Restart perturbations for reversible Markov chains
Trichotomy and pre-cutoff equivalence

Daniel Vial, Vijay Subramanian

ECE Department, University of Michigan
Discrete-time Markov chain (DTMC): a common stochastic model

Fundamental concern: sensitivity of stationary distribution to (bounded) perturbations of transition matrix

- e.g. how much do modeling errors affect long-run behavior?
- e.g. how much damage can adversary do long-term?
Motivation (2/2)

Sensitivity and **mixing time**

- Mixing time: number steps for DTMC to reach equilibrium/stationarity
- If fast mixing, less time for perturbation to take effect
- If slow mixing, small perturbation may have large effect

In particular, connection to **cutoff**

- Cutoff: far from stationary for many steps, then suddenly stationary

<table>
<thead>
<tr>
<th>$k$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$|P^k - \pi|$</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>0.924</td>
<td>0.624</td>
<td>0.312</td>
<td>0.161</td>
<td>0.083</td>
<td>0.041</td>
</tr>
</tbody>
</table>

(from Diaconis 1996)

- Many examples, but little general theory
Overview

1. Perturbation model

2. Mixing times and cutoff

3. Illustrative examples

4. Results (and related work)
**Basic definitions**

DTMC \( \{X_n(t)\}_{t \in \mathbb{Z}^+} \) with states \([n] = \{1, \ldots, n\}\) and transition matrix \(P_n:\)

\[
\mathbb{P}(X_n(t + 1) = j | X_n(t) = i) = P_n(i, j) \quad \forall \, i, j \in [n], \, t \in \mathbb{Z}^+
\]

Assume irreducible, aperiodic \(\Rightarrow\) converge to unique stationary distribution \(\pi_n\) from any starting state:

\[
\lim_{t \to \infty} P_n^t(i, \cdot) = \pi_n = \pi_n P_n \quad \forall \, i \in [n]
\]

Sometimes require reversibility (a.k.a. local balance):

\[
\pi_n(i) P_n(i, j) = \pi_n(j) P_n(j, i) \quad \forall \, i, j \in [n]
\]

Sometimes require laziness (but believe we can drop this):

\[
P_n(i, i) \geq \frac{1}{2} \quad \forall \, i \in [n]
\]
Perturbation model

Given $\alpha_n \in (0, 1)$, define

$$B(P_n, \alpha_n) = \left\{ \tilde{P}_n \in \mathcal{E}_n : \max_{i \in [n]} \| \tilde{P}_n(i, \cdot) - P_n(i, \cdot) \|_{TV} \leq \alpha_n \right\}$$

where

- $\mathcal{E}_n =$ set of $n \times n$ irreducible/aperiodic transition matrices
- $\| \mu - \nu \|_{TV} = \max_{A \subseteq [n]} | \mu(A) - \nu(A) |$ for distributions $\mu, \nu$ on $[n]$.

Denote stationary distribution of $\tilde{P}_n \in \mathcal{E}_n$ by $\tilde{\pi}_n$

**Main focus of this work:** relationship between $\| \tilde{\pi}_n - \pi_n \|_{TV}$ and $\alpha_n$

i.e. how does steady-state error scale with perturbation magnitude?
Illustration and interpretations

<table>
<thead>
<tr>
<th>Interpretation 1</th>
<th>Interpretation 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P_n )</td>
<td>True dynamics</td>
</tr>
<tr>
<td>( \tilde{P}_n )</td>
<td>Modeled dynamics</td>
</tr>
<tr>
<td>( \alpha_n )</td>
<td>Modeling accuracy</td>
</tr>
<tr>
<td>( |\tilde{\pi}<em>n - \pi_n|</em>{TV} )</td>
<td>Effect of modeling errors</td>
</tr>
</tbody>
</table>
Restart perturbations

Given $\alpha_n \in (0, 1)$ and distribution $\sigma_n$ over $[n]$, define

$$P_{\alpha_n, \sigma_n} = (1 - \alpha_n)P_n + \alpha_n 1_n^T \sigma_n \in B(P_n, \alpha_n)$$

and denote corresponding stationary distribution by $\pi_{\alpha_n, \sigma_n}$

Interpretation:

- Flip Bernoulli($\alpha_n$) coin at each step
- If tails, sample next state from $P_n$
- If heads, sample next state from $\sigma_n$

