Sequential decomposition of dynamic games with asymmetric information and dependent states

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Abstract

We consider a general finite-horizon dynamic game with asymmetric information with \( N \) selfish players, where there exists an underlying state of the system that is a controlled Markov process, controlled by players’ actions. In each period, a player makes a common observation of the state together with all the other players, and a private observation, and gets an instantaneous reward which is a function of the state and everyone’s actions. The players’ private observations are conditionally independent across time, conditioned on the system state and players’ previous actions, however, they are potentially correlated among players in each period. An appropriate notion of equilibrium for such games is Perfect Bayesian Equilibrium (PBE) which consists of a strategy and a belief profile of the players which is coupled across time and as a result, the complexity of finding such equilibria grows double-exponentially in time. In this paper, we first present structural results for the optimum policies of the team version of this problem where players are cooperative i.e. have the same objective. Using these results as motivation for the definition of information state, we present a sequential decomposition methodology to compute structured perfect Bayesian equilibria (SPBE) of this game, introduced in [1]. This methodology computes SPBE in linear time. In general, these equilibria exhibit signaling behavior, i.e. players’ actions reveal part of their private information that is payoff relevant to other players.

I. Introduction

Information asymmetry among strategic agents is an important topic, which has seen some very influential works such as [2] and [3]. Akerlof in [2] and Spence in [3] modeled a market of cars and a job market, respectively, as instances of information asymmetry in a game, and show interesting behavior of strategic agents derived from these models. Specifically, Akerlof in [2] showed that in a market of cars, where the quality of car is known only to the seller, lower prices can drive out good cars from the market. Spence in [3] showed that in equilibrium in a

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job market, a candidate can ‘signal’ her higher productivity to a potential employer by opting for higher education credentials. While these works showed very interesting and relevant phenomena for static information asymmetry, in the real world however, there exists many such, and even more complicated decision making scenarios which involves strategic decision makers with dynamically evolving information asymmetry. Some instances of such systems include: (a) in cyber-physical systems, many cyber and physical devices are connected to each other which have different information and they make a decision to optimize their performance objectives; (b) in a wind energy market a wind energy producer observes its own wind production privately and publicly observes the output of other producers which also determine the prices, and its objective is to generate output that maximizes its revenue; (c) in a social network, people have private opinions about a topic and also publicly observe actions of others, based on which they make a decision to maximize their utility. All such scenarios can be modeled as a dynamic game of asymmetric information.

Dynamical systems with strategic players are modeled as dynamic stochastic games, introduced by Shapley in \[5\]. Discrete-time dynamic games with Markovian structure have been studied extensively to model many practical applications, in engineering as well as economics literature \[6\], \[7\]. In dynamic games with perfect and symmetric information, subgame perfect equilibrium (SPE) is an appropriate equilibrium concept and there exists a backward recursive algorithm to find all the SPEs of these games (refer to \[8\]–\[10\] for a more elaborate discussion). Maskin and Tirole in \[11\] introduced the concept of Markov perfect equilibrium (MPE) for dynamic games with symmetric information, where equilibrium strategies are dependent on some payoff relevant Markovian state of the system, rather than on the entire history. In dynamic games of asymmetric information, there are strategic players who are affected by an underlying process that is dynamically evolving, and the players make asymmetric observations about that process. In such games and more generally in any dynamic multi-agent decision problem with asymmetric information, a player’s action not just either **explore** or **exploit** the system\[2\] as it happens in a single agent problem, but also **signal** i.e. reveal part of its private information to the other players that is payoff relevant to them\[3\]. Some appropriate notions of equilibrium for such games is Perfect Bayesian Equilibrium (PBE) or Sequential Equilibrium (SE) \[4\], \[8\] which involve an equilibrium strategy profile and an equilibrium belief profile of all the players, among other refinements. In these equilibria, the equilibrium strategies and beliefs are coupled together through a joint fixed-point equation in the space of strategies and beliefs for all players and for all histories of the game\[4\]. Since the history of such games grows exponentially, the complexity of finding equilibria of such game grows double exponentially in time, rendering such problems intractable. We refer the reader to the Introduction section of \[12\] for a thorough introduction and a brief literature survey on dynamic games with

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1. Sometimes also referred to as dynamic games of incomplete information.
2. Exploitation refers to making a decision based on whereas exploration refers to taking action that improves the current estimate of the state of the system even at some cost in the present, but that improves future reward.
3. Equivalently, signaling occurs in such decision problems if players’ beliefs on a payoff relevant state are strategy dependent i.e. they depend on the strategies of one or more players.
4. In comparison, Nash equilibrium for a static game is a fixed-point equation in the space of probability measures on possible actions of the players \[8\].
asymmetric information.

Recently, there have been a number of results on finding an information state for different classes of such games that decomposes these games across time (in an analogous way a dynamic program decomposes a dynamic optimization problem), and thus reduces the complexity of finding these equilibria from double-exponential to linear in time. Authors in [12], [13], and independently, authors in [14], presented such a sequential decomposition for games to find structured perfect Bayesian equilibrium (SPBE) and common information based perfect Bayesian equilibrium (CIB-PBE), respectively, where each player has a type or a state that evolves (conditionally) independently of other players’ types in a Markovian way. Authors in [15] extended those results to LQG games (i.e. with linear state update, quadratic instantaneous costs and Gaussian random variables), and to games with conditionally independent hidden Markovian types in [16], where instead of perfectly observing its own type, each player makes independent noisy observations about it, respectively. Authors in [17] considered such dynamic game with a system state and delayed information sharing pattern where each players learns every agents private observations and actions with delay of $d$–steps. Authors in [18, ch. 5] generalized that dynamic game with a Markovian state where players make a common and private observations of the state of the system, where these observations are conditionally independent among players, conditioned on the current state and previous action. In this paper, we consider a more general model that strictly encompasses the models considered in [12]–[18], where there exists an underlying state of the system that evolves as a conditionally independent Markov process, and players make arbitrary correlated common and private observations of the state.

A. Contributions

In this paper, we first study the team version of the problem where all the players are cooperative and want to jointly maximize the total expected sum reward of all the players. We pose this problem as an instance of a decentralized decision problem. Motivated by the common information approach presented in [19], we present structural results to find the optimum strategies as functions of appropriately defined common information belief state and private information belief states of the players. In this case, the belief states depend on the strategies of the players and thus there is signaling in the system where players’ actions reveal part of the private information of the players. We note that these results extend the results of [19], where the authors in [19] consider a model with conditionally independent private observations of the players, conditioned on the common information.

We then consider the case when the players are strategic. Similar to [12], we present a backward-forward algorithm to compute SPBE of the game, where each player’s strategy is a function of the same common information belief state and private information belief states computed in the team version of the problem. Using similar forward inductive arguments used in [12], it can be shown that such structured strategies form a rich class where any expected reward profile of the players that can be generated from any general strategy profile can also be generated using such structured strategy profile. These equilibria of the game are analogous to Markov Perfect equilibria (MPE) [11] of symmetric information games.
The paper is structured as follows. In Section II, we present the general model. In Section III, we consider the team problem with cooperative players for which we define information belief state, which further acts as the motivation for the structure of the equilibrium policies of the game version of the problem. In Section IV, we consider the finite horizon version of the game and present the backward-forward algorithm to compute SPBE of the game. We conclude in section VIII. All proofs are presented in Appendices.

B. Notation

We use uppercase letters for random variables and lowercase for their realizations. For any variable, subscripts represent time indices and superscripts represent player identities. We use notation $-i$ to represent all players other than player $i$ i.e. $-i = \{1, 2, \ldots i - 1, i + 1, \ldots, N\}$. We use notation $a_{t:t'}$ to represent vector $(a_t, a_{t+1}, \ldots a_{t'})$ when $t' \geq t$ or an empty vector if $t' < t$. We use $a^{-i}_t$ to mean $(a_1^t, a_2^t, \ldots, a_{i-1}^t, a_{i+1}^{t-1}, \ldots, a_N^t)$. We remove superscripts or subscripts if we want to represent the whole vector, for example $a_t$ represents $(a_1^t, \ldots, a_N^t)$. In a similar vein, for any collection of finite sets $(\mathcal{X}^i)_{i \in N}$, we denote $\times_{i=1}^N \mathcal{X}^i$ by $\mathcal{X}$. We denote the indicator function of any set $A$ by $I_A(\cdot)$. For any finite set $S$, $\Delta(S)$ represents space of probability measures on $S$ and $|S|$ represents its cardinality. We denote by $P^g$ (or $E^g$) the probability measure generated by (or expectation with respect to) strategy profile $g$. We denote the set of real numbers by $\mathbb{R}$. For a probabilistic strategy profile of players $(\beta^i_t)_{i \in N}$ where probability of action $a^i_t$ conditioned on $(a_1^{t-1}, w_1^{t-1})$ is given by $\beta^i_t(a^i_t|a_1^{t-1}, w_1^{t-1})$, we use the short hand notation $\beta^i_t(a_t^{-i}|a_1^{t-1}, w_1^{t-1})$ to represent $\prod_{j \neq i} \beta^j_t(a_t^j|a_1^{t-1}, w_1^{t-1})$. All equalities and inequalities involving random variables are to be interpreted in $a.s.$ sense.

