

# Outline of a Theory of Quantification

Dustin Tucker  
Department of Philosophy  
University of Michigan  
dusttuck@gmail.com

August 9, 2011

## **Abstract**

Historically, Russell's ramified theory of types has been largely neglected. As a foundation of mathematics, it required the axiom of reducibility, which was unpopular at the time. As a solution to paradoxes, it seemed unnecessary in light of Ramsey's distinction between semantical and set-theoretical paradoxes. As a general theory, Russell motivated it with the vicious circle principle, which Gödel attacked. But several authors, including Alonzo Church, David Kaplan, and Rich Thomason, have suggested ramification as a resolution of a family of paradoxes that fall through the cracks of Ramsey's division. These paradoxes have been largely neglected as well, and what little existing work there is often leaves much to be desired. But so does ramification. Like Tarski's hierarchy of languages, it is much too heavy-handed to do justice to the paradoxes. I provide a refinement of ramification's restrictions on propositional quantification, analogous to Kripke's refinement of Tarski's truth predicates. Along the way, I develop a flexible logic of propositions and an alternate resolution of the paradoxes using truth-value gaps.

# 1 Ramification and paradoxes

Ramification<sup>1</sup> is fundamentally a theory of quantification. It says that no proposition can quantify over itself (or over propositions that can quantify over it, etc.).<sup>2</sup> Slightly more carefully, so as to not assume that propositions themselves contain quantifiers, it says that there is an infinite hierarchy of orders of propositions, and that if a sentence (or, even more carefully, a formula  $\mathbf{P}$ ) denotes a proposition of order  $n$ , quantifiers in the sentence ( $\mathbf{P}$ ) can range over only orders  $m < n$ . I often speak loosely of propositions themselves quantifying with the understanding that such talk can be avoided if necessary. I also assume, contrary to Russell’s version of ramification but in line with Church’s (Chu76), that orders are cumulative, so that propositions of order  $n$  also appear in all orders  $m > n$ .

## 1.1 A paradox

I am not concerned here with the details of Russell’s motives for introducing the ramified theory of types. But paradoxes were at the heart of his original developments, and a handful of authors (Chu93; Kap95; Tho88) have proposed ramification as a resolution of certain neglected intensional paradoxes—paradoxes that crucially involve propositions.<sup>3</sup> Suppose, for instance, that (1)–(4) are true.

I fear that everything you hope is false. (1)

You hope that everything I fear is true. (2)

Everything else I fear is true. (3)

Everything else you hope is false. (4)

---

<sup>1</sup>Throughout, I work with Church’s formulation of ramified type theory (Chu76). I do not mean to claim that this is a perfectly accurate reconstruction of Russell’s own theory, but it is close enough for my purposes.

<sup>2</sup>It also concerns quantification over propositional functions, which I address in Section 4.

<sup>3</sup>Other authors (Kne72; Par74; Gla04; Bea82; Lin03b) have also considered restricting propositional quantification to avoid paradoxes. But their suggestions are consistently either too narrow in their focus or not detailed enough to guarantee the right results in the trickier multi-agent cases that I discuss below. Sten Lindström provides more detail in (Lin03a), but the resulting theory is similar enough to ramification that the concerns I raise below about the latter apply to his proposal as well.

(1) says that I am related somehow to the proposition denoted by (5) in virtue of my fears and (2) says that you are related somehow to the proposition denoted by (6) in virtue of your hopes.

Everything you hope is false. (5)

Everything I fear is true. (6)

Adapting notation from later in the paper, I abbreviate “the proposition denoted by (5)” with “[[(5)]]” and so on, and in the interest of simplicity, I use ‘fear’ and ‘hope’ as slightly awkward transitive verbs. Thus, I say things like “I fear [[(5)]] and you hope [[(6)]]” with the understanding that we can, if we wish, rewrite them so as to not presuppose a potentially simplistic and archaic understanding of attitudes.

A situation in which (1)–(4) are true seems to be easy to imagine.<sup>4</sup> Perhaps (1) I am afraid that all your hopes are bound to be disappointed, (2) you don’t like me much and hope that I’m living in a nightmare, and both (4) my fear and (3) your hope are at least *almost* correct. Unfortunately for our imaginations, however, we can prove from these four assumptions that (the propositions denoted by) (5) and (6) are each both true and false.

Suppose that [[(5)]] (the proposition denoted by (5)) is true—that everything you hope is false. By (2), you hope [[(6)]], so it must then be false—something I fear must be false. By (3), the only possible witness is [[(5)]]. Thus, on the supposition that [[(5)]] is true, we have proved that it is false. Anything whose truth implies its falsity must be false,<sup>5</sup> and so we know that [[(5)]] is false—that something you hope is true. By (4), the only possible witness is [[(6)]], which says that everything I fear is true. By (1), I fear [[(5)]], and so we have proved that [[(5)]] is both true and false.<sup>6</sup>

## 1.2 Ramification

Ramification avoids this paradox. It was crucial in the above argument that [[(5)]] was in the range of the quantifier in (6) and [[(6)]] was in the range

---

<sup>4</sup>Of course, the particular attitudes and agents are unimportant.

<sup>5</sup>Barring truth-value gaps.

<sup>6</sup>I have been careful to talk about propositions denoted by sentences rather than the sentences themselves because these are propositional, not sentential, attitudes. It will not do to simply prohibit sentences like (5) and (6) or insist that they do not denote propositions. As long as the propositions themselves are there to be the objects of attitudes and the bearers of truth, we can derive the contradiction.

of the quantifier in (5). But this is not possible in a ramified theory of propositions. Ramifications requires that all quantifiers be restricted to an order, so we must replace (5) and (6) with (5') and (6').

Everything of order  $n$  you hope is false. (5')

Everything of order  $m$  I fear is true. (6')

$\llbracket(5')\rrbracket$  and  $\llbracket(6')\rrbracket$  are of at least orders  $n + 1$  and  $m + 1$  respectively, and so one will definitely be outside the range of the quantifier in the other. Suppose, for instance, that  $n > m$  (the other two cases are equally straightforward). Since I am letting orders be cumulative, anything of order  $m + 1$  will be of order  $n$ , and so the quantifier in (5') ranges over  $\llbracket(6')\rrbracket$ . When we suppose that  $\llbracket(5')\rrbracket$  is true, then, we can still conclude that  $\llbracket(6')\rrbracket$  is false—that something I fear of order  $m$  is false. But the proof can go no farther:  $\llbracket(5')\rrbracket$  cannot be in order  $m$  (because it must be of at least order  $n + 1$  and  $n > m$ ).

I want to pursue ramification as a resolution of these paradoxes. But ramification is too heavy-handed. It prohibits all circular quantification all the time, and most circular quantification is unproblematic in most circumstances. This observation is analogous to one of Kripke's (Kri75, §II), and my proposal stands to traditional ramification in much the same way that Kripke's single truth predicate stands to Tarski's hierarchy. One of the troubles with Tarski's hierarchy of languages is that it prohibits all talk about truth from within an object language. It is, in a way, the ultimate truth-value gap theory: within a language, the truth predicate for that language is *entirely* gappy. This makes it impossible to capture the contingent nature of certain paradoxical circumstances that Kripke discusses, and his solution is to compress the hierarchy of truth predicates into one, leaving gaps only as necessary.

Similarly, ramification is the ultimate restriction on propositional quantification: within an order, *nothing* can quantify over that order. This makes it impossible to capture the contingent nature of the present paradox: the fear and hope reported in (1) and (2) are *risky* but not inherently paradoxical. Any restrictions on propositional quantification that are motivated by paradoxes like this one ought to be similarly responsive to the facts; we should not block (5) and (6) from quantifying over each others' denotations except when we must to avoid contradictions—when (1)–(4) are true. My solution is to compress the orders as much as possible, forcing propositions

up the hierarchy only when leaving them lower in it leads to contradictions.<sup>7</sup>

Unfortunately, Kripke’s actual constructions are of no use in the present case, because they rely on syntactic features of sentences that might have no propositional analogue. For instance, he relies on being able to distinguish the sentences “Snow is white” and “‘Snow is white’ is true” from each other, but it is not so obvious that the proposition *that snow is white* is different from the proposition *that the proposition that snow is white is true*. In any event, I do not think that we must make such substantive metaphysical assumptions to resolve the paradoxes. Thus, although the spirit and motivation of my paper are similar to Kripke’s, my compressed orders must be developed in a different way than his truth predicate.

### 1.3 Preliminaries

In Section 2, I explain how the compressed orders should look and how they relate to ramification in its original form. In Section 3, I show how we can systematically identify the contents of the compressed orders. This begins with a basic, flexible intensional logic. I then describe how we can resolve the paradoxes using truth-value gaps, which help identify the propositions that must be kicked up to a higher order. Finally, I construct the orders, eliminating the truth-value gaps in the process. In Section 4, I discuss the paradox from the end of Appendix B of (Rus03), highlighting some important similarities and differences between uncompressed and compressed ramification. I end with a pair of paradoxes that seem to not require propositional quantification in Section 5 and some alternative approaches to the intensional paradoxes as a whole in Section 6.

Before moving on, I want to set aside some worries about ramification. Even with the compressed orders I develop, it could very well be impossible to quantify over every proposition—there might be no universal order, which contains every proposition. This should be cause for concern, because we seem to be able to make claims about every proposition. According to ramification, for instance, every proposition has an order. But if there is no universal order, then the preceding sentence cannot have expressed what I intended, and the object of that intention is not what you might have thought,

---

<sup>7</sup>This is not quite analogous to Kripke’s approach, as he has only one truth predicate, while I still have an infinite hierarchy of orders. The true analogue is the special case of my proposal in which we restrict ourselves to quantification over order 0 and ignore (or even eliminate) all other orders.

and the content of that thought is not what you tried to make it, and so on.

This is not a new objection to ramification, but it is still an important one. It is, in a way, the problem of providing a universal metalanguage for our theory combined with the challenges facing unrestricted quantification.<sup>8</sup> But the problems are compounded when we are dealing with propositions, supposedly the fundamental bearers of truth.

One might also worry about the orders. Where do the  $n$  and  $m$  in (5') and (6') come from? Are they contextually supplied? If so, what is contextually sensitive? Something in our mental state? And what can propositions look like if they can differ only in the orders over which they quantify (more carefully: only in the orders over which quantifiers in the sentences that best express them quantify)?

Finally, everything in this paper assumes that we can have many beliefs, fears, hopes, etc., and that such attitudes are directed at particular contents, whatever those contents look like. This is a common assumption, but it is not universal, and one might take these paradoxes as evidence that our attitudes do not actually function in this way. Perhaps, for instance, some sort of nominalistic story is correct, and talk about propositions is merely one useful but imperfect way of describing certain mental states. And perhaps that opens the door for other analyses of the paradoxes.