Note: $\pi_{\alpha_n, \pi_n} = \pi_n$ (perturbations need not change stationary distribution)
Properties of restart perturbations

1. **Power iteration**: Solving $\pi_{\alpha_n,\sigma_n} = \pi_{\alpha_n,\sigma_n}P_{\alpha_n,\sigma_n}$ gives

   $$\pi_{\alpha_n,\sigma_n} = \alpha_n\sigma_n(I_n - (1 - \alpha_n)P_n)^{-1} = \alpha_n\sigma_n\sum_{t=0}^{\infty}(1 - \alpha_n)^t P_n^t$$

2. **Linearity**: For distributions $\sigma_{n,1}, \sigma_{n,2}$ and $\lambda \in (0, 1)$,

   $$\pi_{\alpha_n,\lambda\sigma_{n,1} + (1 - \lambda)\sigma_{n,2}} = \lambda\pi_{\alpha_n,\sigma_{n,1}} + (1 - \lambda)\pi_{\alpha_n,\sigma_{n,2}}$$

3. **Perfect sampling**: Letting $T \sim \text{Geometric}(\alpha_n)$,

   $$\pi_{\alpha_n,\sigma_n}(i) = \mathbb{P}(X_n(T) = i|X_n(0) \sim \sigma_n)$$
Sidebar: (Personalized) PageRank

When $P_n$ is random walk on $G = ([n], E)$ . . .

- $\pi_{\alpha_n, 1/n}$ called PageRank, measure of importance/centrality of nodes
- $\pi_{\alpha_n, e_i}$ called Personalized PageRank for $i$, measure of relevance to $i$

Applications: Internet search (Page et al. 1999), recommendation systems (Gupta et al. 2013), community detection (Andersen, Chung, Lang 2006; Kloumann, Ugander, Kleinberg 2017), graph similarity (Koutra, Vogelstein, Faloutsos 2013), bioinformatics (Morrison et al. 2005; Freschi 2007), . . .

Rich literature: efficient PageRank estimation (Jeh, Widom 2003; Avrachenkov et al. 2007; Andersen et al. 2008; Lofgren, Banerjee, Goel 2016; Vial, Subramanian 2017), implementation e.g. MapReduce (Tong, Faloutsos, Pan 2006; Kang, Tsourakakis, Faloutsos 2009; Whang et al. 2015), behavior on random graphs (Avrachenkov et al. 2015; Chen, Litvak, Olvera-Cravioto 2017; Caputo, Quattropani 2019; Vial, Subramanian 2019), . . .
Mixing times

Define \( t \)-step distance from stationarity and \( \epsilon \)-mixing time as

\[
d_n(t) = \max_{i \in [n]} \| P_n^t(i, \cdot) - \pi_n \|_{TV}, \quad t_{\text{mix}}^{(n)}(\epsilon) = \min \{ t \in \mathbb{Z}_+ : d_n(t) \leq \epsilon \}
\]

Also define mixing time \( t_{\text{mix}}^{(n)} = t_{\text{mix}}^{(n)} \left( \frac{1}{4} \right) \)

- Well-known fact: \( d_n(kt_{\text{mix}}^{(n)}(\epsilon)) \leq (2\epsilon)^k \); \( \epsilon = \frac{1}{4} \) gives \( d_n(kt_{\text{mix}}^{(n)}) \leq 2^{-k} \)

Typically desire “Big-O” behavior of \( t_{\text{mix}}^{(n)} \) for sequence \( \{P_n\}_{n \in \mathbb{N}} \)

- e.g. \( t_{\text{mix}}^{(n)} = \Theta(n^2) \) for random walk on \( n \)-cycle

More refined question: how “sharply” does graph of \( d_n(t) \) decay to 0?
(Pre-)cutoff

Sequence of chains exhibits \textit{cutoff} if

\[
\lim_{{n \to \infty}} \frac{t_{\text{mix}}^{(n)}(\epsilon)}{t_{\text{mix}}^{(n)}(1 - \epsilon)} = 1 \quad \forall \ \epsilon \in \left(0, \frac{1}{2}\right)
\]

Weaker condition of \textit{pre-cutoff} only requires

\[
\sup_{\epsilon \in (0, \frac{1}{2})} \limsup_{{n \to \infty}} \frac{t_{\text{mix}}^{(n)}(\epsilon)}{t_{\text{mix}}^{(n)}(1 - \epsilon)} < \infty
\]
Many examples of chains with cutoff

- Classic examples (Diaconis 1996): card shuffling (e.g. Diaconis, Shahshahani 1981; Bayer, Diaconis 1992), diffusion models (e.g. Aldous 1983; Diaconis, Shahshahani 1987),...

