SUMMARY OF PAPERS

MATTHEW GENTRY DURHAM

Abstract. The papers described here are grouped via topic and, within each group, listed in the order in which they were written.

1. THE COARSE GEOMETRY OF TEICHMÜLLER SPACE WITH THE TEICHMÜLLER METRIC


I build an augmentation of the Masur-Minsky marking complex by Groves-Manning combinatorial horoballs to obtain a graph I call the augmented marking complex, \(\mathcal{AM}(S)\). Adapting work of Masur-Minsky, I show this augmented marking complex is quasiisometric to Teichmüller space \(\mathcal{T}(S)\) with the Teichmüller metric, and also completely integrate the Masur-Minsky hierarchy machinery to \(\mathcal{AM}(S)\) to build flexible families of uniform quasi-geodesics in Teichmüller space, called hierarchy paths. As an application, I give a new proof of Rafi’s distance formula for \(\mathcal{T}(S)\) with the Teichmüller metric.

The Masur-Minsky marking complex was a central tool in the proofs of the geometric rank and quasiisometric rigidity theorems for the mapping class group. The augmented marking complex has played the same role, appearing in every proof of the corresponding theorems for the Teichmüller metric. While a similar construction was simultaneously and independently discovered by Eskin-Masur-Rafi, this is the only formal account of this machinery in the literature.

(2) Elliptic actions on Teichmüller space. 2014; to appear in Groups, Geometry, and Dynamics.

Let \(H < \text{MCG}(S)\) be a finite subgroup of the mapping class group and consider the convex subset of \(\mathcal{T}(S)\) fixed by \(H\), \(\text{Fix}(H) \subset \mathcal{T}(S)\), which Kerckhoff famously proved is always nonempty. In this paper, I study the geometry of the set of almost fixed points with respect to the Teichmüller metric.

For any \(R > 0\), I prove that the set of points whose \(H\)-orbits have diameter bounded by \(R\), \(\text{Fix}_R^T(H)\), is contained in a bounded neighborhood of \(\text{Fix}(H)\). Roughly, this says that almost fixed points are close to fixed points. As an application, I show that the orbit of any point in \(\mathcal{T}(S)\) has a fixed coarse barycenter, a property satisfied by any metric space of coarse nonpositive curvature, e.g. hyperbolic or CAT(0). By contrast, I build an explicit family of examples showing that \(\text{Fix}_R^T(H)\) need not be quasiconvex, despite the fact that \(\text{Fix}(H)\) is convex. As an application of the barycenter theorem, I prove that there is an exponential-time algorithm to solve the conjugacy problem for finite order subgroups of \(\text{MCG}(S)\), recovering a theorem of Tao. The main tools are the combinatorial machinery I described in the previous paper, and finer tools from Teichmüller and Weil-Petersson geometry. These results help with issues related to torsion in the semihyperbolicity project below.
(3) The asymptotic geometry of the Teichmuller metric: Dimension and rank. 2014; submitted.

I analyze the asymptotic cones of Teichmüller space with the Teichmüller metric, \((T(S), d_T)\). I give a new proof of a theorem of Eskin-Masur-Rafi which bounds the dimension of quasi-isometrically embedded flats in \((T(S), d_T)\). My approach is an application of the ideas of Behrstock and Behrstock-Minsky to the quasiisometry model we built for \((T(S), d_T)\) in the first paper above. The key observation here is that the analysis Behrstock and Minsky perform on the asymptotic cones of the Masur-Minsky marking graph transfers to the augmented marking graph, defined in paper (1) above.

2. Mapping class groups


Hierarchically hyperbolic spaces provide a common framework for studying mapping class groups of finite type surfaces, Teichmüller space, and all cubical groups, including right-angled Artin and Coxeter groups. Given such a space \(\mathcal{X}\), we build a bordification of \(\mathcal{X}\) compatible with its hierarchically hyperbolic structure. If \(\mathcal{X}\) is proper, e.g. a hierarchically hyperbolic group such as the mapping class group, we get a compactification of \(\mathcal{X}\); we also prove that our construction generalizes the Gromov boundary of a hyperbolic space and is a model for the Poisson boundary for random walks for any hierarchically hyperbolic group.