All these issues are important, and a proponent of ramification must ultimately address them. But my purpose in this paper is only to show that we can give ramification a lighter, more discerning touch. I do not mean to argue that ramification is definitely the correct approach to the paradoxes. Whether any form of ramification can ultimately be developed into a satisfactory theory of intensionality is well beyond the scope (and length) of this paper.

## 2 Compressed orders

For compressed ramification, I still insist that quantifiers be restricted to particular orders, but allow the contents of the orders to be contingent and variable, with each proposition appearing (with a few exceptions explained below) in the lowest order it can. Most propositions will thus be of order 0,

---

<sup>8</sup>For an early discussion of universal metalanguages, see (Fit64). For the problems with quantification, see, e.g., (RU06).

the lowest possible order. This includes propositions that involve propositional quantification, such as the proposition that every proposition of order 17 is self-identical. It also includes  $\llbracket(5')\rrbracket$  and  $\llbracket(6')\rrbracket$  whenever at least one of (1)–(4) is false.

Speaking very loosely about propositions and quantification, here is one way to think about these orders: in a standard ramified hierarchy, if a proposition  $x$  is in a higher order, it means both (i) that  $x$  can quantify over more propositions and (ii) that fewer propositions can quantify over  $x$ . In compressed ramification, only the second is retained. The domains  $x$  can quantify over are no longer tied to  $x$ 's own order;  $x$  being in order 0 means only that every proposition can unproblematically quantify over  $x$ .

Even when (1)–(4) are true, compressed ramification will usually be different from traditional ramification. According to the latter,  $\llbracket(5')\rrbracket$  is of order  $n + 1$  and  $\llbracket(6')\rrbracket$  is of order  $m + 1$ . But this is not actually necessary to resolve the paradoxes. Recall how ramification avoided the paradox when  $n > m$ :  $\llbracket(5')\rrbracket$  could not be in the domain of the quantifier in (6'), and so we could not carry the argument out all the way. But this requires only that  $\llbracket(5')\rrbracket$  be of order  $m + 1$ , not  $n + 1$ . More generally, irrespective of the relationship between  $n$  and  $m$ , to resolve this paradox we need to require only that  $\llbracket(5')\rrbracket$  and  $\llbracket(6')\rrbracket$  both be of order  $\min(n, m) + 1$ ; this is what compressed ramification ensures.

Actually, there are other ways we can avoid this paradox with compressed orders; this is why I had to qualify my above gloss of the compressed orders as making each proposition appear “in the lowest order it can.” The derivation of a contradiction will be blocked as long as one of  $\llbracket(5')\rrbracket$  and  $\llbracket(6')\rrbracket$  is kept out of the range of the other sentence's quantifier, so any of the three following options will work (along with infinitely many uninteresting others).

- (i)  $\llbracket(5')\rrbracket$  and  $\llbracket(6')\rrbracket$  are both of order  $\min(n, m) + 1$ .
- (ii)  $\llbracket(5')\rrbracket$  is of order 0 and  $\llbracket(6')\rrbracket$  is of order  $n + 1$ .
- (iii)  $\llbracket(5')\rrbracket$  is of order  $m + 1$  and  $\llbracket(6')\rrbracket$  is of order 0.

Of these three, I think (i) is clearly the best option. After all, (5') and (6') are perfectly symmetrical; it would be strange if our resolution allowed us to treat  $\llbracket(5')\rrbracket$  and  $\llbracket(6')\rrbracket$  differently.

Still, we must be careful because of other, asymmetrical cases, such as those Arthur Prior raises (Pri61). Consider, for instance, a situation in which

I think *that everything of order  $n$  I am now thinking is false just in case somebody else bears a propositional attitude towards something of order  $m$  at some point in the future.*<sup>9</sup> We have a contradiction as long as the object of my thought is of order  $n$  or less and somebody else bears a propositional attitude towards something or other of order  $m$  or less. Thus, as before, we can avoid the paradox in either of two asymmetrical ways and one symmetrical way. Here, though, making the object of my thought be order  $n + 1$  is preferable to making the objects of everybody else’s attitudes, which might include such harmless propositions as the proposition that snow is white, be order  $m + 1$ —here we *want* an asymmetrical treatment.

Getting these results—respecting the symmetry in the fear/hope case and the asymmetry in the Prior-inspired case—is, I think, crucial for any satisfactory refinement of ramification. This is why we cannot do something simple, like begin with the ramified orders and then push every proposition down as far as it can go (that process would, it seems, select (iii) when  $n > m$ ). And it is why the next section takes a detour through truth-value gaps.

### 3 Constructing the orders

I do not want to assume anything about the nature of propositions. I do not want to assume that they are structured, as Russell thought, or that they are sets of possible worlds, or anything else. As I said in Section 1, I think that avoiding the paradoxes by restricting domains of quantification should not commit us to one or another metaphysical theory, and the construction I develop below is compatible with all the standard ones. Even a possible-worlds semantics will work—compressed ramification is not immediately committed to hyperintensionality, unlike traditional ramification.

Of course at the end of the day it would be nice to know where the orders come from and how domains of quantification are actually restricted, but, again, that goes beyond my present aims.

---

<sup>9</sup>I actually did have this thought, minus the explicit orders, sometime in 2007. It was inspired by (Tho88), which was itself inspired by (Pri61).

### 3.1 An overview of the logic

I use a slightly simplified form of Church’s Russellian Simple Type Theory (Chu74). It is an intensional logic in the loose sense that the formulas we are ultimately interested in are of a type  $p$  of propositions rather than a type  $t$  of truth values: I translate every English sentence with a formula of type  $p$ , but I also translate explicit references to propositions, such as that-clauses, with formulas of type  $p$ . The same formulas can thus appear both as arguments to connectives and as arguments to predicates representing propositional attitudes. For instance, I translate (5) and (6) as

$$\forall x[Hx \rightarrow \neg x] \text{ and} \tag{7}$$

$$\forall x[Fx \rightarrow x] \tag{8}$$

respectively.

One could, following (Tho80), use different symbols for this logic. This would be especially helpful if one wished to include an extensional logic along with the intensional part—if one wished to have both a type  $p$  and a type  $t$ . Elsewhere (Tuc10) I have used, e.g.,  $\rightsquigarrow$  alongside  $\rightarrow$  for this purpose. Since, however, I need only an intensional logic, I do not bother with such a notational distinction; following Church, I use the familiar symbols with the understanding that they do not have their familiar truth-functional interpretations.

One consequence of this is that some of the constructions involving truth values are slightly long-winded. Notions of satisfaction and consistency, for instance, must derive from restrictions placed on  $\mathcal{T}$  and  $\mathcal{F}$ , the sets of true and false propositions in the models.

Finally, the logic I use does not include an explicit truth or falsity predicate. I thus make no distinction between, e.g., the proposition *that snow is white* and the proposition *that the proposition that snow is white is true*—I used the bare ‘ $x$ ’ in (7) and (8) rather than something like ‘ $Tx$ ’. This treatment of truth is not novel: it is the one Arthur Prior uses (Pri61), and it has been defended (GCB74) as the appropriate treatment of (at least many instances of) the English ‘true’. But I think that my decision to translate English sentences in this way is not a substantial one; the constructions I develop below should work just as well with an explicit truth predicate.

I begin with an unramified logic, introducing multiple domains of propositional quantification in Section 3.6.

## 3.2 Syntax

The set  $TS$  of *type symbols* of our language  $\mathfrak{L}$  is the smallest set such that  $p \in TS$  and for all  $\tau, \sigma \in TS$ ,  $\langle \tau, \sigma \rangle \in TS$ . Intuitively,  $p$  is the type of propositions,  $\langle p, p \rangle$  is the type of functions from propositions to propositions, and so on.<sup>10</sup> The *proper symbols* are the constants  $\neg^{\langle p, p \rangle}$ ,  $\wedge^{\langle p, \langle p, p \rangle \rangle}$ ,  $\vee^{\langle p, \langle p, p \rangle \rangle}$ ,  $\rightarrow^{\langle p, \langle p, p \rangle \rangle}$ ,  $\leftrightarrow^{\langle p, \langle p, p \rangle \rangle}$ ; for each  $\tau \in TS$ , the constants  $=^{\langle \tau, \langle \tau, p \rangle \rangle}$  and an infinite alphabet of variables  $x^\tau$ ,  $y^\tau$ , etc.; and several additional constants with superscript  $\tau \in TS$  that I introduce with the paradoxes. The *improper symbols* are  $\forall$ ,  $\exists$ ,  $[$ , and  $]$ .<sup>11</sup> I omit superscripts when no ambiguity thereby arises.

Any proper symbol with superscript  $\tau$  is a *well-formed formula* of type  $\tau$ . If  $\mathbf{P}$  is a well-formed formula of type  $\langle \tau, \sigma \rangle$  and  $\mathbf{Q}$  is a well-formed formula of type  $\tau$ , then  $\mathbf{P}\mathbf{Q}$  (often written  $\mathbf{P}(\mathbf{Q})$ ) is a well-formed formula of type  $\sigma$ . If  $\mathbf{P}$  is a well-formed formula of type  $p$  and  $\mathbf{x}$  is a variable,  $\forall \mathbf{x}[\mathbf{P}]$  and  $\exists \mathbf{x}[\mathbf{P}]$  are well-formed formulas of type  $p$ .  $\Phi$  is the set of well-formed formulas of  $\mathfrak{L}$ . I employ standard abbreviations, using  $\mathbf{P}(\mathbf{Q}, \mathbf{R})$  for  $\mathbf{P}^{\langle \tau, \langle \sigma, \rho \rangle \rangle} \mathbf{Q}^\tau \mathbf{R}^\sigma$  and, given a binary connective or relation symbol  $\mathcal{C}$ ,  $\mathbf{P} \mathcal{C} \mathbf{Q}$  for  $\mathcal{C}(\mathbf{P}, \mathbf{Q})$ . I sometimes insert brackets to disambiguate scope; such disambiguations are necessary only because of my abbreviations. When I omit brackets and parentheses, I assume that juxtaposition has the narrowest scope possible, followed by relation symbols like  $=$ . Thus, for instance, given constants  $a^{\langle p, p \rangle}$ ,  $b^p$ , and  $c^p$  we should read  $ab = ac \wedge c$  as  $[a(b) = a(c)] \wedge c$ , not something like  $a(b = a(c \wedge c))$ .