- Recently, cutoff for generative models, e.g. random walks on random regular graphs (Lubetzky, Sly 2010), non-backtracking walks on sparse random graphs (Ben-Hamou, Salez 2017), general models of sparse chains (Bordenave, Caputo, Salez 2019), ...

Example of pre-cutoff but no cutoff (Levin, Peres, Wilmer 2009, Ch. 18):
Necessary condition for pre-cutoff (and thus cutoff): inverse spectral gap vanishes relative to $t^{(n)}_{\text{mix}}$ (Levin, Peres, Wilmer 2009, Prop. 18.4)

Spectral gap condition not sufficient (Levin, Peres, Wilmer 2009, Ex. 18.7)

No known necessary/sufficient condition in terms of eigenvalues alone (Diaconis 1996 conjectured high multiplicity of second eigenvalue)

Basu, Hermon, Peres 2015: cutoff equivalent to “hitting time cutoff”

Necessary/sufficient conditions for pre-cutoff?
Winning streak reversal (WSR) from Levin, Peres, Wilmer 2009

\[ P_n(i, j) = \begin{cases} 
2^{-j}, & i = 1, j \in \{1, \ldots, n-1\} \\
2^{-(n-1)}, & i = 1, j = n \\
1, & i \in \{2, \ldots, n-1\}, j = i - 1 \\
2^{-1}, & i = n, j \in \{n-1, n\} 
\end{cases} \]

\[ \pi_n = [2^{-1} \ 2^{-2} \ \cdots \ 2^{-(n-1)} \ 2^{-(n-1)}], \text{ since} \]

\[ \sum_{j=1}^{n} \pi_n(j)P_n(j, i) = 2^{-1} \times 2^{-i} + 2^{-(i+1)} \times 1 = 2^{-i} = \pi_n(i) \]
WSR mixing behavior

Stationary by step $n - 1$, i.e. $t_{\text{mix}}^{(n)}(1 - \epsilon)$, $t_{\text{mix}}^{(n)}(\epsilon)$ \leq n - 1

- $P_n(1, \cdot) = \pi_n$ by previous slide
- For $i \in \{2, \ldots, n - 1\}$, move to state 1 in $i - 1$ steps, so $P^i_n(i, \cdot) = \pi_n$
- More subtly, $P^{n-1}_n(n, \cdot) = \pi_n$

Lower bound: $t_{\text{mix}}^{(n)}(1 - \epsilon)$, $t_{\text{mix}}^{(n)}(\epsilon) \geq n - 1 + \log_2(\epsilon)$, since

$$d_n(t) = \max_{i \in [n], A \subset [n]} |P^t_n(i, A) - \pi_n(A)|$$

$$\geq P^t_n(n - 1, \{n - 1 - t\}) - \pi_n(\{n - 1 - t\}) = 1 - 2^{-n+1+t}$$

Combining gives strong form of cutoff:

$$1 \leq \frac{t_{\text{mix}}^{(n)}(\epsilon)}{t_{\text{mix}}^{(n)}(1 - \epsilon)} = 1 + O\left(\frac{1}{n}\right)$$
WSR perturbation behavior

Intuitively, “worst” perturbation is restart at \( n \), i.e. \( P_{\alpha_n, \epsilon_n} \)

Suppose \( \alpha_n t_{\text{mix}}^{(n)}(\epsilon) \to \infty \) (e.g. \( \alpha_n = 1/\sqrt{n} \)):

- Restart at \( n \) every \( \sqrt{n} \) steps \( \Rightarrow \) rarely visit \( [n/2] \) \( \Rightarrow \pi_{\alpha_n, \epsilon_n}([n/2]) \approx 0 \)
- Also \( \pi_n([n/2]) = \sum_{i=1}^{n/2} 2^{-i} \approx 1 \) (for large \( n \))
- Taken together, \( \|\pi_n - \pi_{\alpha_n, \epsilon_n}\|_{TV} \geq \pi_n([n/2]) - \pi_{\alpha_n, \epsilon_n}([n/2]) \approx 1 \)

If instead \( \alpha_n t_{\text{mix}}^{(n)}(\epsilon) \to 0 \) (e.g. \( \alpha_n = 1/n^2 \)):