We introduce a notion of geometrical finiteness for hierarchically hyperbolic subgroups of hierarchically hyperbolic groups in terms of boundary embeddings, and our first main examples are in the mapping class group. In particular we prove that the natural inclusions into the mapping class group of finitely generated Veech groups and the Leininger-Reid combination subgroups extend to continuous embeddings of their Gromov boundaries into the boundary of the mapping class group. Such embeddings extensions do not exist for their orbit maps in \(T(S)\) with the Thurston compactification, which was previously the standard model for a compactification of the mapping class group.

Our second main set of applications are dynamical and structural, built upon our classification of automorphisms of hierarchically hyperbolic spaces and analysis of how the various types of automorphisms act on the boundary. We prove a generalization of the Handel-Mosher “omnibus subgroup theorem” for mapping class groups to all hierarchically hyperbolic groups, and obtain a new proof of the Caprace-Sageev rank-rigidity theorem for all proper cocompact CAT(0) cube complexes.


We develop an organizing principle called finite invariance for determining when an infinite-type surface admits a graph whose vertices are curves which admits an action of the mapping class group with infinite diameter orbits. We prove that finite invariance is sufficiently robust to determine when a surface admits such a graph in all but a few classes of infinite type surfaces, significantly extending work of Araymayona-Valdez, Aramayona-Fossas-Parlier, and Bavard.
(6) **The mapping class group and Teichmüller space are semihyperbolic.** In progress. Joint with Daniel Groves.

We prove that $\mathcal{MCG}(S)$ and $\mathcal{T}(S)$ with the Teichmüller metric admit $\mathcal{MCG}(S)$-equivariant synchronized bounded bicombings by uniform quasigeodesics, i.e. they are semihyperbolic in the sense of Alonso-Bridson. Roughly, a bicombing on a space is a transitive family of paths which behave as if they are in a CAT(0) space. Semihyperbolicity is a strong nonpositive curvature property shared by hyperbolic spaces, CAT(0) spaces, biautomatic groups, and any group acting geometrically on a manifold with nonpositive curvature.

A bicombing for $\mathcal{MCG}(S)$ has been claimed by Hamenstädt in an unpublished preprint using train tracks and announced for $\mathcal{T}(S)$ by Kapovich-Rafi, who slightly tweak Teichmüller geodesics to build their bicombing. Our approach is quite different and more general; in fact, it should be generalizable to a broad class of HHSes.

Our approach to this problem involves bicombing the marking and augmented marking graphs. The key difficulty in bicombing these graphs can be seen in the failure of hierarchy paths to do so: any procedure which produces paths without strict control over the order in which subsurfaces are traversed will fail to produce a bicombing, for how subsurfaces are traversed can change with the endpoints. Instead, for each pair of markings, we use a process called *betweenness* and work of Bestvina-Bromberg-Fujiwara to curate a collection of subsurfaces over which we have strong control when the markings are perturbed. With these subsurfaces in hand, we then build new combinatorial machinery to construct the paths. This involves significant issues with torsion, which we deal with in part using results from my thesis work in papers (1) and (2) above.

3. **Stability and convex cocompactness**


A Kleinian group $\Gamma < \text{Isom}(\mathbb{H}^3)$ is called *convex cocompact* if any orbit of $\Gamma$ in $\mathbb{H}^3$ is quasiconvex or, equivalently, $\Gamma$ acts cocompactly on the convex hull of its limit set in $\partial \mathbb{H}^3$.

We introduce a strong notion of quasiconvexity in finitely generated groups, which we call *stability*, which generalizes the above quasiconvexity characterization of convex cocompactness. Stability also generalizes quasiconvexity from hyperbolic groups and is preserved under quasi-isometry for finitely generated groups.