I have already used ‘ $\mathbf{P}$ ’, ‘ $\mathbf{Q}$ ’, etc. as metavariables over well-formed formulas and ‘ $\mathbf{x}$ ’ as a metavariable over variables, and I continue to do so, sometimes with superscripts to restrict their ranges. Also as I already have, I allow symbols and formulas to name themselves, omitting corner quotes. But I avoid using formulas to name their denotations; I speak of not the proposition  $\mathbf{P} \wedge \mathbf{Q}$  but the proposition denoted by  $\mathbf{P} \wedge \mathbf{Q}$  or the proposition

<sup>10</sup>One could, of course, easily include other types, such as a type  $i$  of individuals, but we will not need them.

<sup>11</sup>One could make  $\mathfrak{L}$  more general by including  $\lambda$  as an improper symbol and replacing  $\forall$  and  $\exists$  with, for each  $\tau \in TS$ , constants  $\forall^{\langle \langle \tau, p \rangle, p \rangle}$  and  $\exists^{\langle \langle \tau, p \rangle, p \rangle}$ . But we will not need  $\lambda$ -abstraction, so I omit it for simplicity’s sake. I have taken both quantifiers and all the connectives as primitive, rather than defining some in terms of others, because we are dealing with propositions, not truth values. I do not want to assume that, for instance, conjunctive propositions are identical to negations of certain disjunctive ones, although I do not rule out such identities.

$\llbracket \mathbf{P} \wedge \mathbf{Q} \rrbracket$ , where  $\llbracket \cdot \rrbracket$  is the interpretation function introduced in Section 3.3.

### 3.3 Semantics

A *model*  $\mathfrak{M}$  is a quadruple  $\langle \mathcal{D}, \mathcal{T}, \mathcal{F}, \llbracket \cdot \rrbracket \rangle$ , where

- $\mathcal{D}$  is a set of sets (*domains*)  $\mathcal{D}_\tau$ , one for each  $\tau \in TS$ ;
- $\mathcal{T}$  and  $\mathcal{F}$  are disjoint subsets of  $\mathcal{D}_p$ , whose purpose I explain below; and
- $\llbracket \cdot \rrbracket$  is an interpretation function, a function  $\Phi \rightarrow \bigcup \mathcal{D}$  ( $\Phi$ , recall, is the set of well-formed formulas of our language  $\mathfrak{L}$ ) taking each well-formed formula  $\mathbf{P}^\tau$  to an element of  $\mathcal{D}_\tau$ .

When  $f$  is a partial function  $\mathcal{D}_p \rightarrow \{0, 1\}$ , I sometimes write ‘ $\mathfrak{M}_f$ ’ for a model in which  $\mathcal{T} = \{x \in \mathcal{D}_p : f(x) = 1\}$  and  $\mathcal{F} = \{x \in \mathcal{D}_p : f(x) = 0\}$ . (I use ‘ $x$ ’, ‘ $y$ ’, etc. throughout as variables over elements of  $\bigcup \mathcal{D}$ .)

As I explained above, I make no assumptions about the nature of propositions; this translates into placing no restrictions on  $\mathcal{D}_p$ .<sup>12</sup> The other  $\mathcal{D}_\tau$ , as well as  $\llbracket \cdot \rrbracket$ , are entirely standard. For any  $\langle \tau, \sigma \rangle \in TS$ ,  $\mathcal{D}_{\langle \tau, \sigma \rangle} = \mathcal{D}_\sigma^{\mathcal{D}_\tau}$ , the set of functions from  $\mathcal{D}_\tau$  to  $\mathcal{D}_\sigma$ . When  $\mathbf{P}$  is a lone proper symbol,  $\llbracket \mathbf{P} \rrbracket$  is unrestricted. When  $\mathbf{P}$  has the form  $\forall x[\mathbf{Q}]$  or  $\exists x[\mathbf{Q}]$ ,  $\llbracket \mathbf{P} \rrbracket$  is mostly unrestricted: we need to ensure only that identity is preserved under alphabetic change of bound variables and substitution of identicals, so that, e.g., we have  $\llbracket \forall x[Ax] \rrbracket = \llbracket \forall y[By] \rrbracket$  if we also have  $\llbracket A \rrbracket = \llbracket B \rrbracket$ . Of course, we must have  $\llbracket \mathbf{P}(\mathbf{Q}) \rrbracket = \llbracket \mathbf{P} \rrbracket(\llbracket \mathbf{Q} \rrbracket)$ ;  $\llbracket \cdot \rrbracket$  is thus entirely determined by its (mostly arbitrary) values for proper symbols and quantified formulas.

$\mathcal{T}$  and  $\mathcal{F}$  can be thought of as containing the true and false propositions respectively. In assuming that  $\mathcal{T}$  and  $\mathcal{F}$  are disjoint, I assume that there are no truth-value gluts; this requirement could be relaxed if one wanted to pursue paraconsistent resolutions. In not requiring  $\mathcal{T}$  and  $\mathcal{F}$  to jointly exhaust  $\mathcal{D}_p$ , I allow truth-value gaps, which, as I have said, help us construct the compressed orders. It is, however, safe and reasonable to require that identity propositions never be gappy and always have the correct truth values—to

---

<sup>12</sup>Beyond assuming that it is big enough. This is practically trivial for most of my (very modest) interests in this paper, though it becomes less trivial as one begins looking harder at paradoxes like the proposition Yablo from Section 5, with its infinite hierarchy of propositions.

require that for any  $\tau \in TS$  and  $\mathbf{x}, \mathbf{y} \in \mathcal{D}_\tau$  we have  $\llbracket =^{\langle \tau, \langle \tau, p \rangle \rangle} \rrbracket(\mathbf{x})(\mathbf{y}) \in \mathcal{T}$  if  $\mathbf{x} = \mathbf{y}$  and  $\llbracket =^{\langle \tau, \langle \tau, p \rangle \rangle} \rrbracket(\mathbf{x})(\mathbf{y}) \in \mathcal{F}$  otherwise.

Finally, we must place some restrictions on our models to ensure that our truth values and truth-value gaps are well-behaved. I require that  $\mathcal{T}$  and  $\mathcal{F}$  follow one direction of the strong Kleene scheme (extended to quantification following (Kri75))—that if a conjunction is in  $\mathcal{T}$ , then both conjuncts must be in  $\mathcal{T}$ , and if it is in  $\mathcal{F}$ , then one of the conjuncts must be in  $\mathcal{F}$ ; that if a universal quantification is in  $\mathcal{T}$ , then every instance must be in  $\mathcal{T}$ , and if it is in  $\mathcal{F}$ , then an instance must be in  $\mathcal{F}$ ; and so on. Since this goes in only one direction, it does not require that, e.g., a conjunction be in  $\mathcal{T}$  if both its conjuncts are.

We can state this restriction explicitly but much more tediously as follows. For any variable  $\mathbf{x}^\tau$  and  $\mathbf{z} \in \mathcal{D}_\tau$ , let  $\llbracket \cdot \rrbracket^{\mathbf{x}/\mathbf{z}}$  be that interpretation function just like  $\llbracket \cdot \rrbracket$  except that  $\llbracket \mathbf{x} \rrbracket^{\mathbf{x}/\mathbf{z}} = \mathbf{z}$ .<sup>13</sup> Then for all  $\mathbf{x}, \mathbf{y} \in \mathcal{D}_p$ , I require that

- (a) if  $\llbracket \neg \rrbracket(\mathbf{x}) \in \mathcal{T}$ , then  $\mathbf{x} \in \mathcal{F}$ ;
- (b) if  $\llbracket \neg \rrbracket(\mathbf{x}) \in \mathcal{F}$ , then  $\mathbf{x} \in \mathcal{T}$ ;
- (c) if  $\llbracket \vee \rrbracket(\mathbf{x})(\mathbf{y}) \in \mathcal{T}$ , then  $\mathbf{x} \in \mathcal{T}$  or  $\mathbf{y} \in \mathcal{T}$ ;
- (d) if  $\llbracket \vee \rrbracket(\mathbf{x})(\mathbf{y}) \in \mathcal{F}$ , then  $\mathbf{x}, \mathbf{y} \in \mathcal{F}$ ;
- (e) if  $\llbracket \wedge \rrbracket(\mathbf{x})(\mathbf{y}) \in \mathcal{T}$ , then  $\mathbf{x}, \mathbf{y} \in \mathcal{T}$ ;
- (f) if  $\llbracket \wedge \rrbracket(\mathbf{x})(\mathbf{y}) \in \mathcal{F}$ , then  $\mathbf{x} \in \mathcal{F}$  or  $\mathbf{y} \in \mathcal{F}$ ;
- (g) if  $\llbracket \rightarrow \rrbracket(\mathbf{x})(\mathbf{y}) \in \mathcal{T}$ , then  $\mathbf{x} \in \mathcal{F}$  or  $\mathbf{y} \in \mathcal{T}$ ;
- (h) if  $\llbracket \rightarrow \rrbracket(\mathbf{x})(\mathbf{y}) \in \mathcal{F}$ , then  $\mathbf{x} \in \mathcal{T}$  and  $\mathbf{y} \in \mathcal{F}$ ;
- (i) if  $\llbracket \leftrightarrow \rrbracket(\mathbf{x})(\mathbf{y}) \in \mathcal{T}$ , then  $\mathbf{x}, \mathbf{y} \in \mathcal{T}$  or  $\mathbf{x}, \mathbf{y} \in \mathcal{F}$ ;
- (j) if  $\llbracket \leftrightarrow \rrbracket(\mathbf{x})(\mathbf{y}) \in \mathcal{F}$ , then either  $\mathbf{x} \in \mathcal{T}$  and  $\mathbf{y} \in \mathcal{F}$  or  $\mathbf{x} \in \mathcal{F}$  and  $\mathbf{y} \in \mathcal{T}$ ;
- (k) if  $\llbracket \forall \mathbf{x}^\tau \rrbracket[\mathbf{P}] \in \mathcal{T}$ , then  $\llbracket \mathbf{P} \rrbracket^{\mathbf{x}/\mathbf{z}} \in \mathcal{T}$  for all  $\mathbf{z} \in \mathcal{D}_\tau$ ;
- (l) if  $\llbracket \forall \mathbf{x}^\tau \rrbracket[\mathbf{P}] \in \mathcal{F}$ , then  $\llbracket \mathbf{P} \rrbracket^{\mathbf{x}/\mathbf{z}} \in \mathcal{F}$  for some  $\mathbf{z} \in \mathcal{D}_\tau$ ;
- (m) if  $\llbracket \exists \mathbf{x}^\tau \rrbracket[\mathbf{P}] \in \mathcal{T}$ , then  $\llbracket \mathbf{P} \rrbracket^{\mathbf{x}/\mathbf{z}} \in \mathcal{T}$  for some  $\mathbf{z} \in \mathcal{D}_\tau$ ; and
- (n) if  $\llbracket \exists \mathbf{x}^\tau \rrbracket[\mathbf{P}] \in \mathcal{F}$ , then  $\llbracket \mathbf{P} \rrbracket^{\mathbf{x}/\mathbf{z}} \in \mathcal{F}$  for all  $\mathbf{z} \in \mathcal{D}_\tau$ .

