.3	1.25	0.5	1.

1 : stochastic

<1 : worse than stochastic

>1 : better than stochastic

W-rate ≈ 0.675*4E-6

GHZ-rate ≈1.0*4E-6

$$|\Psi_W\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle)$$

$$|\Psi_{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

*acceptance angle of 5° → 4E-6

Evaluating Tripartite Classifications

The classification of a given tripartite state can be evaluated rapidly by measuring the entropy of each qubit. For tripartite entangled states there is an additional measure, the tangle, which determines if the state is GHZ or W like.

	S_A	S_B	S_C	τ
Product (A-B-C)	0	0	0	0
Bipartite (A-BC)	0	>0	>0	0
Bipartite (B-AC)	>0	0	>0	0
Bipartite (C-AB)	>0	>0	0	0
W	>0	>0	>0	0
GHZ	>0	>0	>0	>0

W. Dür, G. Vidal, and J. I. Cirac, Phys. Rev. A **62,** 062314 (2000)

Compute Final State Vectors

Eight Element Tripartite Wavefunction:

$$\Psi = \sum_{m=-\frac{1}{2}}^{\frac{1}{2}} \sum_{j=\hat{\theta},\hat{\phi}} \sum_{k=\hat{\theta},\hat{\phi}} \xi_{mjk} \left|\frac{1}{2}, m; \hat{\varepsilon_k}^{(1)}; \hat{\varepsilon_j}^{(2)}\right\rangle$$

Tripartite State Amplitudes:

$$\xi_{mjk} = \sum_{q=-1}^{1} \sum_{m''=-\frac{1}{2}}^{\frac{1}{2}} A_{|F'',m''\rangle} \langle \frac{1}{2}, m | \hat{\varepsilon_k}^{(1)} \cdot \hat{D}^{(1)} | F', m''+q \rangle \times \langle F', m''+q | \hat{\varepsilon_j}^{(2)} \cdot \hat{D}^{(2)} | F'', m'' \rangle$$

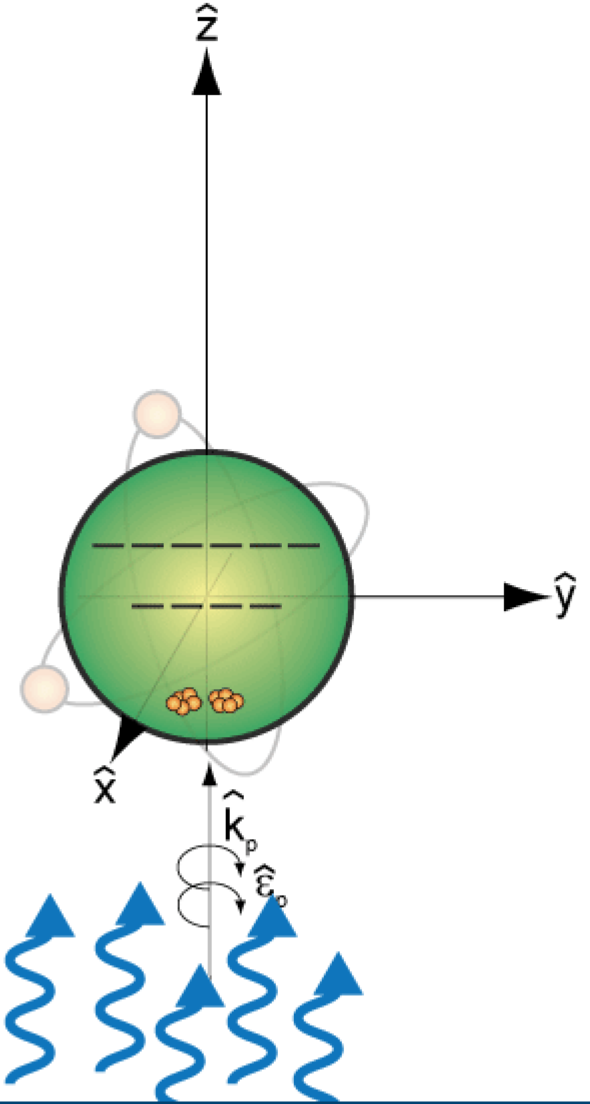
I. I. Sobelman, *Atomic spectra and radiative transitions*
(Springer-Verlag, Berlin ; New York :, 1992), ISBN 0387545182

Sample Entanglement Cascade – ZXY

Prepare the Ground State

Optically pump the atom so the ground state population is all in $|\frac{1}{2}, -\frac{1}{2}\rangle$.

We designate this as the **z-direction**.



Other Tripartite Entangling Schemes

Ions

- K. Kim, et al. (2010). `Quantum simulation of frustrated Ising spins with trapped ions'. *Nature* **465**(7298)

Photons

- M. Bourennane, et al. (2004). `Experimental Detection of Multipartite Entanglement using Witness Operators'. *Physical Review Letters* **92**(8):087902+.
- A. S. Coelho, et al. (2009). `Three-Color Entanglement'. *Science* **326**(5954):823-826.

Positronium

- A. Acin, et al. (2001). `Three-party entanglement from positronium'. *Physical Review A* **63**(4)

Feasibility Example

For a stochastic radiation pattern, if we include photons with an acceptance angle of 5°, we would have a scattering probability of 4E-6

W and GHZ type cascades will maintain 0.9 overlap with acceptance angles up to **12°**, this creates scattering probabilities = 1E-4

An atomic system with 1E6 scattering rates is plausible, so 1-100 Hz entangling event rates are possible.