II. GENERAL MODEL

We consider a discrete-time dynamical system with $N$ strategic players in the set $\mathcal{N} \triangleq \{1, 2, \ldots N\}$. We consider the finite horizon $\mathcal{T} \triangleq \{1, 2, \ldots T\}$ model with perfect recall. The system state is $x_t$, where $x_t \in \mathcal{X}$, which evolves as conditionally independent, controlled Markov process such that

$$P(x_t|x_{1:t-1}, a_{1:t-1}) = Q_t^x(x_t|x_{t-1}, a_{t-1})$$  \hspace{0.5cm} (1a)

where $a_t = (a_1^t, \ldots, a_N^t)$, and $a_i^t$ is the action taken by player $i$ at time $t$. In each period, players jointly observe $z_t \in \mathcal{Z}$, and make private observations $w_t$, where $w_i^t \in \mathcal{W}^i$ is the private observation of player $i$, where all observations are conditionally independent across time, however, conditionally dependent across the players given $(x_t, x_{t-1}, a_{t-1})$ in the following way. $\forall t \in 1, \ldots, T,$

$$P(z_{1:t}, w_{1:t}|x_{1:t}, a_{1:t-1}) = \prod_{n=1}^t Q_t^w(z_n, w_n|x_n, x_{n-1}, a_{n-1}),$$  \hspace{0.5cm} (2)

Our rationale for including $x_{t-1}$ in the observation model is that it easily incorporates the possibility that the rewards are observed by the players at the end of each period. This is because the reward observed by player $i$ at the end of period $t - 1$ and equivalently at the beginning of period $t$ is $R_i^t(x_{t-1}, a_{t-1}).$
where we also define player $i$’s observations $z_t, w^i_t$ through the measures $P^i,w_t(z_t, w^i_t|x_t, x_{t-1}, a_{t-1})$, and $P^s(z_t, w^i_t|x_t, x_{t-1}, a_{t-1})$ which are the marginals from the above kernel. We note that this includes the case where players observe actions of each other, in which case $a_{t-1} \subseteq z_t$. Player $i$ takes action $a^i_t \in A^i$ at time $t$ upon observing $z_{1:t}$ which is common information among players, and $w^i_{1:t}$ which is player $i$’s private information.

The sets $A^i, X, W^i, Z$ are assumed to be finite. Let $g^i = (g^i_t)_t$ be a probabilistic strategy of player $i$ where $g^i_t: Z^t \times (W^i)^t \rightarrow \mathcal{P}(A^i)$ such that player $i$ plays action $a^i_t$ according to $A^i_t \sim g^i_t(\cdot|z_{1:t}, w^i_{1:t})$. Let $g := (g^i)_{i \in \mathcal{N}}$ be a strategy profile of all players. At the end of interval $t$, player $i$ gets an instantaneous reward $R^i_t(x_t, a_t)$. For the finite-horizon problem, the objective of player $i$ is to maximize its total expected reward

$$J^g := \mathbb{E}^g \left[ \sum_{t=1}^T R^i_t(X_t, A_t) \right].$$

Although this model considers all $N$ players acting in all periods of the game, it can accommodate as cases where at each time $t$, players are chosen through an endogenously defined (controlled) Markov process. This can be done by a player selection process defined through $X_t$. For instance, let there exists a set $S \subset X, \mathcal{N}_s \subset \mathcal{N}$ such that for all $i \in \mathcal{N}_s$, $R^i_t(x_t, a_t) = 0$ if $x_t \in S$ and $Q^i_t(x_{t+1}|x_t, a_t) = Q^i_t(x_{t+1}|x_t, a^i_t|_{\mathcal{N} \setminus \mathcal{N}_s})$. Similarly for observation kernels of $w$, and $z$, they don’t depend on actions of player in $\mathcal{N}_s$. Here, whenever the state process enters $S$, players in the set $\mathcal{N}_s$ don’t get any reward and do not affect the state transitions, and thus are inactive as long as the process $\{X_t\}_t$ stays in the set $S$.

### III. Team Problem

In this section, we consider the finite-horizon case where all the players are cooperative and have the same objective of maximizing

$$J^g := \mathbb{E}^g \left[ \sum_{t=1}^T R^i_t(X_t, A_t) \right],$$

where we define $R_t(X_t, A_t) := \sum_{i \in \mathcal{N}} R^i_t(X_t, A_t)$. We assume the same information structure in the problem as defined in Section II such that a player’s strategy is of the form $a^i_t \sim g^i_t(\cdot|z_{1:t}, w^i_{1:t})$. Such decentralized problems can not be solved using classical tools from the theory of Markov Decision Processes (MDP) [20]. However, there exists some key ideas in the literature such as agent-by-agent approach [21] and common information approach [19] that present structural results of the optimum policies for a class of such systems with non-classical information structure. Such techniques have been used in dynamic team problems such as [22]–[24]. For this problem, we use an approach inspired by [19]. We note that while the results of this section motivate the structure of the equilibrium policies in the next section, these results are not used later in the paper for the analysis of the dynamic games, with the exception of the definitions of private beliefs $\xi_i$ and common beliefs $\pi_i$ and their updates in Lemma II.

Thus the reader interested in SPBE can skip Section III-B.

For every agent $i \in \mathcal{N}$ and any policy profile $g$, let $\xi^i_t \in \Delta(X')$ denote the private belief of agent $i$ on current
state of the system given its information,
\[ \xi^i_t(x_t) := P^g(X_t = x_t|z_{1:t}, w_{1:t}) \]  
(5)

Let \( \pi_t \in \Delta(\mathcal{X} \times (\Delta(\mathcal{X}))^N) \) be a common belief on \((x_t, \xi_t)\) conditioned on the common information \((z_{1:t})\), defined as follows.
\[ \pi_t(x_t, \xi_t) := P^g(X_t = x_t, \Xi_t = \xi_t|z_{1:t}) \]  
(6)

### A. Common agent approach

An alternative way to view the problem is as follows. As is done in common information approach \([19]\), at time \( t \), a fictitious common agent observes the common information \((z_{1:t})\) and generates prescription functions \( \gamma_t = (\gamma^i_t)_{i \in \mathcal{N}} = \psi[z_{1:t}] \). Player \( i \) uses these prescription functions \( \gamma^i_t \) to operate on its private information \( w^i_{1:t} \) to produce its action \( a^i_t \), i.e. \( \gamma^i_t : \Delta(\mathcal{X}) \rightarrow \Delta(\mathcal{A}^i) \) and \( a^i_t \sim \gamma^i_t(\cdot|w^i_{1:t}) = \psi[z_{1:t}](\cdot|w^i_{1:t}) \). It is clear that for any \( g \) policy profile of the players there exists an equivalent \( \psi \) profile of the common agent (and vice versa) that generates the same control actions for every realization of the information of the players. This approach transforms the decentralized control problem where multiple players use different strategies to control their actions to a centralized control problem of the fictitious common agent. Thus an optimal policy found for the fictitious common agent using standard tools from MDP Theory \([20]\) can be implemented distributedly by the agents.

We define a special type of common agent’s policy as follows. We call a common agent’s policy be of type \( \theta \) if the common agent observes the common belief \( \pi_t \) derived from the common observation \((z_{1:t})\), and generates prescription functions \( \gamma_t = (\gamma^i_t)_{i \in \mathcal{N}} = \theta_t[\pi_t] \). Player \( i \) uses these prescription function \( \gamma^i_t \) to operates on its private beliefs \( \xi^i_t \) to produce its action \( a^i_t \), i.e. \( \gamma^i_t : \Delta(\mathcal{X}) \rightarrow \Delta(\mathcal{A}^i) \) and \( a^i_t \sim \gamma^i_t(\cdot|\xi^i_t) = \theta[\pi_t](\cdot|\xi^i_t) \). Equivalently, we call a common agent’s policy be of type \( \theta \) if for all \( i \in \mathcal{N} \) and for all time \( t \), player \( i \)’s action \( a^i_t \) depends on its information \((z_{1:t}, w^i_{1:t})\) through the belief states \( \pi_t = P^g(X_t = x_t, \Xi_t = \xi_t|z_{1:t}) \) and \( \xi^i_t = P^g(X_t = x_t|z_{1:t}, w^i_{1:t}) \).

In the next lemma we show that for any given \( \theta \) policy, the belief states \( \pi_t \) and \( \xi^i_t \) can be updated recursively and jointly as follows. Let \( \xi^i_t(x_1) := Q^i_t(x_1) \) and \( \pi_1(x_1, \xi_1) := Q^1_t(x_1) \prod_{i=1}^N \delta_{\delta^i}(\xi^i_1) \).

**Lemma 1:** For any given policy of type \( \theta \), there exists update functions \( G^i_t \), independent of \( \theta^i \), such that
\[ \xi^{i+1}_{t+1} = G^i_t(\xi^i_t, \pi_t, w^i_{t+1}, z_{t+1}, \gamma^{i-1}) \]  
(7)
and update functions \( F_i \), independent of \( \theta \), such that
\[ \pi_{t+1} = F_i(\pi_t, \gamma_t, z_{t+1}) \].  
(8)

**Proof:** Please see Appendix A
Based on the above definitions of the belief states, in the next subsection, we present a dynamic program for the common agent to compute its optimum policies which equivalently computes the optimum decentralized policies of the players.

B. Dynamic program for the common agent’s problem

We define a common agent’s policy \( \theta = (\theta_i)_{i \in N, t \in T} \) and a sequence of functions \((V_t)_{t \in \{1, 2, \ldots, T+1\}}\), where \( V_t : \Delta(\mathcal{X} \times (\mathcal{P}(\mathcal{X}))^N) \rightarrow \mathbb{R} \), in a backward recursive way as follows.

1. Initialize \( \forall \pi_{T+1} \in \Delta(\mathcal{X} \times (\mathcal{P}(\mathcal{X}))^N) \),
   
   \[ V_{T+1}(\pi_{T+1}) := 0. \]  
   (9)

2. For \( t = T, T-1, \ldots, 1 \), \( \forall \pi_t \in \Delta(\mathcal{X} \times (\mathcal{P}(\mathcal{X}))^N) \), let \( \theta_t[\pi_t] \) be generated as follows. Set \( \hat{\gamma}_t = \theta_t[\pi_t] \), where \( \hat{\gamma}_t \) is the solution of the following optimization equation,

   \[ \hat{\gamma}_t \in \arg \max_{\gamma_t} \mathbb{E} \left\{ R_t(X_t, A_t) + V_{t+1}(F_t(\pi_t, \gamma_t, Z_{t+1})) \middle| \pi_t, \gamma_t \right\}, \]  
   (10)

   where expectation in (25) is with respect to random variables \((X_t, A_t, Z_{t+1})\) through the measure

   \[ \sum_{\xi_t} \pi_t(x_t, \xi_t) \gamma_t(a_t|\xi_t) \sum_{z_{t+1}} Q_{t+1}(x_{t+1}|x_t, a_t) P(z_{t+1}|x_{t+1}, x_t, a_t) F \]  

   Furthermore, set

   \[ V_t(\pi_t) := \mathbb{E} \left\{ R_t(X_t, A_t) + V_{t+1}(F_t(\pi_t, \hat{\gamma}_t, Z_{t+1})) \middle| \pi_t, \hat{\gamma}_t \right\}. \]  
   (11)

The following theorem shows that the policy \( \theta \) computed from the above dynamic program is optimum.