---

<sup>13</sup>We are sure to have a unique such function because an interpretation function is entirely determined by the arbitrary values it assigns to the proper symbols and quantificational formulas.

### 3.4 Paradoxes

At this point, we have the resources to reconstruct the paradoxes in our system. We can capture the essence of (1)–(4) with (9) and (10), letting (3) and (4) be vacuously satisfied. (7) and (8) are reproduced from above (with a change in variable).

$$\forall y[Hy \rightarrow \neg y] \tag{7}$$

$$\forall y[Fy \rightarrow y] \tag{8}$$

$$\forall x[Fx \leftrightarrow x = \forall y[Hy \rightarrow \neg y]] \tag{9}$$

$$\forall x[Hx \leftrightarrow x = \forall y[Fy \rightarrow y]] \tag{10}$$

The paradox is that if we suppose  $\llbracket(9)\rrbracket, \llbracket(10)\rrbracket \in \mathcal{T}$  and eliminate truth-value gaps, requiring  $\mathcal{T} \cup \mathcal{F} = \mathcal{D}_p$ , then we can prove that  $\llbracket(7)\rrbracket$  and  $\llbracket(8)\rrbracket$  are in both  $\mathcal{T}$  and  $\mathcal{F}$ . Given that  $\mathcal{T}$  and  $\mathcal{F}$  must be disjoint, this amounts to saying that there are no gapless models with  $\llbracket(9)\rrbracket, \llbracket(10)\rrbracket \in \mathcal{T}$ .

The proof parallels the informal derivation in Section 1. Given  $\llbracket(9)\rrbracket, \llbracket(10)\rrbracket \in \mathcal{T}$ , suppose that  $\llbracket(7)\rrbracket \in \mathcal{T}$ . Then by clauses (k), (g), and (b) of the restriction above, we know that for all  $z \in \mathcal{D}_p$ , if  $\llbracket H \rrbracket(z) \in \mathcal{T}$ , then  $z \in \mathcal{F}$ . Since  $\llbracket(10)\rrbracket \in \mathcal{T}$ , we know by clauses (k) and (i) and our insistence that identity statements be well-behaved that  $\llbracket H \rrbracket(\llbracket(8)\rrbracket) \in \mathcal{T}$ , and so we have  $\llbracket(8)\rrbracket \in \mathcal{F}$ . By clauses (l) and (h), we thus know that for some  $z \in \mathcal{D}_p$ ,  $\llbracket F \rrbracket(z) \in \mathcal{T}$  and  $z \in \mathcal{F}$ . Since  $\llbracket(9)\rrbracket \in \mathcal{T}$ , we know by clauses (k) and (i) and the well-behavedness of identity that  $\llbracket(7)\rrbracket$  is the only  $z \in \mathcal{D}_p$  for which we have  $\llbracket F \rrbracket(z) \in \mathcal{T}$ , and so we have  $\llbracket(7)\rrbracket \in \mathcal{F}$ , which contradicts our initial supposition, given that  $\mathcal{T}$  and  $\mathcal{F}$  are disjoint.

$\llbracket(7)\rrbracket$ , then, cannot be in  $\mathcal{T}$ , and so must be in  $\mathcal{F}$  if we do not allow truth-value gaps. But from here, through similar reasoning, we can prove that it is in  $\mathcal{T}$ , contra the disjointness of  $\mathcal{T}$  and  $\mathcal{F}$  but this time with no suppositions.

Truth-value gaps, of course, block this proof by prohibiting the inference from  $\llbracket(7)\rrbracket \notin \mathcal{T}$  to  $\llbracket(7)\rrbracket \in \mathcal{F}$ . Ramification blocks the proof by restricting clauses (k)–(n), as we will see below.

### 3.5 Truth-value gaps

Call a model  $\mathfrak{M}'$  an *extension* of a model  $\mathfrak{M}$  iff  $\mathfrak{D} = \mathfrak{D}'$ ,  $\llbracket \rrbracket = \llbracket \rracket'$ ,  $\mathcal{T} \subseteq \mathcal{T}'$ , and  $\mathcal{F} \subseteq \mathcal{F}'$ , and call an extension  $\mathfrak{M}'$  of  $\mathfrak{M}$  *proper* iff either  $\mathcal{T} \subset \mathcal{T}'$  or  $\mathcal{F} \subset \mathcal{F}'$ . Recalling from Section 3.3 that  $\mathfrak{M}_f$  is a model in which  $\mathcal{T}$  and  $\mathcal{F}$  are defined

by a partial function  $f$ ,  $\mathfrak{M}_g$  is then a proper extension of  $\mathfrak{M}_f$  iff  $\mathfrak{D}_f = \mathfrak{D}_g$ ,  $\llbracket \cdot \rrbracket_f = \llbracket \cdot \rrbracket_g$ , and  $f \subset g$ . Call  $\mathfrak{M}_f$  *maximal* iff it has no proper extensions—iff for any  $g \supset f$ ,  $\mathfrak{M}_g$  would violate one of clauses (a)–(n) above. In effect, a model is maximal iff no more propositions can be made true or false without violating our restrictions on truth-value assignments. Any maximal model is, according to a truth-value gap resolution, one possible way the world can be.

That maximal models exist, if any models do, is almost immediate. (This should not be a surprise, as they are closely analogous to maximal consistent sets of sentences.) Let  $F$  be a set of partial functions  $f : \mathcal{D}_p \rightarrow \{0, 1\}$  such that for all  $f \in F$ ,  $\mathfrak{M}_f$  is a model. It suffices to show that if  $F$  is totally ordered by  $\subset$  (which guarantees that  $\bigcup F$  is a function), then  $\mathfrak{M}_{\bigcup F}$ , abbreviated  $\mathfrak{M}_F$ , is also a model—also treats truth values as clauses (a)–(n) require.

This is straightforward. Suppose, for instance, that for some  $\mathbf{x}, \mathbf{y} \in \mathcal{D}_p$ ,  $\llbracket \wedge \rrbracket(\mathbf{x})(\mathbf{y}) \in \mathcal{T}_F$ . We need to show  $\mathbf{x}, \mathbf{y} \in \mathcal{T}_F$  in order to show that  $\mathfrak{M}_F$  satisfies clause (e). But we have this: if  $\llbracket \wedge \rrbracket(\mathbf{x})(\mathbf{y}) \in \mathcal{T}_F$ , then for some  $f \in F$  we have  $\llbracket \wedge \rrbracket(\mathbf{x})(\mathbf{y}) \in \mathcal{T}_f$ , whence, since  $\mathfrak{M}_f$  satisfies clause (e) by supposition, we have  $\mathbf{x}, \mathbf{y} \in \mathcal{T}_f$ , whence we have  $\mathbf{x}, \mathbf{y} \in \mathcal{T}_F$  immediately. Every other case proceeds in exactly the same fashion: if something is in  $\mathcal{T}_F$  or  $\mathcal{F}_F$ , then it is in some  $\mathcal{T}_f$  or  $\mathcal{F}_f$  respectively, whence whatever the relevant clauses require of  $\mathfrak{M}_F$  holds of  $\mathfrak{M}_f$ , whence it holds of  $\mathfrak{M}_F$  as well.

Knowing that maximal models exist is a step in the right direction, but it does not quite get us where we want to be. Ultimately, we need maximal models in which our paradoxical assumptions are true—models for which we have  $\llbracket (9) \rrbracket, \llbracket (10) \rrbracket \in \mathcal{T}$ . Truth values are preserved in extensions—taking extensions is monotonic—so this amounts to needing just one such model, maximal or otherwise.

We cannot in general be certain that there are models in which  $\llbracket (9) \rrbracket, \llbracket (10) \rrbracket \in \mathcal{T}$  because we cannot in general be certain that we do not have bizarre propositional identities. If, for instance, we have  $\llbracket (9) \rrbracket = \llbracket (7) \rrbracket$ , then we cannot make  $\llbracket (9) \rrbracket$  true. Thus, to ensure that it is possible to make  $\llbracket (9) \rrbracket$  and  $\llbracket (10) \rrbracket$  true, we must make some minimal assumptions about the nature of propositions. But all we need is to usually<sup>14</sup> have  $\llbracket \mathbf{P} \rrbracket \neq \llbracket \mathbf{Q} \rrbracket$  when  $\mathbf{P}$  and  $\mathbf{Q}$  are truth-functionally inequivalent. This makes  $\llbracket (9) \rrbracket$  and  $\llbracket (10) \rrbracket$  relatively innocuous: all  $\llbracket (9) \rrbracket \in \mathcal{T}$  requires, for instance, is that propositions of the

<sup>14</sup>I consider possible exceptions in Section 5.

form  $\llbracket F \rrbracket(\mathbf{x})$  be true or false, and  $F(\mathbf{P})$  is not truth-functionally equivalent to very much.<sup>15</sup>

### 3.6 Compressed orders formally

The idea is to begin with a maximal, gappy model from the previous section; restrict the domain of quantification to just those propositions that have been assigned truth values; and push the other propositions up to a higher order, where we can (eventually) assign them truth values. The truth-value gaps disappear at the end of the day, but along the way, they help us get exactly the results, both symmetrical and asymmetrical, described in Section 2.

To begin, we must enrich our language and models to make room for infinitely many domains of propositional quantification. Let  $\mathcal{I}$  be a non-empty set of ordinals with no greatest element. The natural numbers will do, although transfinite orders should pose no problems. The only change to our language is to replace each variable  $\mathbf{x}^p$  with the variables  $\mathbf{x}^i$ ,  $i \in \mathcal{I}$ . These new variables are treated as having superscript  $p$  for syntactic purposes.