- Visit state 1 before restart, reach original stationary distribution
- Once “locked into” this distribution, can’t escape
- Consequently, \( \|\pi_n - \pi_{\alpha_n, \epsilon_n}\|_{TV} \approx 0 \)
Complete graph bijection (CGB)

Construction:

- Draw complete graphs / cliques on $\{1, \ldots, \frac{n}{2}\}$ and $\{1 + \frac{n}{2}, \ldots, n\}$
- Add edges $\{(1, 1 + \frac{n}{2}), (2, 2 + \frac{n}{2}), \ldots, (\frac{n}{2}, n)\}$

We study lazy simple random walk on resulting graph

By symmetry, uniform distribution is stationary
CGB mixing behavior

“Half-mixed” in one step, i.e. \( t_{\text{mix}}^{(n)}(1 - \epsilon) = \Theta(1) \)

- \( P_n(X_n(0), \cdot) \approx \text{uniform on } X_n(0)’s \text{ clique} \Rightarrow \|P_n(X_n(0), \cdot) - \pi_n\|_{TV} \approx \frac{1}{2} \)

To fully mix, require \( n \) steps, i.e. \( t_{\text{mix}}^{(n)}(\epsilon) = \Theta(n) \)

- At each step, \( \Theta(\frac{1}{n}) \) mass escapes \( X_n(0)’s \text{ clique} \)
- Thus, \( \Theta(n) \) steps for constant mass to escape \( X_n(0)’s \text{ clique} \)

“Opposite” of cutoff, since

\[
\frac{t_{\text{mix}}^{(n)}(\epsilon)}{t_{\text{mix}}^{(n)}(1 - \epsilon)} = \Theta(n)
\]

which is maximal (up to constants) among \( t_{\text{mix}}^{(n)}(\epsilon) = \Theta(n) \) chains
By symmetry, “worst” (restart) perturbation is $P_{\alpha_n, e_1}$

One step after restart, uniform on $\{1, \ldots, \frac{n}{2}\}$

“Half-mixed”, can’t fully escape this distribution

More specifically, can show that for any $\alpha_n \to 0$ and any $\tilde{P}_n \in B(P_n, \alpha_n)$,

$$\limsup_{n \to \infty} \|\tilde{\pi}_n - \pi_n\|_{TV} \leq \frac{1}{2}$$
Summary and key insights

WSR: strong form of cutoff, sensitive to $\alpha_n$-magnitude perturbations

- Assuming $\alpha_n \gg 1/t_{\text{mix}}^{(n)}(\epsilon)$

CGB: “opposite” of cutoff, robust against perturbation

Do these phenomena hold more generally?
**Theorem (Vial, Subramanian 2019)**

Let $P_n \in \mathcal{E}_n, \alpha_n \in (0, 1), \epsilon \in (0, 1)$. Assume $\lim_{n \to \infty} \alpha_n t_{\text{mix}}^{(n)}(\epsilon) = c \in [0, \infty]$, $P_n$ is lazy and reversible $\forall$ $n$, and $\{P_n\}$ has cutoff.

- **If $c = 0$, then $\forall \{\tilde{P}_n\}$ with $\tilde{P}_n \in B(P_n, \alpha_n) \forall$ $n$,**
  \[
  \lim_{n \to \infty} \|\tilde{\pi}_n - \pi_n\|_{TV} = 0.
  \]

- **If $c = \infty$, then $\exists \{\sigma_n\}$ with $\sigma_n \in \Delta_{n-1} \forall$ $n$ s.t.**
  \[
  \lim_{n \to \infty} \|\pi_{\alpha_n, \sigma_n} - \pi_n\|_{TV} = 1.
  \]

- **If $c \in (0, \infty)$, then $\forall \{\sigma_n\}$ with $\sigma_n \in \Delta_{n-1} \forall$ $n$,**
  \[
  \limsup_{n \to \infty} \|\pi_{\alpha_n, \sigma_n} - \pi_n\|_{TV} \leq 1 - e^{-c}.
  \]
  Also, the bound is tight, i.e. $\exists \{\sigma_n\}$ with $\sigma_n \in \Delta_{n-1} \forall$ $n$ s.t.
  \[
  \liminf_{n \to \infty} \|\pi_{\alpha_n, \sigma_n} - \pi_n\|_{TV} \geq 1 - e^{-c}.
  \]
Notes on theorem