Theorem 1: For all strategies \( \psi \) of the common agent and \( \forall t, z_{1:t} \),

\[ \mathbb{E}^\theta \left\{ \sum_{n=1}^T R_n(X_n, A_n) \middle| z_{1:t} \right\} \geq \mathbb{E}^\psi \left\{ \sum_{n=1}^T R_n(X_n, A_n) \middle| z_{1:t} \right\}. \]  
   (12)

Proof 1: See Appendix B.

We note that in this problem, the optimal strategies derived by the above procedure depend on the beliefs that are strategy dependent (since, as shown in Lemma 1, the evolution of \( \pi_t, \xi_t \) depend on the strategies of the players), and thus there is signaling i.e. players’ actions reveal part of their private information.

IV. DYNAMIC GAMES WITH ASYMMETRIC INFORMATION

In this section, we study the problem presented in Section II i.e. when agents are strategic.
A. Solution concept: PBE

We introduce perfect Bayesian equilibrium (PBE) as an appropriate equilibrium concept for the game considered. Any history of this game at which players take action is of the form \( h_t = (x_{1:t}, w_{1:t}, z_{1:t}) \). Let \( H_t \) be the set of such histories at time \( t \). At any time \( t \) player \( i \) observes \( h^i_t = (w^i_{1:t}, z_{1:t}) \) and all players together observe \( h^*_t = (z_{1:t}) \) as common history. Let \( H^*_t \) be the set of observed histories of player \( i \) at time \( t \) and \( H^*_t \) be the set of common histories at time \( t \). An appropriate concept of equilibrium for such games is PBE [9] which consists of a pair \((\beta^*, \mu^*)\) of strategy profile \( \beta^* = (\beta^*_t, i) \in T, i \in N \) where \( \beta^*_t, i : H^*_t \rightarrow \Delta(A^i) \) and a belief profile \( \mu^* = (\mu^*_t, i) \in T, i \in N \) where \( \mu^*_t : H^*_t \rightarrow \Delta(H_t) \) that satisfy sequential rationality so that \( \forall i \in N, t \in T, h^*_t \in H^*_t, \beta^i \)

\[
\mathbb{E}(\beta^*|\beta^*_{t-1}, \mu^*) \left\{ \sum_{n=t}^{T} R^i_n(X_n, A_n) | h^*_t \right\} \\
\geq \mathbb{E}(\beta^*|\beta^*_{t-1}, \mu^*) \left\{ \sum_{n=t}^{T} R^i_n(X_n, A_n) | h^*_t \right\}
\]

and the beliefs are updated using Bayes’ rule, whenever possible.

In this paper, instead of defining beliefs \( i \mu^*_t \) as beliefs on every history \( h_t \) of the game, we define such beliefs on an appropriately defined information state of the game that is payoff relevant to the players. More specifically, we first define a common belief \( \pi^*_t \) as a belief on \( (x, \xi_t) \), where \( \xi_t = (\xi^1_t, \xi^2_t, \ldots, \xi^N_t) \) is a vector of private beliefs of the players on state \( x \). Then player \( i \) derives its equilibrium belief \( i \mu^*_t \) on \( (x, \xi_t^{-i}) \) by conditioning \( \pi^*_t \) on its private belief \( \xi^i_t \) i.e. \( i \mu^*_t(x, \xi_t^{-i}) = \pi^*_t(x, | \xi^{-i}_t, \xi^i_t) \).

In the following we present a backward-forward algorithm to compute SPBE of this game.

1) Backward Recursion: In this section, we define an equilibrium generating function \( \theta = (\theta^*_t)_{i \in N, t \in T} \) and a sequence of functions \( \left( V^i_t \right)_{i \in N, t \in \{1, 2, \ldots, T + 1\}} \), where \( V^i_t : \Delta(\mathcal{X} \times (\Delta(\mathcal{X}))^N) \times \Delta(\mathcal{X}) \rightarrow \mathbb{R} \), in a backward recursive way, as follows.

1. Initialize \( \forall \pi_{T+1} \in \Delta(\mathcal{X} \times (\Delta(\mathcal{X}))^N), \xi^i_{T+1} \in \Delta(\mathcal{X}) \),

\[
V^i_{T+1}(\pi_{T+1}, \xi^i_{T+1}) := 0.
\]

2. For \( t = T, T-1, \ldots, 1, \forall \pi_t \in \Delta(\mathcal{X} \times (\Delta(\mathcal{X}))^N) \), let \( \theta^i_t[\pi_t] \) be generated as follows. Set \( \gamma^i_t = \theta^i_t[\pi_t] \), where \( \gamma^i_t \) is the solution, if it exists\(^6\), of the following fixed-point equation, \( \forall i \in N, \xi^i_t \in \Delta(\mathcal{X}) \),

\[
\gamma^i_t(\xi^i_t) = \arg \max_{\gamma^i_t(\xi^i_t)} \mathbb{E}^{\gamma^i_t(\xi^i_t)} \left\{ R^i_t(X_t, A_t) + V^i_{t+1}(F_t(\pi_t, \gamma^i_t, Z_{t+1}), G^i_t(\xi^i_t, \pi_t, W^i_{t+1}, Z_{t+1}, \gamma^{-i}_t)) \right\}.
\]

where expectation in \( (25) \) is with respect to random variables \( (X_t, A_t, W^i_{t+1}, Z_{t+1}) \) through the measure \( \sum_{\xi^{-i}_{t-1}, x_{t+1}} \xi^i_t(x_t) \pi_t(\xi^{-i}_{t-1} | x_t) \gamma^{-i}_t(a^i_t | \xi^i_t) \gamma^{-i}_t(a^{-i} | \xi^{-i}_{t-1}) Q^i_t(x_{t+1} | x_t, a_t) P^t_{w^i} (z_{t+1}, w^i_{t+1} | x_{t+1}, x_t, a_t), F \) and \( G^i \) are

\(^6\)Existence of general solution of this per stage fixed-point equation is an open question. We briefly discuss this in Remark\(^3\).
∀ is a PBE of the game, i.e.

\[ V^i_t(\pi_t, \xi_t^i) := \mathbb{E}^{{}\mathbb{E}_t^i(\mathbb{E}^{\xi_t^i}G_t^i)\pi_{t+1}^{i, t}, \pi_t, \xi_t^i} \left\{ R^i_t(X_t, A_t) + V^i_{t+1}(F_t(\pi_t, \tilde{\gamma}_t, Z_{t+1}), G_t^i(\xi_t^i, \pi_t, W_t^{i^i}, Z_{t+1}, \tilde{\gamma}_t^{-i}))|\xi_t^i \right\} . \]

(16)

It should be noted that (25) is a fixed-point equation where the maximizer \( \tilde{\gamma}_t^i \) appears in both, the left-hand-side and the right-hand-side of the equation. However, it is not the outcome of the maximization operation as in a best response equation, similar to that of a Bayesian Nash equilibrium.

2) **Forward Recursion:** Based on \( \theta \) defined above in [24–26], we now construct a set of strategies \( \beta^* \) and beliefs \( \mu^* \) for the game \( \mathcal{D} \) in a forward recursive way, as follows. We define the updates of equilibrium common beliefs \( \pi_t^* \) and player \( i \)'s equilibrium private belief \( \xi_t^i, \), where player \( i \)'s equilibrium belief \( \mu_t^i \) is constructed by conditioning \( \pi_t^i \) on its private belief \( \xi_t^i \).

1. Initialize at time \( t = 0, \forall i \in \mathcal{N} \),

\[
\pi_1^i[\phi](x, \xi_1) := Q(x) \prod_{i=1}^N \delta_{\theta}(\xi_i^1) \quad (17)
\]

\[
\xi_1^i[\phi] = Q^x \quad (18)
\]

2. For \( t = 1, 2 \ldots T, i \in \mathcal{N}, \forall z_{1:t}, w_{1:t}^i \)

\[
\beta_t^i(a_t^i|z_{1:t}, w_{1:t}^i) := \theta_t^i[\pi_t^i[z_{1:t}]](a_t^i|\xi_t^i[z_{1:t}, w_{1:t}]) \quad (19a)
\]

\[
i_t^i(x, \xi_t^{-i}|z_{1:t}, w_{1:t}) := \pi_t^i[z_{1:t}](x, \xi_t^{-i}|\xi_t^i[z_{1:t}, w_{1:t}]) \quad (19b)
\]

and

\[
\pi_{t+1}^i[z_{1:t+1}] := F_t(\pi_t^i[z_{1:t}], \theta_t^i[\pi_t^i[z_{1:t}]], z_{t+1}) \quad (19c)
\]

\[
\xi_{t+1}^i[z_{1:t+1}, w_{1:t+1}^i] := G_t(\xi_t^i[z_{1:t}, w_{1:t}^i], \pi_t^i[z_{1:t}, w_{1:t+1}^i, z_{t+1}, \theta_t^{-i}[\pi_t^i[z_{1:t}]]]) \quad (19d)
\]

where \( F \) and \( G \) are defined in Lemma [1]

In the following theorem, we show that the equilibrium strategy and belief profile \( (\beta^*, \mu^*) \) defined above constitute a PBE of the game considered.

**Theorem 2:** A strategy and belief profile \( (\beta^*, \mu^*) \), constructed through backward/forward recursive algorithm is a PBE of the game, i.e. \( \forall i \in \mathcal{N}, t \in \mathcal{T}, (z_{1:t}, w_{1:t}^i), \beta_{t:T}^i \),

\[
\mathbb{E}^{\beta_{t:T}^i, \beta_{t-1:t}^{-i}, \mu_t^i} \left\{ \sum_{n=t}^T R_n^i(X_n, A_n)|z_{1:t}, w_{1:t}^i \right\} \geq \mathbb{E}^{\beta_{t:T}^i, \beta_{t-1:t}^{-i}, \mu_t^i} \left\{ \sum_{n=t}^T R_n^i(X_n, A_n)|z_{1:t}, w_{1:t}^i \right\} . \quad (20)
\]

**Proof:** The proof is provided in Appendix [D].
V. REMARKS

A few remarks are in order.