We now need to add orders to the variables in (7)–(10). In the interest of not (further) overworking italic Latin letters, I use ‘ $\alpha$ ’, ‘ $\beta$ ’, etc. as variables over elements of  $\mathcal{I}$  when we will need to refer to the orders later.

$$\forall y^\alpha [Hy \rightarrow \neg y] \tag{7'}$$

---

<sup>15</sup>One way to demonstrate that it is possible to prohibit such problematic identities is to take a cue from Anthony Anderson’s models of Church’s Logic of Sense and Denotation (And80). Let  $\mathcal{D}_p$  be the set of equivalence classes of closed formulas of type  $p$  for a particular equivalence relation  $\mathcal{R}$ , given the familiar definition of ‘closed’. For the most restrictive account of identity that is of interest for our purposes,  $\mathcal{R}$  is (the reflexive, transitive closure of) the relation that holds between all and only (i) formulas that vary only in their bound variables (suitably and familiarly restricted to avoid relating problematic pairs like  $\forall x[x = \forall y[x = y]]$  and  $\forall y[y = \forall y[y = y]]$ ) and (ii) formulas  $\mathbf{P}$  and  $\mathbf{P}'$  where (a)  $\mathbf{P}'$  is the result of replacing a part  $\mathbf{Q}$  of  $\mathbf{P}$  with  $\mathbf{R}$  and (b) either  $\mathcal{R}(\mathbf{Q}, \mathbf{R})$  or  $\mathcal{R}(\mathbf{R}, \mathbf{Q})$ . Our predicates then denote functions from sets of formulas to sets of formulas.  $\llbracket F \rrbracket$ , for instance, is the function that takes a set of formulas and returns the set of formulas constructed by prepending  $F$  to the elements of the argument. That is (momentarily insisting on corner quotes and recalling that officially there are no parentheses in  $\mathfrak{L}$ ), for a set of formulas  $\mathbf{x}$ ,  $\llbracket F \rrbracket(\mathbf{x}) = \{\ulcorner F\mathbf{P} \urcorner : \mathbf{P} \in \mathbf{x}\}$ . This fine-grained treatment of propositions opens the door for a violation of Cantor’s theorem: if we have a unique formula of type  $p$  for every  $\mathbf{x} \in \mathcal{D}_{\langle p, p \rangle}$ , then this construction falls apart. But our current language  $\mathfrak{L}$  is safe, because it can express only three of the (infinitely many) elements of  $\mathcal{D}_{\langle p, p \rangle} - \llbracket F \rrbracket$ ,  $\llbracket H \rrbracket$ , and  $\llbracket \neg \rrbracket$ .

$$\forall y^\beta [Fy \rightarrow y] \quad (8')$$

$$\forall x^\gamma [Fx \leftrightarrow x = \forall y^\alpha [Hy \rightarrow \neg y]] \quad (9')$$

$$\forall x^\delta [Hx \leftrightarrow x = \forall y^\beta [Fy \rightarrow y]] \quad (10')$$

(9') says that I fear (and fear only, through order  $\gamma$ ) that everything you hope of order  $\alpha$  or less is false (i.e., I fear  $\llbracket(7')\rrbracket$ ), and (10') says that you hope (and hope only, through order  $\delta$ ) that everything I fear of order  $\beta$  or less is true (i.e., you hope  $\llbracket(8')\rrbracket$ ). For simplicity, suppose  $\alpha < \beta < \gamma < \delta$ . The goal of compressed ramification is then to have  $\llbracket(7')\rrbracket$  and  $\llbracket(8')\rrbracket$  in order  $\alpha + 1$ ;  $\beta$ ,  $\gamma$ , and  $\delta$  can mostly drop out of the picture.

Notice in particular that I do not impose orders on arguments or outputs of functions— $\llbracket F \rrbracket$ , for instance, can still take and return propositions of any order. This is different from traditional ramification. But it is natural if one does not want to assume in advance that propositions have particular orders.

The changes to the models are only slightly more involved. A model  $\mathfrak{M}$  is now a quintuple  $\langle \mathfrak{D}, \mathfrak{Q}, \mathcal{T}, \mathcal{F}, \llbracket \cdot \rrbracket \rangle$ , where  $\mathfrak{Q}$  is a set of domains of propositional quantification  $\mathcal{Q}_i$ ,  $i \in \mathcal{I}$ .<sup>16</sup> To make the  $\mathcal{Q}_i$  actually function as domains of quantification, we need to change clauses (k)–(n) of our restriction on  $\mathcal{T}$  and  $\mathcal{F}$  when the superscript  $\tau$  on  $\mathbf{x}$  is some  $i \in \mathcal{I}$  (when  $\tau \notin \mathcal{I}$ , the original clauses suffice).

- (k<sup>i</sup>) If  $\llbracket \forall \mathbf{x}^i [\mathbf{P}] \rrbracket \in \mathcal{T}$ , then  $\llbracket \mathbf{P} \rrbracket^{\mathbf{x}/z} \in \mathcal{T}$  for all  $z \in \mathcal{Q}_i$ ;
- (l<sup>i</sup>) if  $\llbracket \forall \mathbf{x}^i [\mathbf{P}] \rrbracket \in \mathcal{F}$ , then  $\llbracket \mathbf{P} \rrbracket^{\mathbf{x}/z} \in \mathcal{F}$  for some  $z \in \mathcal{Q}_i$ ;
- (m<sup>i</sup>) if  $\llbracket \exists \mathbf{x}^i [\mathbf{P}] \rrbracket \in \mathcal{T}$ , then  $\llbracket \mathbf{P} \rrbracket^{\mathbf{x}/z} \in \mathcal{T}$  for some  $z \in \mathcal{Q}_i$ ; and
- (n<sup>i</sup>) if  $\llbracket \exists \mathbf{x}^i [\mathbf{P}] \rrbracket \in \mathcal{F}$ , then  $\llbracket \mathbf{P} \rrbracket^{\mathbf{x}/z} \in \mathcal{F}$  for all  $z \in \mathcal{Q}_i$ .

### 3.6.1 First attempt

It is tempting to think that constructing the orders is almost trivial. Suppose we have a maximal model  $\mathfrak{M}$  from Section 3.5. In effect, this is a model in which  $\mathcal{Q}_i = \mathcal{D}_p$  for all  $i \in \mathcal{I}$ . Thus, we know we can assign truth values to the propositions in  $\mathcal{T}$  or  $\mathcal{F}$  even when everything quantifies over them, and so they can be in order 0 without trouble.

<sup>16</sup>We need not require that we have  $\mathcal{Q}_i \subseteq \mathcal{D}_p$  or  $\llbracket \mathbf{x}^i \rrbracket \in \mathcal{Q}_i$ . The former is covered by the construction of the models, and the latter is not necessary: the order of a variable only matters when it is bound, and then its initial value is unimportant.

Why, then, can we not simply let  $\mathcal{T} \cup \mathcal{F}$  be  $\mathcal{Q}_0$ ? We could repeat the process for all the subsequent orders: keeping our new  $\mathcal{Q}_0$  fixed, we simply find another maximal model, with orders 1 and up unrestricted; make the new  $\mathcal{T} \cup \mathcal{F}$  our  $\mathcal{Q}_1$ ; and so on.

The trouble is with the first step. When we cut  $\mathcal{Q}_0$  down to  $\mathcal{T} \cup \mathcal{F}$  from  $\mathcal{D}_p$ , we cannot retain the original truth-value assignment. To see this, consider the proposition  $\llbracket \exists x[Fx] \rrbracket$ . When  $\mathcal{Q}_0 = \mathcal{D}_p$ , we have  $\llbracket \exists x[Fx] \rrbracket \in \mathcal{T}$ . That is, it is true that I fear something. But when we move to  $\mathcal{Q}_0 = \mathcal{T} \cup \mathcal{F}$ , this proposition must be false, because the only thing I fear,  $\llbracket (7') \rrbracket$ , is now outside the domain of quantification.

This would not be troublesome on its own—we could simply adjust the truth-value assignment and then move on to constructing  $\mathcal{Q}_1$ . But there may be propositions that can be assigned truth values only when other propositions are in the domain of quantification. That is, moving to  $\mathcal{Q}_0 = \mathcal{T} \cup \mathcal{F}$  might introduce new truth-value gaps.

We can see this by looking at the second paradox in Section 2, in which I thought something about the objects of both my thought and everybody else's attitudes. My thought was problematic only in situations in which somebody else bore an attitude towards a proposition. But it is no more difficult to make a paradox contingent on any other fact, such as whether I fear something. More formally, consider the following two formulas. Glossing over the orders, (12) says that the only  $A$ -proposition is the proposition *that every  $A$ -proposition is true iff I fear something*, and (11) denotes that proposition.

$$\forall y^{\alpha'} [Ay \rightarrow y] \leftrightarrow \exists y^0 [Fy] \quad (11)$$

$$\forall x^{\beta'} [Ax \leftrightarrow x = [\forall y^{\alpha'} [Ay \rightarrow y] \leftrightarrow \exists y^0 [Fy]]] \quad (12)$$

Let a model  $\mathfrak{M}_{f,\mathcal{Q}}$  be the model like  $\mathfrak{M}_f$  with  $\mathcal{Q}_0 = \mathcal{Q}$ . With this notation, our initial maximal, gappy model is  $\mathfrak{M}_{f,\mathcal{D}_p}$ , and the model after we have cut  $\mathcal{Q}_0$  down to  $\mathcal{T} \cup \mathcal{F}$  would be  $\mathfrak{M}_{f,\text{Dom}(f)}$ , where  $\text{Dom}(f)$  is the domain of  $f$ . The trouble is that  $\mathfrak{M}_{f,\text{Dom}(f)}$  is not guaranteed to be a model. If we have  $\llbracket (12) \rrbracket \in \mathcal{T}$ , then  $\llbracket (11) \rrbracket$  is truth-valueless whenever we have  $\llbracket \exists y^0 [Fy] \rrbracket \in \mathcal{F}$ —whenever we have  $\llbracket (7') \rrbracket \notin \mathcal{Q}_0$ . And we have this in  $\mathfrak{M}_{f,\text{Dom}(f)}$  so long as we have  $\llbracket (9') \rrbracket, \llbracket (10') \rrbracket \in \mathcal{T}$ —so long as the main paradox gets off the ground. Thus,  $\mathfrak{M}_{f,\text{Dom}(f)}$  will not do if  $\llbracket (12) \rrbracket \in \mathcal{T}$ ; we must instead find some maximal  $\mathfrak{M}_{g,\text{Dom}(f)}$ , which does not assign a truth value to  $\llbracket (11) \rrbracket$ . This means we need to cut  $\mathcal{Q}_0$  down further to  $\text{Dom}(g)$ , so that it does not include  $\llbracket (11) \rrbracket$ . And, of

course, once we have done so, we may need to cut it down still further due to other propositions, which were assigned truth values in  $\mathfrak{M}_{f, \mathcal{D}_p}$  and  $\mathfrak{M}_{g, \text{Dom}(f)}$  but cannot be assigned truth values in any  $\mathfrak{M}_{h, \text{Dom}(g)}$ . And so on.<sup>17</sup>

### 3.6.2 Second attempt

Luckily, following through on the “and so on” at the end of the last paragraph can be monotonic in the sense that the domain of quantification never grows, and so we can be guaranteed to have fixed points that can serve as our order 0. We must, however, change our approach slightly:  $\mathfrak{M}_{f, \mathcal{Q}}$  must be the model like  $\mathfrak{M}_f$  with  $\mathcal{Q}_i = \mathcal{Q}$  for all  $i \in \mathcal{I}$ . The plan is thus to cut every domain down, opening the later ones back up to  $\mathcal{D}_p$  only after we have fixed  $\mathcal{Q}_0$ .