Tight characterization: given any $c' \in [0, 1]$, can choose $\alpha_n, \sigma_n$ s.t.

$$\lim_{n \to \infty} \| \pi_{\alpha_n, \sigma_n} - \pi_n \|_{TV} = c'.$$

Some bounds require weaker assumptions (see Lemmas 1-2 in paper)

- Upper bounds don’t need laziness, reversibility, or cutoff
- Lower bounds only need pre-cutoff in the case $\alpha_n t_{mix}^{(n)}(\epsilon) \to \infty$

Notes on assumptions for lower bound:

- Believe we can drop laziness
- Reversibility common in mixing times literature (real eigenvalues)
- Necessity of (pre-)cutoff – next theorem
Basu, Hermon, Peres 2015: if pre-cutoff holds, \( \exists x_n \in [n], A_n \subset [n] \) s.t.

\[
\pi_n(A_n) \approx 1, \quad t_{\text{mix}}^{(n)}(\epsilon) \propto \text{time to reach } A_n \text{ from } x_n
\]

i.e. “mixing times are hitting times of large sets” (also Peres, Sousi 2015)
Related results in literature (1/2)

Caputo, Quattropani 2019 shows distance to stationarity \(d_{\alpha_n,\sigma_n}(t)\) on \(P_{\alpha_n,\sigma_n}\) follows trichotomy when \(P_n = \) random walk on configuration model (CM):

- If \(\alpha_n t_{\text{mix}}(\epsilon) \to 0\), \(d_{\alpha_n,\sigma_n}\) is step function, like \(d_n\) (left)
- If \(\alpha_n t_{\text{mix}}(\epsilon) \to \infty\), \(d_{\alpha_n,\sigma_n}(t)\) decays exponentially (right)
- If \(\alpha_n t_{\text{mix}}(\epsilon) \to (0, \infty)\), interpolation (middle)

Avena et al. 2018: similar result for dynamic CM
Vial, Subramanian 2019 studies matrix $\Pi = \{\pi_{\alpha_n,e_i}\}_{i \in [n]}$ for CM

- If $\alpha_n t_{\text{mix}}(\epsilon) \to 0$, $\text{rank}(\Pi) = \Theta(1)$
- If $\alpha_n t_{\text{mix}}(\epsilon) \to \infty$, $\text{rank}(\Pi) = \Omega(\frac{n}{\log n})$ (conjecture)
- If $\alpha_n t_{\text{mix}}(\epsilon) \to (0, \infty)$, $\text{rank}(\Pi) = O(n^c), c \in (0, 1)$

These four results have similar flavor:

- If $\alpha_n t_{\text{mix}}(\epsilon) \to 0$, some property unchanged
- If $\alpha_n t_{\text{mix}}(\epsilon) \to \infty$, this property maximally changed
- If $\alpha_n t_{\text{mix}}(\epsilon) \to (0, \infty)$, intermediate behavior

Suggests (ill-posed) question: characterizing this class of properties
Some notion of cutoff assumed for all of our lower bounds – is it necessary?

**Sensitivity Condition (SC)**

For certain sequences \( \{\alpha_n, \epsilon\}_{n \in \mathbb{N}, \epsilon \in (0, \frac{1}{2})} \subset (0, 1) \), \( \exists \{\sigma_n, \epsilon\}_{n \in \mathbb{N}, \epsilon \in (0, \frac{1}{2})} \) such that

\[
\lim_{n \to \infty} \| \pi_n - \pi_{\alpha_n, \epsilon, \sigma_n, \epsilon} \|_{TV} = 1 \quad \forall \ \epsilon \in (0, \frac{1}{2}).
\]

**Theorem (Vial, Subramanian 2019)**

Assume each \( P_n \) is lazy and reversible \( \forall \ n \). If pre-cutoff holds, then SC holds. If

\[
\sup_{\epsilon \in (0, \frac{1}{2})} \liminf_{n \to \infty} \frac{t_{mix}^{(n)}(\epsilon)}{t_{mix}^{(n)}(1 - \epsilon)} = \infty,
\]

then SC fails.