Remark 1: We emphasize that even though the backward-forward algorithm presented above finds a class of equilibrium strategies that are structured i.e. depend on the common and private belief states, the unilateral deviations of players in (29) are considered in the space of general strategies, i.e., the algorithm does not make any bounded rationality assumptions.

Remark 2: Intuition of the proof: In such games, one could use the one-shot deviation principle [25] to argue that sequential rationality reduces to showing that no player wants to deviate unilaterally in \( \beta^i_t \) at any time \( t \), keeping the rest of the strategy \( \beta^i_{t+1:T} \) as the equilibrium strategy. We argue that this is equivalent to (25) i.e. for a given \((\pi^*_i, \xi^*_i)\), a player’s unilateral deviation in its strategy \( \beta^i_{t+1} \) is the same as unilaterally deviation in its measure \( \gamma^i_t(\cdot|\xi^i_t) \) on its action \( a^i_t \) (and not on the whole function \( \gamma^i_t(\cdot) \)). This is because under such unilateral deviations, a player uses the same future information states \((\pi^*_{t+1}, \xi^*_{t+1})\) as it would have done under an equilibrium strategy, whose update depends on equilibrium \( \gamma^i_t \). This is so because player uses the same equilibrium \( \pi^*_{t+1} \) to predict other player’s actions, and update of its equilibrium private belief \( \xi^*_{t+1} \) does not depend on \( \gamma^i_t \).

Remark 3: While it is known that for any finite dynamic game with asymmetric information and perfect recall, there always exists a PBE [8, Prop. 249.1], existence of SPBE is not guaranteed. It is clear from our algorithm that existence of SPBE boils down to existence of a solution to the fixed-point equation (25) at every stage. Specifically, at each time \( t \) given the functions \( V^i_{t+1} \) for all \( i \in N \) from the previous round (in the backwards recursion) equation (25) must have a solution \( \tilde{\gamma}^i_t \) for all \( i \in N \). Generally, existence of equilibria is shown through Kakutani’s fixed point theorem, as is done by proving existence of a mixed strategy Nash equilibrium of a finite game [8], [26]. This is done by showing existence of fixed point of the best-response correspondences of the game. Among other conditions, it requires the closed graph property of the correspondences, which is usually implied by the continuity property of the utility functions involved. For (25) establishing existence is not straightforward due to: (a) potential discontinuity of the \( \pi_t \) update function \( F \) when the denominator in the Bayesian update is 0 and (b) potential discontinuity of the value functions, \( V^i_{t+1} \).

Remark 4: The second sub-case in (36f) dictates how beliefs are updated for histories with zero probability. The particular expression used is only one of many possible updates than can be used here. Dynamics that govern the evolution of public beliefs at histories with zero probability of occurrence affect equilibrium strategies. Thus, the construction proposed for calculating PBEs in this paper will produce a different set of equilibria if one changes the second sub-case above. The most well-known example of another such update is the intuitive criterion proposed in [27] for Nash equilibria, later generalized to sequential equilibria in [28]. The intuitive criterion assigns zero probability to states that can be excluded based on data available to all players (in our case action profile history \( a_{1:t-1} \)). Another example of belief update is universal divinity, proposed in [29].

Remark 5: This model also allows to incorporate many bounded rationality models. Some examples include using a discount factor \( \delta \) or by restricting the search for optimum \( \gamma^i_t(\cdot|\xi^i_t) \) in (25) in the space of functions that
are linear in private information variables.

VI. SPECIAL CASE: REPEATED GAMES WITH OBSERVED TYPES

In this section, we specialize the results presented in Section IV to repeated games where there exist \( N \) players with types \( x = (x^1, \ldots, x^N) \) and each player observes its type perfectly as its private information, and observe each others' actions jointly. At time \( t \), player \( i \) has private information \( x^i \) and observes common information \( a_{1:t} \). For this model, the forward/backward methodology presented in Section IV can be simplified as follows. For this problem, we redefine the common belief \( \pi_t \in X^N \) as

\[
\pi_t(x) = P(\theta | a_{1:t})
\]

and player \( i \) takes action through prescription function \( \gamma_i^{t} : \mathcal{X}^i \rightarrow \mathcal{A}^i \) as \( A^i \sim \gamma_i^t(x^i) \) i.e. player \( i \) takes its action based on its current private type \( x^i \). The common belief is updated as

\[
\pi_{t+1}(x_{t+1}) = \frac{\sum_{x,a} \pi_t(x) \left( \prod_{i=1}^N \gamma_i^t(a_i^t | x^i) \right)}{\sum_{\tilde{x},a} \pi_t(\tilde{x}) \left( \prod_{i=1}^N \gamma_i^t(a_i^t | \tilde{x}^i) \right)}
\]

(21)

if the denominator is not 0 and as

\[
\pi_{t+1}(x) = \pi_t(x)
\]

(22)

otherwise.

A. Backward Recursion

The equilibrium generating function \( \theta_i \in \mathcal{N}, \pi \in \mathcal{T} \) and a sequence of functions \( (V^i_t)_{i \in \mathcal{N}, t \in \{1,2,..., T+1\}} \), where \( V^i_t : \Delta(\mathcal{X}) \times \mathcal{X}^i \rightarrow \mathbb{R} \) are defined in a backward recursive way as follows.

1. Initialize \( \forall \pi_{T+1} \in \Delta(\mathcal{X}), \)

\[
V^i_{T+1}(\pi_{T+1}, x^i) := 0.
\]

(23)

2. For \( t = T, T-1, \ldots, 1 \), \( \forall \pi_t \in \Delta(\mathcal{X}) \), let \( \theta_t[\pi_t] \) be generated as follows. Set \( \tilde{\gamma}_t = \theta_t[\pi_t] \), where \( \tilde{\gamma}_t \) is the solution, if it exists, of the following fixed-point equation, \( \forall i \in \mathcal{N}, x \),

\[
\tilde{\gamma}_t^i(x^i) \in \arg \max_{\gamma_t^i(\cdot)} \mathbb{E}_{\gamma_t^i(\cdot)} \left\{ R^i_t(X, A_t) + V^i_{t+1}(F_t(\pi_t, \tilde{\gamma}_t, A_t), x^i) \right\},
\]

(24)

where expectation in (25) is with respect to random variables \( (X, A_t) \) through the measure

\[
\sum_{x} \pi_t(x) \gamma_t^i(a_i^t | x^i) \tilde{\gamma}_t^{-1}(a_i^t | x^i).
\]

Furthermore, set

\[
V^i_t(\pi_t, x^i) := \mathbb{E}_{\gamma_t^i(\cdot) \tilde{\gamma}_t^{-1}, \pi_t} \left\{ R^i_t(X, A_t) + V^i_{t+1}(F_t(\pi_t, \tilde{\gamma}_t, A_t), x^i) \right\}.
\]

(25)
B. Forward Recursion

Based on $\theta$ defined above in (24)–(26), the set of strategies $\beta^*$ and beliefs $\mu^*$ for the game $\mathcal{D}$ are constructed in a forward recursive way as follows. We define the updates of equilibrium common beliefs $\pi_t^*$, where player $i$’s equilibrium belief $i^*\mu_t^*$ is constructed by conditioning $\pi_t^*$ on its private observed type $x^i$.

1. Initialize at time $t = 0, \forall i \in \mathcal{N}$,

$$\pi_t^*[(x_1)] := Q(x_1)$$

(26)

2. For $t = 1, 2 \ldots T, i \in \mathcal{N}, \forall a_{1:t-1}, x^i$

$$\beta_t^*|a_{1:t-1}, x^i := \theta_t[\pi_t^*[a_{1:t-1}]](a_t|x^i)$$

(27a)

$$i^*\mu_t^*(x^{-i}|a_{1:t-1}, x^i) := \pi_t^*[a_{1:t-1}](x^{-i}|x^i)$$

(27b)

and

$$\pi_{t+1}^*[z_{1:t+1}] := F_t(\pi_t^*[a_{1:t-1}], i^*\mu_t^*, a_t)$$

(27c)

where $F_t$ is defined in (22).

**Corollary 1:** A strategy and belief profile $(\beta^*, \mu^*)$, constructed through backward/forward recursive algorithm is a PBE of the game, i.e. $\forall i \in \mathcal{N}, t \in T, (a_{1:t-1}, x^i), \beta_t^{|_{T}, T}$,

$$\mathbb{E}^{\beta_t^*, \beta_{t:T}^*, \mu_t^*} \left\{ \sum_{n=t}^{T} R_n^i(X_n, A_n)| a_{1:t-1}, x^i \right\} \geq$$

$$\mathbb{E}^{\beta_t^*, \beta_{t:T}^*, \mu_t^*} \left\{ \sum_{n=t}^{T} R_n^i(X_n, A_n)| a_{1:t-1}, x^i \right\}.$$ 

(28)

**Proof:** The result is implied by Theorem 2.

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VIII. CONCLUSION

In this paper, we considered a general framework of decision making with both strategic and non strategic players, where there is an underlying state of the system that is dynamically evolving and players observe a common observation and potentially correlated private observations of the state of the system. Each player receives a reward that is a function of the state and actions of all the players. We define a common information belief state and private information belief states of the players based on which we presented structural results of optimum policy when the players are cooperative. Using these as motivation, for the strategic players, we presented a backward-forward
methodology similar to the one presented in [1] to compute its structured perfect Bayesian equilibria (SPBE).
We specialize our results to the case when each players has a type associated with it which together evolve as a
controlled Markov process and player observes its type perfectly. Future work includes proving such a methodology
for discounted infinite-horizon case and specializing results to many practical settings such as games on graph,
where players who are connected on the graph have correlated private information. Some practical applications of
interest include security games for cyber-physical systems and Bayesian learning games in a social network with
fully rational and potentially adversarial agents.