To see why this is necessary, imagine that the superscript 0 in (11) and (12) were a 1, and suppose that  $\min(\alpha + \beta) \geq 1$ . This ensures that  $\llbracket(12)\rrbracket \in \mathcal{T}$  is paradoxical during the construction of  $\mathcal{Q}_1$ , and so we must have  $\llbracket(11)\rrbracket \notin \mathcal{Q}_1$ . Now, if we were not cutting every subsequent order down while constructing  $\mathcal{Q}_0$ , we would have  $\llbracket\exists y^1[Fy]\rrbracket \in \mathcal{T}$  during that construction, whence we would be able to assign a truth value to  $\llbracket(11)\rrbracket$ , whence we would have  $\llbracket(11)\rrbracket \in \mathcal{Q}_0$ . But the orders are supposed to be cumulative, so that is a contradiction. Ultimately, the idea behind this change is that we cannot know in advance where a proposition will first enter the hierarchy of orders, and so if we discover that a proposition must be kept out of the order we are constructing, the only safe approach (in light of these Prior-inspired paradoxes) is to assume, for the duration of that construction, that it cannot be in any subsequent order either.<sup>18</sup>

With that change made, call a model  $\mathfrak{M}_{f, \mathcal{Q}}$  *intermediate* if  $\text{Dom}(f) \subseteq \mathcal{Q}$ : intermediate models are models in which every proposition that has a truth value is also in the domain of quantification.<sup>19</sup> The argument for the existence of maximal models from Section 3.5 carries over without alteration: we can

<sup>17</sup>There may also be propositions that can be assigned truth values only after the domain has been cut down. If we prepend a  $\neg$  to  $\exists y^0[Fy]$  in (12) and (11), then  $\llbracket(11)\rrbracket$  will be such a proposition when  $\llbracket(12)\rrbracket \in \mathcal{T}$ . In the interest of simplicity, I do not try to expand  $\mathcal{Q}_0$  to include such propositions.

<sup>18</sup>Notice also that in light of this, truth values will change as the orders grow: for any  $i \in \mathcal{I}$  and  $x \in \mathcal{Q}_{i+1} \setminus \mathcal{Q}_i$ , we must have  $\llbracket\exists y[y = x]\rrbracket^{x/x} \in \mathcal{F}$  during the construction of  $\mathcal{Q}_i$  but  $\in \mathcal{T}$  during the construction of  $\mathcal{Q}_{i+1}$ .

<sup>19</sup>I think that focusing on models of this sort is not necessary for avoiding the paradoxes, but it is simplifying. It guarantees, for instance, that there is no expansion of the sort described in 17.

be certain that if there are any intermediate models for some  $\mathcal{Q}$ , then there is a maximal intermediate  $\mathfrak{M}_{f,\mathcal{Q}}$ .

Now let  $\mathcal{G}$  be a function on subsets of  $\mathcal{D}_p$  which, given some set  $\mathcal{Q} \subseteq \mathcal{D}_p$ , returns  $\text{Dom}(f)$ , where  $\mathfrak{M}_{f,\mathcal{Q}}$  is a maximal intermediate model (and returns  $\emptyset$  if there are no intermediate models at all with domain of quantification  $\mathcal{Q}$ ).<sup>20</sup> Thus, for instance, working with the  $f$  and  $g$  from the end of Section 3.6.1, we could have  $\mathcal{G}(\mathcal{D}_p) = \text{Dom}(f)$  and  $\mathcal{G}(\text{Dom}(f)) = \text{Dom}(g)$ .

$\mathcal{G}$  is guaranteed to have fixed points—we always have  $\mathcal{G}(\mathcal{Q}) \subseteq \mathcal{Q}$ —but the most common is bound to be  $\emptyset$ . It thus remains to show that if we begin with  $\mathcal{D}_p$  and require only that we have  $\llbracket(9')\rrbracket, \llbracket(10')\rrbracket \in \mathcal{T}$ ,  $\mathcal{G}$  will always return a  $\mathcal{Q}$  for which there are intermediate models. The challenge here is analogous to that of showing that there are models with  $\llbracket(9)\rrbracket, \llbracket(10)\rrbracket \in \mathcal{T}$  in Section 3.5. As with that case, we cannot guarantee that, given arbitrary true propositions (or arbitrary propositional identities),  $\mathcal{G}$  never returns  $\emptyset$ . For instance, if we insist on looking at models with  $\llbracket\exists x[Ax]\rrbracket \in \mathcal{T}$ , then we will have  $\mathcal{G}(\mathcal{G}(\mathcal{D}_p)) = \emptyset$ . But also as in Section 3.5, a simple examination of  $\llbracket(9')\rrbracket$  and  $\llbracket(10')\rrbracket$  should put these fears to rest: those propositions require only that certain propositions of the form  $\llbracket F \rrbracket(\mathbf{x})$ ,  $\llbracket H \rrbracket(\mathbf{x})$  be assigned particular truth values, and truth-value assignments to such propositions are never jeopardized as the domain of quantification shrinks, because (once again barring bizarre identities) our clauses (a)–(n) place no substantive restrictions on them.

The proposal is to use fixed points of  $\mathcal{G}$ , starting from  $\mathcal{G}(\mathcal{D}_p)$ , for our orders. A (non-empty) fixed point of  $\mathcal{G}$  is a domain of quantification  $\mathcal{Q}$  such that there is a maximal intermediate model  $\mathfrak{M}_{f,\mathcal{Q}}$  with  $\text{Dom}(f) = \mathcal{Q}$ . We can take such a  $\mathcal{Q}$  as our order 0—it is a set of propositions whose presence in every domain of quantification never causes problems. Then we begin anew, working on order 1. The notation quickly becomes unwieldy, but let  $\mathfrak{M}_{f,\mathcal{Q},\mathcal{Q}'}$  be a model with  $\mathcal{Q}_0 = \mathcal{Q}$  and  $\mathcal{Q}_i = \mathcal{Q}'$  for all  $i > 0$ . The idea is that, given our fixed point  $\mathcal{Q}$  from above, we start over with a maximal  $\mathfrak{M}_{f,\mathcal{Q},\mathcal{D}_p}$ , in which all the orders aside from 0 are again unrestricted. We then cut each  $\mathcal{Q}_i$ ,  $i > 0$ , down to  $\text{Dom}(f)$ , and then to  $\text{Dom}(g)$  for some maximal intermediate  $\mathfrak{M}_{g,\mathcal{Q},\text{Dom}(f)}$ , and so on until we reach another fixed point.<sup>21</sup> That gives us

<sup>20</sup>It turns out that maximal models can disagree about the location of truth-value gaps—we can have maximal models  $\mathfrak{M}_{f,\mathcal{Q}}$  and  $\mathfrak{M}_{g,\mathcal{Q}}$  with  $\text{Dom}(f) \neq \text{Dom}(g)$ . Constructing a suitable  $\mathcal{G}$  thus requires the axiom of choice. Since we are ultimately interested in fixed points, it is probably best to always choose  $\mathcal{Q}$  if possible.

<sup>21</sup>Technically, we need a new  $\mathcal{G}$ , which holds order 0 fixed and varies only subsequent

order 1. And then we do it again for every subsequent order, letting any transfinite order be the union of every lower order as usual.

As we proceed through the orders, we have progressively fewer truth-value gaps. If the only paradoxical assumptions that we care about are (9') and (10'), then we could very well have orders  $0$ – $\alpha$  identical and orders  $\alpha + 1$  and up identical, so long as our truth-value assignment didn't happen to make any other paradoxical assumptions true accidentally (recall that we assumed  $\alpha < \beta < \gamma < \delta$ ). In any event, as far as the fear/hope paradox is concerned, as we build up through order  $\alpha$ ,  $\llbracket(7')\rrbracket$  and  $\llbracket(8')\rrbracket$  will lack truth values and thus be kept outside the orders. Once we hit  $\alpha + 1$ , however, we will be able to assign them both truth values without trouble:  $\llbracket(7')\rrbracket$  can be vacuously true, because  $\llbracket(8')\rrbracket$  is the only proposition you hope (at least through order  $\delta$ ) and it cannot be in order  $\alpha$ . And then  $\llbracket(8')\rrbracket$  can be true, because  $\llbracket(7')\rrbracket$  is the only proposition I fear (at least through order  $\gamma$ , and thus certainly through  $\beta$ ) and we just saw that it can true.

## 4 The Appendix B paradox

I have so far characterized ramification as a theory of only propositional quantification, but it is also a theory of propositional functional quantification—of quantification over propositional functions.<sup>22</sup> This allows it to resolve a general form of the paradox Russell presented in Appendix B of (Rus03). Suppose that we have a constant  $m^{\langle(p,p),p\rangle}$  for which the following is true.

$$\forall x^{\langle(p,p)\rangle} \forall y^{\langle(p,p)\rangle} [mx = my \rightarrow x = y] \quad (13)$$

Such a supposition should be worrying, because if it is correct, then for every function from propositions to propositions, there is a unique proposition, in violation of Cantor's theorem. Indeed, if we also suppose

$$\forall x^p [wx \leftrightarrow \exists y[x = my \wedge \neg yx]], \quad (14)$$

we can prove that  $\llbracket w \rrbracket(\llbracket mw \rrbracket)$  is both true and false. Suppose that it is true—suppose  $\llbracket w \rrbracket(\llbracket mw \rrbracket) \in \mathcal{T}$ . Then by (14) we know that for some  $\mathbf{y}$ ,  $\llbracket mw \rrbracket = \llbracket m \rrbracket(\mathbf{y})$  and  $\mathbf{y}(\llbracket mw \rrbracket) \in \mathcal{F}$ . By the former and (13) we have  $\mathbf{y} = \llbracket w \rrbracket$ ,

---

orders.