We show that the stable subgroups of mapping class groups are precisely the convex cocompact subgroups, which is the important class of subgroups which determine hyperbolic surface group extensions by work of Farb-Mosher and Hamenstädt. This generalizes a well-known result of Behrstock that pseudo-Anosov mapping classes are Morse to subgroups and is related to questions asked by Farb-Mosher and Farb.

I believe that stability is the correct generalization of the classical notion of convex cocompactness from Kleinian groups to all finitely generated groups.

(8) **Boundary convex cocompactness and stability for subgroups of finitely generated groups.** 2016; submitted. Joint with Matthew Cordes.

Using the Morse boundary, we develop an equivalent characterization of subgroup stability which generalizes the above boundary characterization from Kleinian groups. Namely,
we prove that a finitely generated subgroup $H < G$ of a finitely generated group is stable if and only if $H$ has a compact limit set $\Lambda(H)$ in the Morse boundary of $G$ and acts cocompactly on the weak hull of $\Lambda(H)$.

We take this as strong evidence for my above stated claim: that stability is the correct generalization of convex cocompactness to finitely generated groups.

(9) **Middle recurrence and pulling back stability under proper actions.** 2016; submitted. Joint with Tarik Aougab and Samuel Taylor.

We prove that stability pulls back under proper actions on proper metric spaces. This result has several applications, including that convex cocompact subgroups of both mapping class groups and outer automorphism groups of free groups are stable. We also characterize stability in relatively hyperbolic groups whose parabolic subgroups have linear divergence.

Our main tool is a new, effective characterization of stability via a property called *middle recurrence*, which behaves nicely under Lipschitz maps. Our notion of middle recurrence generalizes a notion due to Drutu-Mozes-Sapir and the effectiveness of the relationship to stability that we establish is key for the pulling back theorem.

Perhaps the main application of this theorem is to characterizing so-called convex cocompact subgroups of $\text{Out}(F_n)$. Namely, we prove that if a subgroup $G < \text{Out}(F_n)$ admits a quasisometrically embedded orbit into the free factor graph—a curve graph analogue—then it is a stable subgroup. This result gives the first known examples of non-virtually cyclic stable subgroups of $\text{Out}(F_n)$.

(10) **Stability and universal acylindricity for hierarchically hyperbolic groups.** In preparation. Joint with Carolyn Abbott and Jason Behrstock.

In this paper, we study strong hyperbolic features of hierarchically hyperbolic groups (HHGs). Our main results are a simple characterization of subgroup stability in HHGs and a proof that every HHG admits and HH structure such that its action on its top level curve graph is a maximal acylindrical action, in both cases generalizing the situation in the mapping class group and right-angled Artin groups. We also prove a similar statement about acylindricity actions for certain relative HHGs. These results are new for all cubical groups which are not right-angled Artin groups. Roughly speaking, characterizing the maximal acylindrical action means that every cubical group now has a preferred acylindrical action on a hyperbolic space. For general cubical groups, this is not the contact graph, but for RAAGs it is and, as a consequence, we prove that subgroup stability is equivalent to qi-embedding into the contact graph, which was previously shown by Koberda-Mangahas-Taylor for RAAGs.

As an application of the stability results, we prove that the Morse boundary of any HHG embeds into the Gromov boundary of its top level curve graph $\mathcal{C}S$ in the HH structure from above. This allows us to conclude that the asymptotic dimension of any stable subspace of any HHG is bounded by the asymptotic dimension of $\mathcal{C}S$, an improvement on the previous bound which was the asymptotic dimension of the HHG itself, which is finite but bounded above super-exponentially by the asymptotic dimension of $\mathcal{C}S$. 

Department of Mathematics, University of Michigan, 530 Church St, Ann Arbor, MI 48109, U.S.A, 
E-mail address: durhamma@umich.edu