**APPENDIX A**

**Proof:** Now for any \( \theta \) we have,

\[
\begin{align*}
\xi^i_{t+1}(x_{t+1}) &= P^\theta(x_{t+1} | z_{1:t+1}, w^i_{1:t+1}) \\
&= \frac{\sum_{x_t, a_t} P^\theta(x_t, a_t, x_{t+1}, w^i_{1:t+1}, z_{1:t+1} | z_{1:t}, w^i_{1:t})}{\sum_{\tilde{x}_t, x_{t+1}, a_t} P^\theta(\tilde{x}_t, a_t, w^i_{1:t}, z_{1:t+1}, \tilde{x}_{t+1} | z_{1:t}, w^i_{1:t})} \\
&= \frac{\sum_{x_t, a_t} \xi^i_t(x_t) P^\theta(a^i_t \mid z_{1:t}, w^i_{1:t}, x_t) Q^i_t(x_{t+1} | a_t, x_t) P^i_w(z_{t+1}, w^i_{t+1} | x_{t+1}, x_t, a_t)}{\sum_{\tilde{x}_t, a_t} \xi^i_t(\tilde{x}_t) P^\theta(a^{-i}_t \mid z_{1:t}, w^i_{1:t}, \tilde{x}_t) Q^i_t(x_{t+1} | \tilde{x}_t, a_t) P^i_w(z_{t+1}, w^i_{t+1} | \tilde{x}_{t+1}, \tilde{x}_t, a_t)},
\end{align*}
\]

where (30e) is true because \( a^i_t \) is a function of \( (z_{1:t}, w^i_{1:t}) \) and thus the term involving \( a^i_t \) can be cancelled in the
numerator and denominator.

\[
\begin{align*}
P^\theta(a^{-i}_t \mid w^i_{1:t}, x_t) &= \sum_{\xi^{-i}_t} P^\theta(\xi^{-i}_t, a^i_t \mid w^i_{1:t}, z_{1:t}, x_t) \\
&= \sum_{\xi^{-i}_t} P^\theta(\xi^{-i}_t \mid w^i_{1:t}, z_{1:t}, x_t) \prod_{j \neq i} P^\theta(a^j_t \mid x_{t+1}, \xi^j_t) \\
&= \sum_{\xi^{-i}_t} \pi_t(\xi^{-i}_t \mid x_t) \prod_{j \neq i} \gamma^j_t(a^j_t \mid \xi^j_t)
\end{align*}
\]

Thus,

\[
\begin{align*}
\xi^i_{t+1}(x_{t+1}) &= \frac{\sum_{x_t, a_t} \xi^i_t(x_t) \left( \sum_{\xi^{-i}_t} \pi_t(\xi^{-i}_t \mid \xi^i_t, x_t) \prod_{j \neq i} \gamma^j_t(a^j_t \mid \xi^j_t) \right) Q^i_t(x_{t+1} \mid x_t, a_t) P^i_w(z_{t+1}, w^i_{t+1} \mid x_{t+1}, x_t, a_t)}{\sum_{\tilde{x}_t, a_t} \xi^i_t(\tilde{x}_t) \left( \sum_{\xi^{-i}_t} \pi_t(\xi^{-i}_t \mid \xi^i_t, \tilde{x}_t) \prod_{j \neq i} \gamma^j_t(a^j_t \mid \xi^j_t) \right) Q^i_t(\tilde{x}_{t+1} \mid x_t, a_t) P^i_w(z_{t+1}, w^i_{t+1} \mid \tilde{x}_{t+1}, \tilde{x}_t, a_t)}. \\
&= G^i_t(\xi^i_t, \pi_t, w^i_{t+1}, z_{t+1}, \gamma^{-i}_t)
\end{align*}
\]
With some abuse of notation, we also define the joint update function derived from above,

$$\xi_{t+1} = G_t(\xi_t, \pi_t, w_{t+1}, z_{t+1}, \gamma_t)$$  \hspace{1cm} (33)$$

**Lemma 2:** There exists an update function \( F \) of \( \pi_t \), independent of \( \psi \)

$$\pi_{t+1} = F_t(\pi_t, \gamma_t, z_{t+1})$$  \hspace{1cm} (34)$$

**Proof:**

\[
\begin{align*}
\pi_{t+1}(x_{t+1}, \xi_{t+1}) &= P^0(x_{t+1}, \xi_{t+1} | z_{1:t+1}, \gamma_{1:t+1}) \\ 
&= P^0(x_{t+1}, \xi_{t+1} | z_{1:t+1}, \gamma_{1:t}) \\ 
&= \frac{\sum_{x_t, \xi_t, a_t, w_{t+1}} P^0(x_t, \xi_t, a_t, x_{t+1}, w_{t+1}, z_{t+1}, \xi_{t+1} | z_{1:t}, \gamma_{1:t})}{\sum_{\xi_t, a_t} P^0(\xi_t, a_t | z_{1:t}, \gamma_{1:t})} \\ 
&= \frac{\sum_{x_t, \xi_t, a_t, x_{t+1}, z_{t+1}, w_{t+1}} \pi_t(x_t, \xi_t) \left( \prod_{i=1}^N \gamma_i^t(a_i^t | \xi_i^t) \right) Q^\pi_t(x_{t+1} | x_t, a_t) Q^\mu_t(z_{t+1}, w_{t+1} | x_{t+1}, x_t, a_t) I_G_t(\xi_t, w_{t+1}, a_t, \pi_t, \gamma_t)(\xi_{t+1})}{\sum_{\xi_t, a_t} \pi_t(\xi_t) \prod_{i=1}^N \gamma_i^t(a_i^t | \xi_i^t)} \\ 
& \quad \text{if the denominator is not 0 and} \\ 
& \pi_{t+1}(x_{t+1}, \xi_{t+1}) = \sum_{\xi_t, x_t, a_t, x_{t+1}, w_{t+1}} \pi_t(x_t, \xi_t) Q^\pi_t(x_{t+1} | x_t, a_t) Q^\mu_t(z_{t+1}, w_{t+1} | x_{t+1}, x_t, a_t) I_G_t(\xi_t, w_{t+1}, a_t, \pi_t, \gamma_t)(\xi_{t+1}) \\ 
& \quad \text{otherwise. Thus we have,} \\ 
& \pi_{t+1} = F_t(\pi_t, \gamma_t, z_{t+1}) \\ 
\end{align*}
\]

**APPENDIX B**

**Proof:** We prove (29) using induction and from results in Lemmas 3 and 4 proved in Appendix C. For the base case at \( t = T \), \( \forall z_{1:T} \)

$$E^\theta \left\{ R_T(X_T, A_T) \bigg| z_{1:T} \right\} = V_T(\pi_T)$$  \hspace{1cm} (36a)$$

$$\geq E^\psi \left\{ R_T(X_T, A_T) \bigg| z_{1:T} \right\}$$  \hspace{1cm} (36b)$$

where (37a) follows from Lemma 4 and (37b) follows from Lemma 3 in Appendix C
Let the induction hypothesis be that for $t + 1$, $z_{t+1}$, $ψ$,

$$E^θ \left\{ \sum_{n=t+1}^{T} R_n(X_n, A_n) \right\} \geq E^ψ \left\{ \sum_{n=t+1}^{T} R_n(X_n, A_n) \right\}.$$  \hfill (37a)

Then $∀z_{t:t}$, $ψ$, we have

$$E^θ \left\{ \sum_{n=t}^{T} R_n(X_n, A_n) \right\} \geq V_t(π_t)$$  \hfill (38a)

$$\geq E^ψ \left\{ R_t(X_t, A_t) + V_{t+1}(F_t(π_t, γ_t, Z_{t+1})) \right\} \hfill (38b)

= E^ψ \left\{ R_t(X_t, A_t) + E^θ \left\{ \sum_{n=t+1}^{T} R_n(X_n, A_n) \right\} \right\} \hfill \hfill (38c)

\geq E^ψ \left\{ R_t(X_t, A_t) + E^ψ \left\{ \sum_{n=t+1}^{T} R_n(X_n, A_n) \right\} \right\} \hfill \hfill (38d)

= E^ψ \left\{ \sum_{n=t}^{T} R_n(X_n, A_n) \right\}.$$  \hfill (38e)

where (39a) follows from Lemma 4, (39b) follows from Lemma 3, (39c) follows from Lemma 4 and (39d) follows from induction hypothesis in (38a).

**APPENDIX C**

**Lemma 3:** $∀t ∈ T$, $z_{1:t}, ψ$

$$V_t(π_t) \geq E^ψ \left\{ R_t(X_t, A_t) + V_{t+1}(F_t(π_t, ψ_t(\cdot | z_{1:t}, \cdot), Z_{t+1})) \right\}.$$  \hfill (39)

**Proof:** We prove this lemma by contradiction.

Suppose the claim is not true for $t$. This implies $∃ψ_t, \hat{z}_{1:t}$ such that

$$E^ψ \left\{ R_t(X_t, A_t) + V_{t+1}(F_t(\hat{π}_t, ψ_t(\cdot | \hat{z}_{1:t}, \cdot), Z_{t+1})) \right\} > V_t(\hat{π}_t).$$  \hfill (40)

where $\hat{π}_t(x_t, ξ_t) = P^ψ(x_t, ξ_t | \hat{z}_{1:t})$. We will show that this contradicts the definition of $V_t$ in (26). Let $ξ_t^i(x_t) = P^ψ(x_t | \hat{z}_{1:t}, w_{1:t}^i)$ i.e. $ξ_t$ and $\hat{π}_t$ are those private and public beliefs of the players that are derived through $\hat{z}_{1:t}$.