1. (1) slightly stronger than converse of pre-cutoff (lim inf instead of lim sup)
2. Thus, theorem says pre-cutoff and SC (almost) equivalent
3. Complements Basu, Hermon, Peres 2015 (cutoff \( \Leftrightarrow \) “hitting time cutoff”)

---

**Pre-cutoff “equivalence”**

---
Illustration of “equivalence”

Parition of lazy/reversible sequences induced by SC:

<table>
<thead>
<tr>
<th>SC holds</th>
<th>SC fails</th>
</tr>
</thead>
</table>
| \[
\sup_{\epsilon \in (0,1/2)} \limsup_{n \to \infty} \frac{t^{(n)}_\text{mix}(\epsilon)}{t^{(n)}_\text{mix}(1 - \epsilon)} < \infty
\] | \[
\sup_{\epsilon \in (0,1/2)} \liminf_{n \to \infty} \frac{t^{(n)}_\text{mix}(\epsilon)}{t^{(n)}_\text{mix}(1 - \epsilon)} = \infty
\] |

Believe gray area only contains “bizarre” sequences
- e.g. alternating between WSR and CGB
“Certain sequences”

Sensitivity Condition (SC)

For any \( \{\alpha_n, \epsilon\}_n \in \mathbb{N}, \epsilon \in (0, 1/2) \subset (0, 1) \) satisfying (2), \( \exists \{\sigma_n, \epsilon\}_n \in \mathbb{N}, \epsilon \in (0, 1/2) \) such that \( \lim_{n \to \infty} \|\pi_n - \pi_{\alpha_n, \epsilon, \sigma_n, \epsilon}\|_{TV} = 1 \ \forall \ \epsilon \in (0, 1/2). \)

\[
\sup_{\epsilon \in (0, 1/2)} \liminf_{n \to \infty} \alpha_n, \epsilon t_{mix}^{(n)}(\epsilon) = \infty, \ \frac{\alpha_n, \epsilon}{\alpha_n, \delta} \in \left[ \frac{t_{mix}^{(n)}(1 - \delta)}{t_{mix}^{(n)}(1 - \epsilon)}, 1 \right] \ \forall \ 0 < \delta \leq \epsilon < \frac{1}{2}. \tag{2}
\]

Proof intuition broadly similar to previous theorem

Little intuition for (2), but yields the following “magic”:

- When pre-cutoff fails, \( \{\{\alpha_n, \epsilon\}_n, \epsilon : (2) \text{ holds}\} \) is sufficiently large that we can find \( \{\sigma_n, \epsilon\}_n, \epsilon \) violating conclusion of SC

- When pre-cutoff holds, \( \{\{\alpha_n, \epsilon\}_n, \epsilon : (2) \text{ holds}\} \) collapses to a set for which conclusion of SC holds
Conclusions

Main takeaways

- Perturbation error depends on perturbation magnitude $\times$ mixing time
- Pre-cutoff almost equivalent to certain notion of “perturbation sensitivity”

Further reading

- Vial, Subramanian, “A Structural Result for Personalized PageRank and its Algorithmic Consequences”, POMACS, June 2019
- Levin, Peres, Wilmer, “Markov Chains and Mixing Times”


Andersen, Reid et al. (2008). “Local computation of PageRank contributions”. In: Internet Mathematics 5.1-2, pp. 23–45.


time” for sparse Markov chains”. In: *Probability Theory and Related Fields* 173.1-2, 
pp. 261–292.

Caputo, Pietro, Matteo Quattropani (2019). “Mixing time of PageRank surfers on 

PageRank on directed configuration networks”. In: *Random Structures & 

Diaconis, Persi (1996). “The cutoff phenomenon in finite Markov chains”. In: 
*Proceedings of the National Academy of Sciences* 93.4, pp. 1659–1664.

with random transpositions”. In: *Probability Theory and Related Fields* 57.2, 
pp. 159–179.

– (1987). “Time to reach stationarity in the Bernoulli–Laplace diffusion model”. In: 

Freschi, Valerio (2007). “Protein function prediction from interaction networks using a 
random walk ranking algorithm”. In: *Bioinformatics and Bioengineering, 2007. BIBE 

Gupta, Pankaj et al. (2013). “WTF: The who to follow service at twitter”. In: 
*Proceedings of the 22nd international conference on World Wide Web*. ACM, 
pp. 505–514.

Jeh, Glen, Jennifer Widom (2003). “Scaling personalized web search”. In: *Proceedings 


Vial, Daniel, Vijay Subramanian. “Restart perturbations for lazy, reversible Markov chains: trichotomy and pre-cutoff equivalence”. In: *(In Preparation)* ()..