<sup>22</sup>This would likely not be Russell's terminology, and the resulting paradoxes are likely not quite what Russell had in mind, but they will do for the present purposes.

whence by the latter we have  $\llbracket w \rrbracket(\llbracket mw \rrbracket) \in \mathcal{F}$ , contra our assumption. Thus  $\llbracket w \rrbracket(\llbracket mw \rrbracket)$  must be false—we must have  $\llbracket w \rrbracket(\llbracket mw \rrbracket) \in \mathcal{F}$ . Then by (14) we know that for no  $\mathbf{y}$  do we have both  $\llbracket mw \rrbracket = \llbracket m \rrbracket(\mathbf{y})$  and  $\mathbf{y}(\llbracket mw \rrbracket) \in \mathcal{F}$ . In particular, we do not have those for  $\mathbf{y} = \llbracket w \rrbracket$ . Of course,  $\llbracket mw \rrbracket = \llbracket m \rrbracket(\llbracket w \rrbracket)$ , so we must not have  $\llbracket w \rrbracket(\llbracket mw \rrbracket) \in \mathcal{F}$ , a contradiction.

Nevertheless, according to some intuitions, (13) seems plausible. If, for instance,  $\llbracket m \rrbracket(\mathbf{x})$  is the proposition *that  $\mathbf{x}$  is my favorite function from propositions to propositions*, it is difficult to see how (13) could be false—surely that proposition is unique to  $\mathbf{x}$ . One can then think of  $\llbracket w \rrbracket$  as that function which is true of a proposition  $\mathbf{x}$  (more carefully: which returns a true proposition when given a proposition  $\mathbf{x}$ ) just in case for some  $\mathbf{y}$ , (i)  $\mathbf{x}$  is the proposition *that  $\mathbf{y}$  is my favorite function from propositions to propositions* and (ii)  $\mathbf{y}$  is not true of  $\mathbf{x}$ — $\mathbf{y}(\mathbf{x}) \in \mathcal{F}$ .

If one can argue independently that one of (13) and (14) is false—if, for instance, one’s preferred theory of propositions ensures that (13) is false, as many do—then of course the paradox dissolves. In fact, it is impossible to make (13) true given the models I have been discussing. The domains of those models are sets, and so  $\mathcal{D}_{\langle p,p \rangle}$ , which is the set of all functions from  $\mathcal{D}_p$  to itself, must be larger than  $\mathcal{D}_p$ . But then it is not possible to have a one-to-one function from  $\mathcal{D}_{\langle p,p \rangle}$  to  $\mathcal{D}_p$ , as  $\llbracket m \rrbracket$  must be according to (13). Nevertheless, one can imagine more flexible models which do not immediately make (13) false, and it is instructive to see how both uncompressed and compressed ramification can make (13) and (14) consistent.

## 4.1 Uncompressed ramification

Traditional ramification resolves this paradox by placing restrictions on quantification over propositional functions. One can think of the original implementation of ramification as replacing every  $p$  in a type symbol with a numeral, as we did before for the variables of type  $p$ . That is, for every function, we restrict both the possible arguments and the possible values to a particular order. Thus, for instance, for  $\alpha, \beta, \gamma \in \mathcal{I}$  and  $\gamma \leq \alpha$ , (13) becomes

$$\forall x^{\langle \alpha, \beta \rangle} \forall y^{\langle \alpha, \beta \rangle} [m^{\langle \langle \alpha, \beta \rangle, \gamma \rangle} x = my \rightarrow x = y]. \quad (13')$$

(We must have  $\gamma \leq \alpha$  so that the output of  $\llbracket m \rrbracket$  can be an argument to its input—so that  $w(mw)$  is well-formed. We do not need to have  $\gamma = \alpha$  because orders are cumulative.)

We can define the order of a function recursively as the sum of the orders of its input and output. Thus, for instance,  $\llbracket m \rrbracket$ <sup>23</sup> is of order  $\alpha + \beta + \gamma$ .<sup>24</sup> It is then easiest to see how the contradiction is blocked by introducing Church's comprehension schema, which I have translated into the present system. For every  $\mathbf{P}^p$  we are guaranteed to have the following true, so long as  $\mathbf{z}$  (does not appear in  $\mathbf{P}$  and) is (i) of higher order than every bound variable in  $\mathbf{P}$  and (ii) of at least as high an order as every free variable and constant in  $\mathbf{P}$  (Chu76, p. 750).<sup>25</sup>

$$\exists \mathbf{z} [\forall \mathbf{x} [\mathbf{z}\mathbf{x} \leftrightarrow \mathbf{P}]] \quad (15)$$

We need the following in order to be certain we have the right sort of  $\llbracket w \rrbracket$ , and thus to derive a contradiction from (13').

$$\exists z^{(\alpha, \beta)} \left[ \forall x^\gamma [zx \leftrightarrow \exists y^{(\alpha, \beta)} [x = my \wedge \neg yx]] \right]$$

However, this is not an instance of (15) for two reasons. First,  $z$  is of the same order as  $y$ , in violation of clause (i) above. Second,  $z$  is of lower order than  $m$ , in violation of clause (ii). (Here it is important that the lowest order in Church's system is 1.) The only true instance is

$$\exists z^{(\alpha, \delta)} \left[ \forall x^\gamma [zx \leftrightarrow \exists y^{(\alpha, \beta)} [x = my \wedge \neg yx]] \right] \quad (14')$$

for some  $\delta \geq \beta + \gamma$  (and thus  $> \beta$ , because, again, Church's lowest order is 1). Thus, although we can be certain from (14') that there is a function very much like the  $\llbracket w \rrbracket$  that the paradox requires, there are two reasons that we cannot take it as  $\llbracket w \rrbracket$  and still prove that  $\llbracket w(mw) \rrbracket$  is both true and false. First,  $\llbracket w \rrbracket$  would then be outside the domain of the quantifiers in (13'). Second,  $w(mw)$  would then be ill-formed because  $mw$  would be ill-formed— $w$  would be not the right type to appear as an argument to  $m$ .

---

<sup>23</sup>Here and throughout I am departing slightly from Church: he uses comprehension schemas, rather than talking directly about the denotations of formulas, but this shift is harmless for my purposes.

<sup>24</sup>This is not quite the way orders are defined in (Chu76). There, functions of arbitrarily many arguments are considered, so that we can have types like  $\langle \sigma_1, \sigma_2, \dots, \sigma_n, \tau \rangle$ , and the order of such a type is the sum of the order of  $\tau$  and the highest of the orders of  $\sigma_1$ – $\sigma_n$ . Luckily, we do not need to address functions of more than one argument, so the simple definition in the text suffices.

<sup>25</sup>The schema is actually more general, allowing for arbitrarily many free  $\mathbf{x}$  in  $\mathbf{P}$ , but we can make do with the schema for a single  $\mathbf{x}$ .

## 4.2 Compressed ramification

I think it is telling that in both crucial steps above, it was overdetermined that the paradox could not go through. We had two reasons that  $\llbracket w \rrbracket$  could not be of type  $\langle \alpha, \beta \rangle$ , which gave us two reasons that  $\llbracket w(mw) \rrbracket$  was not both true and false. Compressed ramification sets out to retain just the first of each pair of reasons—to retain the one reason in each case that involves quantification. Once we see that those reasons are enough to resolve the paradox, we can greatly simplify our orders. We no longer need to care about the orders of the inputs to a function: we can look exclusively at the orders of its outputs, which can of course vary in a theory of compressed ramified types. The story then remains that when we quantify over propositional functions, we really quantify over only functions of a particular order. The difference is that the order of a function is now simply the highest of the orders of its outputs.

## 4.3 Machinery

Recall that  $\mathcal{I}$  is a set of indices, intuitively corresponding to our orders. Before, when we restricted propositional quantification, we replaced every variable  $\mathbf{x}^p$  with a collection of variables  $\mathbf{x}^i$ . Now we want to restrict quantification over functions as well, so for every variable  $\mathbf{x}$ , if the superscript on  $\mathbf{x}$  ends in a  $p$ , we replace that variable with a collection of variables on which that last  $p$  has been replaced with some  $i \in \mathcal{I}$ . Thus, for instance,  $x^{\langle (p,p), p \rangle}$  is replaced with a collection of variables  $x^{\langle (p,p), i \rangle}$ ,  $y^{\langle p, \langle p, p \rangle \rangle}$  with  $y^{\langle p, \langle p, i \rangle \rangle}$ , and so on.  $i$ s are still treated as  $p$ s for syntactic purposes; as before, they only play a role in restrictions (k)–(n) on truth-values assignments.

Intuitively,  $x^{\langle p, i \rangle}$  ranges over only functions all of whose outputs are of order  $i$  or less. We can capture this more formally by defining a domain of quantification  $\mathcal{Q}_\tau$  for every  $\tau$  that is a variable superscript. It is easiest to recursively define  $\mathcal{Q}_\tau$  more generally for every  $\tau$  that is either a type symbol or a variable superscript. We already have  $\mathcal{Q}_i$  for  $i \in \mathcal{I}$  from our models. Let  $\mathcal{Q}_p = \mathcal{D}_p$ , and then let  $\mathcal{Q}_{\langle \tau, \sigma \rangle} = \mathcal{Q}_\sigma^{\mathcal{Q}_\tau}$ , where  $\mathcal{Q}_\sigma^{\mathcal{Q}_\tau}$ , recall, is the set of functions from  $\mathcal{Q}_\tau$  to  $\mathcal{Q}_\sigma$ . Indices appear only at the ends of variable superscripts, so this amounts to having  $\mathcal{Q}_{\langle \tau, \sigma \rangle} = \mathcal{Q}_\sigma^{\mathcal{D}_\tau}$ . This makes  $\mathcal{Q}_{\langle \tau, \sigma \rangle}$  the set of functions in  $\mathcal{D}_{\langle \tau, \sigma \rangle}$  whose ranges are subsets of  $\mathcal{Q}_\sigma$ , as intended.