For all $i ∈ N$, construct $\hat{ξ}_t^i(a_t^i | \hat{z}_{1:t}, w_{1:t}^i)$ :=

\[
\begin{cases}
\hat{ξ}_t^i(a_t^i | \hat{z}_{1:t}, w_{1:t}^i) & \quad ξ_t^i = ξ_t^{i,s} \\
\text{arbitrary} & \quad \text{otherwise}
\end{cases}
\]

Then for $\hat{z}_{1:t}$, we have
\[ V_t(\tilde{\pi}_t) \]

\[ \begin{align*}
&= \mathbb{E}^{\tilde{\psi}} \left\{ R_t(X_t, A_t) + V_{t+1}(F_t(\tilde{\pi}_t, \tilde{\psi}(\cdot | \tilde{z}_{1:t} \cdot \cdot), Z_{t+1}) \right| \tilde{z}_{1:t} \right\} \\
&= \max_{\gamma_t} \left( R_t(X_t, A_t) + V_{t+1}(F_t(\tilde{\pi}_t, \tilde{\psi}(\cdot | \tilde{z}_{1:t} \cdot \cdot), Z_{t+1}) \right| \tilde{z}_{1:t}, \gamma_t \right),
\end{align*} \tag{41f} \]

Lemma 4: \( \forall t \in \mathcal{T}, \tilde{z}_{1:t} \in \mathcal{H}_t^\omega \)

\[ V_t(\tilde{\pi}_t) = \mathbb{E}^{\theta} \left\{ \sum_{n=t}^{T} R_n(X_n, A_n) \right| \tilde{z}_{1:t} \right\} \tag{42} \]

Proof:

We prove the lemma by induction. For \( t = T \),

\[ \mathbb{E}^{\theta} \left\{ R_T(X_T, A_T) \left| \tilde{z}_{1:T} \right. \right\} = \sum_{x_T, a_T, \xi_T} R_T(x_T, a_T) P^\theta(x_T, a_T \left| \tilde{z}_{1:T} \right. \right) \tag{43a} \]
\[
\sum_{x_T, a_T, \xi T} R_T(x_T, a_T) \pi_T(x_T, \xi T) \theta_T[\pi_T](a_T|\xi_T) = V_T(\pi_T), \tag{43b}
\]

\[
V_{t+1}(\pi_{t+1}) = \mathbb{E}^{\theta} \left\{ \sum_{n=t+1}^{T} R_n(X_n, A_n) \bigg| z_{1:t+1} \right\}. \tag{44}
\]

Then \(\forall t \in \mathcal{T}, z_{1:t} \in \mathcal{H}_t\), we have

\[
\mathbb{E}^{\theta} \left\{ \sum_{n=t}^{T} R_n(X_n, A_n) \bigg| z_{1:t} \right\} = \mathbb{E}^{\theta} \left\{ R_i(X_i, A_i) + \mathbb{E}^{\theta} \left\{ \sum_{n=t+1}^{T} R_n(X_n, A_n) \bigg| z_{1:t}, Z_{t+1} \right\} \bigg| z_{1:t} \right\}, \tag{45a}
\]

\[
= \mathbb{E}^{\theta} \left\{ R_i(X_i, A_i) + V_{t+1}(F_i(\pi_T, \theta_T[\pi_T](\cdot|\cdot), Z_{t+1})) \bigg| z_{1:t} \right\}, \tag{45b}
\]

\[
= V_i(\pi_i), \tag{45c}
\]

where (46a) follows from the induction hypothesis in (45) and update of \(\pi_t\), and (46b) follows from the definition of \(V_i\) in (26).

**APPENDIX D**

*(PROOF OF THEOREM 2)*

In the following theorem, we will assume that the equilibrium strategies and beliefs \((\beta^*, \mu^*)\) are generated using an equilibrium function \(\theta\). Moreover, we will also use \(\pi_t^*\) and \(\xi_t^*\) maps corresponding to \(\mu_t^*\) as defined in (28). With slight abuse of notation, we use both beliefs and belief functions as superscripts on expectations, where the reference is clear from the context. These functions when used as superscripts in expectation denote the belief functions, which when applied on the conditioned random variables, define beliefs on the random variables of interest inside the expectation.

**Proof:** We prove (29) using induction and from results in Lemma 5, 6 and 7 proved in Appendix E. For base case at \(t = T\), \(\forall i \in \mathcal{N}, (z_{1:T}, w_{1:T}) \in \mathcal{H}_T, \beta^i\)

\[
\mathbb{E}^{\beta_{T}^{i, \beta_{T}^{i-1}, \pi_{T}^{i}, \xi_{T}^{i}}^{i} \left\{ R_i(X_T, A_T) \bigg| z_{1:T}, w_{1:T} \right\} \}
\]

\[
= V_T^i(\pi_T^i[z_{1:T}], \xi_T^i[z_{1:T}, w_{1:T}]) \tag{46a}
\]

\[
\geq \mathbb{E}^{\beta_{T}^{i, \beta_{T}^{i-1}, \pi_{T}^{i}, \xi_{T}^{i}}^{i} \left\{ R_i(X_T, A_T) \bigg| z_{1:T}, w_{1:T} \right\} \}
\]

where (47a) follows from Lemma 7 and (47b) follows from Lemma 5 in Appendix E.
Let the induction hypothesis be that for $t + 1$, $\forall i \in \mathcal{N}, (z_{1:t+1}, w_{1:t+1}^i, \beta_t^i) \in \mathcal{H}_t^i$, 
\[
\mathbb{E}^{\beta_{t+1}^i, \beta_{t+1}^i, \pi_{t+1}^i, \xi_{t+1}^i} \left\{ \sum_{n=t+1}^T R_n^i(X_n, A_n) \Big| z_{1:t}, w_{1:t+1}^i \right\} \geq \mathbb{E}^{\beta_{t+1}^i, \beta_{t+1}^i, \pi_{t+1}^i, \xi_{t+1}^i} \left\{ \sum_{n=t+1}^T R_n^i(X_n, A_n) \Big| z_{1:t}, w_{1:t+1}^i \right\}.
\]

Then $\forall i \in \mathcal{N}, (z_{1:t}, w_{1:t}^i) \in \mathcal{H}_t^i, \beta_t^i$, we have 
\[
\mathbb{E}^{\beta_t^i, \beta_t^i, \pi_t^i, \xi_t^i} \left\{ \sum_{n=t}^T R_n^i(X_n, A_n) \Big| z_{1:t}, w_{1:t}^i \right\} = V_t^i(\pi_t^i[z_{1:t}], \xi_t^i[z_{1:t}, w_{1:t}^i]) (48a)
\] 
\[
\geq \mathbb{E}^{\beta_t^i, \pi_t^i, \xi_t^i} \left\{ R_t^i(X_t, A_t) + V_{t+1}^i(\pi_{t+1}^i[z_{1:t}, z_{t+1}], \xi_{t+1}^i[z_{1:t}, z_{t+1}, w_{1:t+1}, W_{t+1}^i]) \Big| z_{1:t}, w_{1:t}^i \right\} (48b)
\] 
\[
= \mathbb{E}^{\beta_t^i, \pi_t^i, \xi_t^i} \left\{ R_t^i(X_t, A_t) + \mathbb{E}^{\beta_t^i, \beta_{t+1}^i, \pi_{t+1}^i, \xi_{t+1}^i} \left\{ \sum_{n=t+1}^T R_n^i(X_n, A_n) \Big| z_{1:t}, z_{t+1}, w_{1:t+1}, W_{t+1}^i \right\} \Big| z_{1:t}, w_{1:t}^i \right\} \right| (48c)
\] 
\[
\geq \mathbb{E}^{\beta_t^i, \pi_t^i, \xi_t^i} \left\{ R_t^i(X_t, A_t) + \mathbb{E}^{\beta_t^i, \beta_{t+1}^i, \pi_{t+1}^i, \xi_{t+1}^i} \left\{ \sum_{n=t+1}^T R_n^i(X_n, A_n) \Big| z_{1:t}, z_{t+1}, w_{1:t+1}, W_{t+1}^i \right\} \Big| z_{1:t}, w_{1:t}^i \right\} \right| (48d)
\] 
\[
= \mathbb{E}^{\beta_t^i, \pi_t^i, \xi_t^i} \left\{ R_t^i(X_t, A_t) + \mathbb{E}^{\beta_t^i, \beta_{t+1}^i, \pi_{t+1}^i, \xi_{t+1}^i} \left\{ \sum_{n=t+1}^T R_n^i(X_n, A_n) \Big| z_{1:t}, z_{t+1}, w_{1:t+1}, W_{t+1}^i \right\} \Big| z_{1:t}, w_{1:t}^i \right\} \right| (48e)
\] 
\[
= \mathbb{E}^{\beta_t^i, \beta_{t}^i, \pi_t^i, \xi_t^i} \left\{ \sum_{n=t}^T R_n^i(X_n, A_n) \Big| z_{1:t}, w_{1:t}^i \right\} \right| (48f)
\]

where (49a) follows from Lemma [7], (49b) follows from Lemma [5], (49c) follows from Lemma [7], (49d) follows from induction hypothesis in (48a) and (49e) follows from Lemma [6]

**APPENDIX E**

As we did in the previous theorem, in the following lemmas, we would assume that the equilibrium strategies and beliefs ($\beta^*, \mu^*$) are generated using an equilibrium function $\theta$. We also use $\pi_t^i$ and $\xi_t^{*, i}$ corresponding to $i^* \mu_t^*$ as defined in (28).

**Lemma 5:** This lemma states that the reward that player $i$ would get on playing equilibrium strategy will be greater or equal to the reward it would get if it deviates only at time $t$, keeping the rest of its strategy as equilibrium strategy, $\forall t \in \mathcal{T}$, $i \in \mathcal{N}, (z_{1:t}, w_{1:t}^i) \in \mathcal{H}_t^i, \beta_t^i$ 
\[
V_t^i(\pi_t^i[z_{1:t}], \xi_t^i[z_{1:t}, w_{1:t}^i]) \geq \mathbb{E}^{\beta_t^i, \beta_t^i, \pi_t^i, \xi_t^i} \left\{ R_t^i(X_t, A_t) + V_{t+1}^i(\pi_{t+1}^i[z_{1:t}, z_{t+1}], \xi_{t+1}^i[z_{1:t}, z_{t+1}, w_{1:t+1}, W_{t+1}^i]) \Big| z_{1:t}, w_{1:t}^i \right\} \right| (49)
\]
\textbf{Proof:} We prove this lemma by contradiction.

Suppose the claim is not true for \( t \). This implies \( \exists i, \hat{\beta}_t^i, \hat{z}_{1:t}, \hat{w}_{1:t}^i \) such that
\[
\mathbb{E}[^{\hat{\beta}_t^i, \hat{z}_{1:t}, \hat{w}_{1:t}^i} \{ R_t^i(X_t, A_t) + V_{t+1}^i(\pi_{t+1}^*, [\hat{z}_{1:t}, Z_{t+1}], \xi_{t+1}^{i*}[\hat{z}_{1:t}, \hat{w}_{1:t}^i, W_{t+1}^i]) \mid \hat{z}_{1:t}, \hat{w}_{1:t}^i \} ] > V_t^i(\pi_t^*, [\hat{z}_{1:t}, \hat{w}_{1:t}^i]).
\] (50)

We will show that this contradicts the definition of \( V_t^i \) in (26).

Construct \( \gamma_t^i(a_t^i \mid \xi_t^i) = \begin{cases} \hat{\gamma}_t^i(a_t^i \mid \hat{z}_{1:t}, \hat{w}_{1:t}^i) & \text{arbitrary} \\ \xi_t^i = \xi_t^{i*} [\hat{z}_{1:t}, \hat{w}_{1:t}^i] & \text{otherwise}. \end{cases} \)

Then for \( \hat{z}_{1:t}, \hat{w}_{1:t}^i \), we have
\[
V_t^i(\pi_t^*, [\hat{z}_{1:t}, \hat{w}_{1:t}^i])
\leq \mathbb{E}[^{\hat{\beta}_t^i, \hat{z}_{1:t}, \hat{w}_{1:t}^i} \{ R_t^i(X_t, A_t) + V_{t+1}^i(\pi_{t+1}^*, [\hat{z}_{1:t}, Z_{t+1}], \xi_{t+1}^{i*}[\hat{z}_{1:t}, \hat{w}_{1:t}^i, W_{t+1}^i]) \mid \hat{z}_{1:t}, \hat{w}_{1:t}^i \}] (51a)
\]
\[
= \sum_{x_t, a_t, z_t, w_{t+1}, \xi_t^{i*}} \left[ R_t^i(x_t, a_t) + V_{t+1}^i(\pi_{t+1}^*, [\hat{z}_{1:t}, Z_{t+1}], \xi_{t+1}^{i*}[\hat{z}_{1:t}, \hat{w}_{1:t}^i, W_{t+1}^i]) \mid \hat{z}_{1:t}, \hat{w}_{1:t}^i \right] \xi_t^{i*}[\hat{z}_{1:t}, \hat{w}_{1:t}^i] (51b)
\]
\[
\leq \mathbb{E}[^{\gamma_t^i(\xi_t^i)} \{ R_t^i(X_t, A_t) + V_{t+1}^i(\pi_{t+1}^*, [\hat{z}_{1:t}, Z_{t+1}], \xi_{t+1}^{i*}[\hat{z}_{1:t}, \hat{w}_{1:t}^i, W_{t+1}^i]) \mid \xi_t^{i*}[\hat{z}_{1:t}, \hat{w}_{1:t}^i] \} ] (51c)
\]
\[
\leq \max_{\gamma_t^i(\xi_t^i)} \mathbb{E}[^{\gamma_t^i(\xi_t^i)} \{ R_t^i(X_t, A_t) + V_{t+1}^i(\pi_{t+1}^*, [\hat{z}_{1:t}, Z_{t+1}], \xi_{t+1}^{i*}[\hat{z}_{1:t}, \hat{w}_{1:t}^i, W_{t+1}^i]) \mid \xi_t^{i*}[\hat{z}_{1:t}, \hat{w}_{1:t}^i] \} ] (51d)
\]
\[
= V_t^i(\pi_t^*, [\hat{z}_{1:t}, \hat{w}_{1:t}^i]) (51e)
\]
where (52a) follows from (51e), (52c) follows from the definition of \( \hat{\gamma}_t^i \), and (52f) follows from the definition of \( V_t^i \) in (26).

\textbf{Lemma 6:} \( \forall i \in \mathcal{N}, t \in T, (z_{1:t+1}, w_{1:t+1}^i) \in \mathcal{H}_{t+1}^i \) and \( \beta_t^i \)
\[
\mathbb{E}[^{\beta_t^i, \beta_{t+1}^i, \pi_t^i, \xi_t^i} \{ \sum_{n=t+1}^{T} R_n^i(X_n, A_n) \mid z_{1:t+1}, w_{1:t+1}^i \} ] = \mathbb{E}[^{\beta_{t+1}^i, \beta_{t+1}^i, \pi_{t+1}^i, \xi_{t+1}^i} \{ \sum_{n=t+1}^{T} R_n^i(X_n, A_n) \mid z_{1:t+1}, w_{1:t+1}^i \} ] .
\] (52)
Since the above expectations involve random variables \(X_{t+1:T}, A_{t+1:T}\), we consider
\[
P^{\beta_t, T, \pi_t, \kappa_t}_i (x_{t+1:T}, a_{t+1:T} | x_t, a_t, w_{i,t+1}).
\]

\[
P^{\beta_t}_{i,T, \pi_t, \kappa_t}_i (x_{t+1:T}, a_{t+1:T} | z_{1:t+1}, w_{i,t+1})
\]
\[
= P^{\beta_t}_{i,T, \pi_t, \kappa_t}_i (x_{t+1} | z_{1:t+1}, w_{i,t+1}) P^{\beta_{t+1}}_{i,T, \pi_{t+1}, \kappa_{t+1}} (x_{t+2:T} | x_{t+1}, z_{1:t+1}, w_{i,t+1})
\]
\[
= \sum_{a_t} P^{\beta_t}_{i,T, \pi_t, \kappa_t}_i (a_t, x_{t+1}, w_{i,t+1}, z_{1:t+1} | z_{1:t}, w_{i,t}) P^{\beta_{t+1}}_{i,T, \pi_{t+1}, \kappa_{t+1}} (a_{t+1:T} | x_{t+2:T} | x_{t+1}, z_{1:t+1}, w_{i,t+1})
\]  
\[
= \sum_{a_t} P^{\beta_t}_{i,T, \pi_t, \kappa_t}_i (a_t, x_{t+1}, w_{i,t+1}, z_{1:t+1} | z_{1:t}, w_{i,t})
\]  
\[
= \sum_{a_t} P^{\beta_t}_{i,T, \pi_t, \kappa_t}_i (a_t, x_{t+1}, w_{i,t+1}, z_{1:t+1} | z_{1:t}, w_{i,t})
\]

We first note that
\[
P^{\beta_t}_{i,T, \pi_t, \kappa_t}_i (a_{t+1:T} | x_{t+2:T} | x_{t+1}, z_{1:t+1}, w_{i,t+1})
\]
\[
= \beta_{t+1}^i (a_t^i | z_{1:t+1}, w_{i,t+1}) \left( \sum_{\xi_{t+1}} \pi_{t+1}^i [z_{1:t+1}] (\xi_{t+1} \mid \xi_{t+1}^i [z_{1:t+1}], w_{i,t+1}, x_{t+1}) \beta_{t+1}^i (a_t^i | z_{1:t+1}, \xi_{t+1}^i) \right) Q_t^i (x_{t+2} | x_{t+1}, a_{t+1})
\]
\[
P^{\beta_t}_{i,T, \pi_t, \kappa_t}_i (a_{t+2:T} | x_{t+3:T} | z_{1:t+1}, x_{t+2}, w_{i,t+1})
\]
\[
= \sum_{a_t} P^{\beta_t}_{i,T, \pi_t, \kappa_t}_i (a_t, \pi_{t+1}^i, z_{1:t+1}, w_{i,t+1}, x_{t+1}) (a_{t+1:T} | x_{t+2:T} | x_{t+1}, z_{1:t+1}, w_{i,t+1})
\]

We consider the numerator and the denominator in the left hand side of the above expression separately. The numerator in (53b) is given by
\[
N_r = \sum_{x_{t, \xi_{t+1}}} P^{\beta_t}_{i,T, \pi_t, \kappa_t}_i (x_{t}, \xi_{t+1} | z_{1:t}, w_{i,t+1}) \beta_{t}^i (a_{t}^i | z_{1:t-1}, w_{i,t}, x_{t}) \beta_{t+1}^i (a_{t}^i | z_{1:t-1}, \xi_{t+1}) Q_t^i (x_{t+1} | x_t, a_t)
\]
\[
P^{i,w}_t (z_{t+1}, w_{i,t+1} | x_{t+1}, x_t, a_t)
\]
\[
= \sum_{x_{t, \xi_{t+1}}} \xi_{t+1}^i [z_{1:t}, w_{i,t+1}] (x_t) \pi_{t}^i [z_{1:t}, \xi_{t+1}^i | z_{1:t}, w_{i,t+1}, x_{t}) \beta_{t}^i (a_{t}^i | z_{1:t-1}, w_{i,t}, x_{t}) \beta_{t+1}^i (a_{t}^i | z_{1:t-1}, \xi_{t+1}) Q_t^i (x_{t+1} | x_t, a_t)
\]
\[
P^{i,w}_t (z_{t+1}, w_{i,t+1} | x_{t+1}, x_t, a_t)
\]

Similarly, the denominator in (54b) is given by
\[
D_r = \sum_{\bar{x}_{t, \bar{\xi}_{t+1}}, \bar{\tilde{x}}_{t+1}} P^{\beta_{t}, \kappa_{t}, \pi_{t}}_{\bar{x}, \bar{\xi}_{t+1}} (\bar{x}_t, \bar{\xi}_{t+1} | z_{1:t}, w_{i,t+1}) \beta_{t}^i (a_{t}^i | z_{1:t-1}, w_{i,t}, x_{t}) \beta_{t+1}^i (a_{t}^i | z_{1:t-1}, \bar{\xi}_{t+1}) Q_t^i (\bar{x}_{t+1} | \bar{x}_t, a_t)
\]
\[
P^{i,w}_t (z_{t+1}, w_{i,t+1} | \bar{x}_{t+1}, \bar{x}_t, a_t)
\]
\[
= \sum_{\bar{x}_{t, \bar{\xi}_{t+1}}, \bar{\tilde{x}}_{t+1}} \xi_{t+1}^i [z_{1:t}, w_{i,t+1}] (\bar{x}_t) \pi_{t}^i [\bar{\xi}_{t+1}^i | \bar{\xi}_{t+1}^i [z_{1:t}, w_{i,t+1}, \bar{x}_{t}) \beta_{t}^i (a_{t}^i | z_{1:t-1}, w_{i,t}, x_{t}) \beta_{t+1}^i (a_{t}^i | z_{1:t-1}, \bar{\xi}_{t+1}) Q_t^i (\bar{x}_{t+1} | \bar{x}_t, a_t)
\]
\[
P^{i,w}_t (z_{t+1}, w_{i,t+1} | \bar{x}_{t+1}, \bar{x}_t, a_t)
\]
By canceling the terms $\beta^i_t(\cdot)$ in the numerator and the denominator,

$$
N_r = \sum_{\xi_t, \xi_{t-1}^{-i}} \xi^{i,*}_{t}[z_{1:t}, w^{i}_{1:t+1}](x_t)\pi^*_t[z_{1:t}](\xi^{-i}_t|\xi^{i,*}_t[z_{1:t}, w^{i}_{1:t+1}], x_t)\beta^{i,i}_{t}^{-1}(a^{-i}_t|z_{1:t}, \xi^{-i}_t)Q^*_t(x_{t+1}|x_t, a_t)P^{i,w}_t(z_{t+1}, w^{i}_{t+1}|x_{t+1}, x_t, a_t)
$$

(53i)

$$
= \sum_{x_t} \xi^{i,*}_{t}[z_{1:t}, w^{i}_{1:t+1}](x_t)Q^*_t(x_{t+1}|x_t, a_t)P^{i,w}_t(z_{t+1}, w^{i}_{t+1}|x_{t+1}, x_t, a_t)\sum_{\xi^{-i}_t} \pi^*_t[z_{1:t}](\xi^{-i}_t|x_t, \xi^{i,*}_t[z_{1:t}, w^{i}_{1:t+1}])\beta^{i,i}_{t}^{-1}(a^{-i}_t|z_{1:t-1}, \xi^{-i}_t)
$$

(53j)

and

$$
D_r = \sum_{\tilde{x}_t, \tilde{x}_{t-1}^{i,1}} \xi^{i,*}_{t}[z_{1:t}, w^{i}_{1:t+1}](\tilde{x}_t)\pi^*_t[z_{1:t}](\tilde{\xi}^{-i}_t|\xi^{i,*}_t[z_{1:t}, w^{i}_{1:t+1}], \tilde{x}_t)\beta^{i,i}_{t}^{-1}(a^{-i}_t|z_{1:t-1}, \tilde{\xi}^{-i}_t)Q^*_t(\tilde{x}_{t+1}|\tilde{x}_t, a_t)P^{i,w}_t(z_{t+1}, w^{i}_{t+1}|\tilde{x}_{t+1}, \tilde{x}_t, a_t)
$$

(53k)

$$
= \sum_{\tilde{x}_t, \tilde{x}_{t-1}^{i,1}} \xi^{i,*}_{t}[z_{1:t}, w^{i}_{1:t+1}](\tilde{x}_t)Q^*_t(\tilde{x}_{t+1}|\tilde{x}_t, a_t)P^{i,w}_t(z_{t+1}, w^{i}_{t+1}|\tilde{x}_{t+1}, \tilde{x}_t, a_t)\sum_{\tilde{\xi}^{-i}_t} \pi^*_t[z_{1:t}](\tilde{\xi}^{-i}_t|\xi^{i,*}_t[z_{1:t}, w^{i}_{1:t+1}], \tilde{x}_t)\beta^{i,i}_{t}^{-1}(a^{-i}_t|z_{1:t-1}, \tilde{\xi}^{-i}_t)
$$

(53l)

And thus

$$
\frac{N_r}{D_r} = \xi^{i}_{t+1}(x_{t+1})
$$

(53m)

Thus using the above equation and (54o), (54b) is given by

$$
\xi^{i}_{t+1}(x_{t+1}) = \xi^{i}_{t}(x_{t})P^{i+1}_{t}(x_{t+1}|x_{t}^{i}, a_{t}^{i+1}, x_{t+2:T}^{i}, w_{1:t+1}^{i})
$$

(53n)

$$
P^{i+1}_{t}(x_{t+1}|x_{t}^{i}, a_{t}^{i+1}, x_{t+2:T}^{i}, w_{1:t+1}^{i}) = \mathbb{E}[T_{i}^i(x_{T}, A_{T})|x_{1:t}, w_{1:t}^{i}]
$$

(53o)

Lemma 7: \forall i \in \mathcal{N}, t \in \mathcal{T}, z_{1:t} \in \mathcal{H}_t, w_{1:t}^{i} \in (\mathcal{V}^i)^t

$$
V^i_t(\pi^*_t[z_{1:t}], \xi^{i,*}_t[z_{1:t}, w_{1:t}^{i}]) = \mathbb{E}[T_{i}^i(x_{T}, A_{T})|x_{1:t}, w_{1:t}^{i}] = \sum_{n=t}^{T} R_{t}^i(X, A)|z_{1:t}, w_{1:t}^{i}
$$

(54)

Proof:

We prove the lemma by induction. For $t = T$,

$$
\mathbb{E}[T_{i}^i(x_{T}, A_{T})|x_{1:t}, w_{1:t}^{i}] = \sum_{x_{T}^{i}, a_{T}} R_{T}^i(x_{T}, a_{T})\xi_T^i(x_{T})\pi^*_T[z_{1:T}](\xi^{-i}_T|x_T, \xi_t^{i,*}[z_{1:t}, w_{1:T}^{i}])\beta_T^{i,i}(a_t^{-i}|z_{1:t}, \xi_T^{-i})
$$

(55a)

$$
= V^i_T(\pi_T^i[z_{1:T}], \xi^{i,*}_T[z_{1:T}, w_{1:T}^i]),
$$

(55b)
where (56b) follows from the definition of \( V^i \) in (26) and the definition of \( \beta^* \) in the forward recursion in (28).

Suppose the claim is true for \( t + 1 \), i.e., \( \forall i \in \mathcal{N}, t \in \mathcal{T}, (z_{1:t}, w^i_{1:t+1}) \in \mathcal{H}_{t+1} \)

\[
V^i_{t+1}(\pi^{i*}_{t+1}[z_{1:t+1}], \xi^{i*}_{t+1}[z_{1:t+1}, w^i_{1:t+1}]) = \\
\mathbb{E}^{\beta^*_t, \beta^*_{t+1}, \pi^{i*}_{t+1}, \xi^{i*}_{t+1}} \left\{ \sum_{n=t+1}^{T} R^i_n(X_n, A_n) \bigg| z_{1:t+1}, w^i_{1:t+1} \right\}.
\]

Then \( \forall i \in \mathcal{N}, t \in \mathcal{T}, (z_{1:t}, w^i_{1:t}) \in \mathcal{H}_t \), we have

\[
\mathbb{E}^{\beta^*_t, \beta^*_{t+1}, \pi^{i*}_{t}, \xi^{i*}_{t}} \left\{ \sum_{n=t+1}^{T} R^i_n(X_n, A_n) \bigg| z_{1:t}, w^i_{1:t} \right\}
\]

\[
= \mathbb{E}^{\beta^*_t, \beta^*_{t+1}, \pi^{i*}_{t}, \xi^{i*}_{t}} \left\{ R^i_t(X_t, A_t) + \mathbb{E}^{\beta^*_t, \beta^*_{t+1}, \pi^{i*}_{t+1}, \xi^{i*}_{t+1}} \left\{ \sum_{n=t+1}^{T} R^i_n(X_n, A_n) \bigg| z_{1:t}, Z_{t+1}, w^i_{1:t}, W^i_{t+1} \right\} \bigg| z_{1:t}, w^i_{1:t} \right\}
\]

\[
= \mathbb{E}^{\beta^*_t, \beta^*_{t+1}, \pi^{i*}_{t}, \xi^{i*}_{t}} \left\{ R^i_t(X_t, A_t) + \mathbb{E}^{\beta^*_t, \beta^*_{t+1}, \pi^{i*}_{t+1}, \xi^{i*}_{t+1}} \left\{ \sum_{n=t+1}^{T} R^i_n(X_n, A_n) \bigg| z_{1:t}, Z_{t+1}, w^i_{1:t}, W^i_{t+1} \right\} \bigg| z_{1:t}, w^i_{1:t} \right\} \tag{57a}
\]

\[
= \mathbb{E}^{\beta^*_t, \beta^*_{t+1}, \pi^{i*}_{t}, \xi^{i*}_{t}} \left\{ R^i_t(X_t, A_t) + V^i_{t+1}(\pi^{i*}_{t+1}[z_{1:t}, Z_{t+1}], \xi^{i*}_{t+1}[z_{1:t}, Z_{t+1}, w^i_{1:t}, W^i_{t+1}]) \bigg| z_{1:t}, w^i_{1:t} \right\} \tag{57b}
\]

\[
= \mathbb{E}^{\beta^*_t, \beta^*_{t+1}, \pi^{i*}_{t}, \xi^{i*}_{t}} \left\{ R^i_t(X_t, A_t) + V^i_{t+1}(\pi^{i*}_{t+1}[z_{1:t}, Z_{t+1}], \xi^{i*}_{t+1}[z_{1:t}, Z_{t+1}, w^i_{1:t}, W^i_{t+1}]) \bigg| z_{1:t}, w^i_{1:t} \right\} \tag{57c}
\]

\[
= \mathbb{E}^{\beta^*_t, \beta^*_{t+1}, \pi^{i*}_{t}, \xi^{i*}_{t}} \left\{ R^i_t(X_t, A_t) + V^i_{t+1}(\pi^{i*}_{t+1}[z_{1:t}, Z_{t+1}], \xi^{i*}_{t+1}[z_{1:t}, Z_{t+1}, w^i_{1:t}, W^i_{t+1}]) \bigg| z_{1:t}, w^i_{1:t} \right\} \tag{57d}
\]

\[
= V^i_t(\pi^{i*}_{t}[z_{1:t}], \xi^{i*}_{t+1}[z_{1:t}, w^i_{1:t}]) \tag{57e}
\]

where (58b) follows from Lemma 6 in Appendix E, (58c) follows from the induction hypothesis in (57), (58d) follows because the random variables involved in expectation, \( X_t, A_t, Z_{t+1}, W^i_{t+1} \) do not depend on \( \beta^*_t, \beta^*_{t+1} \), and (58e) follows from the definition of \( V^i_t \) in (26).

\[\]