Now we can return to nearly the original forms of the last four restrictions on truth-value assignments, changing the  $\mathcal{D}$ s to  $\mathcal{Q}$ s and, of course, widening the range of  $\tau$  from  $TS$  to include all our new variable superscripts:

- (k') if  $\llbracket \forall \mathbf{x}^\tau [\mathbf{P}] \rrbracket \in \mathcal{T}$ , then  $\llbracket \mathbf{P} \rrbracket^{\mathbf{x}/z} \in \mathcal{T}$  for all  $z \in \mathcal{Q}_\tau$ ;
- (l') if  $\llbracket \forall \mathbf{x}^\tau [\mathbf{P}] \rrbracket \in \mathcal{F}$ , then  $\llbracket \mathbf{P} \rrbracket^{\mathbf{x}/z} \in \mathcal{F}$  for some  $z \in \mathcal{Q}_\tau$ ;
- (m') if  $\llbracket \exists \mathbf{x}^\tau [\mathbf{P}] \rrbracket \in \mathcal{T}$ , then  $\llbracket \mathbf{P} \rrbracket^{\mathbf{x}/z} \in \mathcal{T}$  for some  $z \in \mathcal{Q}_\tau$ ; and
- (n') if  $\llbracket \exists \mathbf{x}^\tau [\mathbf{P}] \rrbracket \in \mathcal{F}$ , then  $\llbracket \mathbf{P} \rrbracket^{\mathbf{x}/z} \in \mathcal{F}$  for all  $z \in \mathcal{Q}_\tau$ .

An unintended consequence of this construction is that  $\llbracket \wedge \rrbracket$ ,  $\llbracket \neg \rrbracket$ , etc. never appear in a domain of quantification unless we have a domain of universal propositional quantification—unless for some  $i \in \mathcal{I}$  we have  $\mathcal{Q}_i = \mathcal{D}_p$ . This is because they have outputs of every order. This is not an ideal consequence, but I think that it is also not fatal.

### 4.3.1 Resolution

At this point we have enough to explain how compressed orders resolve the paradox. In fact, the construction from Section 3 can go through unchanged as long as we use our new (k')–(n'). Neither (13) nor (13') is a formula of our language; we must instead have

$$\forall x^{(p,\alpha)} \forall y^{(p,\alpha)} [m^{(p,p)} x = my \rightarrow x = y] \quad (13'')$$

for some  $\alpha \in \mathcal{I}$ . We must also rewrite (14). We can assume that there is a value for  $w$ , rather than going through a comprehension schema, because our compressed orders take care of the contradiction.

$$\forall x^\beta [w^{(p,p)} x \leftrightarrow \exists y^{(p,\gamma)} [x = my \wedge \neg yx]] \quad (14'')$$

To get the paradox off the ground, suppose that we have  $\llbracket (13'') \rrbracket, \llbracket (14'') \rrbracket \in \mathcal{T}$ . For simplicity, I suppose that for all  $\mathbf{x} \neq \llbracket mw \rrbracket$ ,  $\llbracket w \rrbracket(\mathbf{x})$  can be assigned a truth value unproblematically.<sup>26</sup>  $\beta$  matters only insofar as there is no paradox at all if we have some reason to exclude  $\llbracket mw \rrbracket$  from  $\mathcal{Q}_\beta$ , so also suppose that we have  $\llbracket mw \rrbracket \in \mathcal{Q}_\beta$ .<sup>27</sup> In what follows, **let ‘y’ range over only those functions for which we have  $\mathbf{y} \neq \llbracket w \rrbracket$  and  $\llbracket mw \rrbracket = \llbracket m \rrbracket(\mathbf{y})$** . Of course, there may very well be no such  $\mathbf{y}$ . Whether there is such a  $\mathbf{y}$  can be

<sup>26</sup>This is entirely a simplifying assumption. I discuss it below.

<sup>27</sup>It is unlikely that this assumption will be false very often. Orders depend entirely on truth-value assignments and there are no restrictions on the truth values of propositions of the form  $\llbracket m \rrbracket(\mathbf{x})$ , so we have no reason to expect  $\llbracket mw \rrbracket$  to ever be gappy, and thus no reason to expect to have even  $\llbracket mw \rrbracket \notin \mathcal{Q}_0$ , let alone  $\llbracket mw \rrbracket \notin \mathcal{Q}_\beta$ .

determined before we begin constructing the orders, as it is purely a matter of identities, and identities are fixed from the outset.

First, notice that if we have  $\llbracket w(mw) \rrbracket \in \mathcal{T}$  at an order  $\delta$  (more precisely, if we have it during the construction of  $\mathcal{Q}_\delta$ ), then both  $\llbracket w \rrbracket$  and some  $\mathbf{y}$  such that  $\mathbf{y}(\llbracket mw \rrbracket) \in \mathcal{F}$  are in  $\mathcal{Q}_{\langle p, \delta \rangle}$ :  $\llbracket w \rrbracket$  because of our simplifying assumption that every other proposition of the form  $\llbracket w \rrbracket(\mathbf{x})$  has a truth value and such a  $\mathbf{y}$  because we need a witness to the right-hand side of (14''). Thus we cannot make  $\llbracket w(mw) \rrbracket$  true at an order  $\delta \leq \alpha$  (again, more precisely, during the construction of  $\mathcal{Q}_\delta$ ,  $\delta \leq \alpha$ ), on pain of contradiction from (13'') (recall the bold stipulation above).

Now, there are two exhaustive but not exclusive possibilities.

- (i) **If for some  $\mathbf{y}$  we can consistently have  $\mathbf{y} \in \mathcal{Q}_{\langle p, \gamma \rangle}$  and  $\mathbf{y}(\llbracket mw \rrbracket) \in \mathcal{F}$ , then** that  $\mathbf{y}$  can witness the truth of the right-hand side of (14''). For this  $\mathbf{y}$ , **there is a least  $\gamma' \leq \gamma$  such that  $\mathbf{y} \in \mathcal{Q}_{\langle p, \gamma' \rangle}$ ,**<sup>28</sup> **and we can have  $\llbracket w(mw) \rrbracket \in \mathcal{T}$  at order  $\min(\gamma', \alpha + 1)$  but no earlier.**<sup>29</sup> That is, we have models in which  $\llbracket w(mw) \rrbracket \in \mathcal{T}$  and  $\notin \mathcal{Q}_{\min(\gamma', \alpha + 1) - 1}$  but  $\in \mathcal{Q}_{\min(\gamma', \alpha + 1)}$ . We need to be at least as high as  $\gamma'$  to make sure that we have our witness, and we need to be beyond order  $\alpha$  for the reasons in the preceding paragraph.
- (ii) **If we can consistently have  $\mathbf{y}(\llbracket mw \rrbracket) \in \mathcal{T}$  for every  $\mathbf{y} \in \mathcal{Q}_{\langle p, \gamma \rangle}$ , then we can have  $\llbracket w(mw) \rrbracket \in \mathcal{F}$  at order  $\gamma + 1$  but no earlier.** That is, we have models in which  $\llbracket w(mw) \rrbracket \in \mathcal{F}$  and  $\notin \mathcal{Q}_\gamma$  but  $\in \mathcal{Q}_{\gamma + 1}$ . This covers the case in which there are no  $\mathbf{y}$ —in which we do not have a  $\mathbf{y} \neq \llbracket w \rrbracket$  such that  $\llbracket mw \rrbracket = \llbracket m \rrbracket(\mathbf{y})$ . We cannot make  $\llbracket w(mw) \rrbracket$  false before order  $\gamma + 1$ , given our simplifying assumption, because then we would have  $\llbracket w \rrbracket \in \mathcal{Q}_{\langle p, \gamma \rangle}$ , and  $\llbracket w \rrbracket$  itself would witness the truth of the right-hand side of (14''), contra the falsity of  $\llbracket w(mw) \rrbracket$ .

If the antecedent of only (whence exactly, since they are exhaustive) one of (i) and (ii) holds, then the corresponding consequent tells us what the models look like. If both antecedents hold, then we have some of each type of model.<sup>30</sup>

<sup>28</sup>If no  $\mathbf{y}(\mathbf{x})$  is potentially paradoxical, then we should have  $\gamma' = 0$ , but we can harmlessly work with the more general case.

<sup>29</sup>Barring bizarre identities as always.

<sup>30</sup>One might think that the possible models depend on how  $\min(\gamma', \alpha + 1)$  and  $\gamma$  compare:

If we do away with the simplifying assumption that  $\llbracket w(mw) \rrbracket$  is the only proposition of the form  $\llbracket w \rrbracket(\mathbf{y})$  whose truth is problematic, then we add a further layer to our cases: we could make  $\llbracket w(mw) \rrbracket$  false earlier than order  $\gamma + 1$  if doing so didn't force  $\llbracket w \rrbracket$  into  $\mathcal{Q}_{\langle p, \gamma \rangle}$ , and we could make  $\llbracket w(mw) \rrbracket$  true earlier than order  $\alpha + 1$  if doing so didn't force  $\llbracket w \rrbracket$  into  $\mathcal{Q}_{\langle p, \alpha \rangle}$ . But this only multiplies (in relatively uninteresting directions) the ways we can resolve the paradox, so the only result of doing away with this assumption would be more clauses in the already complex possibilities described above; it really is playing only a simplifying role.

### 4.3.2 Summary

This is a reiteration of the bold text above. Let ‘ $\mathbf{y}$ ’ range over only those functions for which we have  $\mathbf{y} \neq \llbracket w \rrbracket$  and  $\llbracket mw \rrbracket = \llbracket m \rrbracket(\mathbf{y})$ . If for some  $\mathbf{y}$  we can consistently have  $\mathbf{y} \in \mathcal{Q}_{\langle p, \gamma \rangle}$  and  $\mathbf{y}(\llbracket mw \rrbracket) \in \mathcal{F}$ , then there is a least  $\gamma' \leq \gamma$  such that  $\mathbf{y} \in \mathcal{Q}_{\langle p, \gamma' \rangle}$ , and we can have  $\llbracket w(mw) \rrbracket \in \mathcal{T}$  at order  $\min(\gamma', \alpha + 1)$ . If we can consistently have  $\mathbf{y}(\llbracket mw \rrbracket) \in \mathcal{T}$  for every  $\mathbf{y} \in \mathcal{Q}_{\langle p, \gamma \rangle}$ , then we can have  $\llbracket w(mw) \rrbracket \in \mathcal{F}$  at order  $\gamma + 1$ .

## 4.4 Related paradoxes

The goal of compressed ramification is to retain traditional ramification's resolutions of paradoxes while allowing for more flexible orders. I think its treatment of the above version of the Appendix B paradox satisfies this goal. But there are other paradoxes that highlight important differences between uncompressed and compressed ramification, including some that compressed ramification simply cannot resolve.

---

that if  $\min(\gamma', \alpha + 1) \leq \gamma + 1$ , then we have models of the sort described in (i); if  $\gamma + 1 \leq \min(\gamma', \alpha + 1)$ , then we have models of the sort described in (ii); and only if  $\min(\gamma', \alpha + 1) = \gamma + 1$  do we have models of both sorts. But this assumes that once a proposition has been assigned a truth value during the construction of an order, it retains that truth value through the construction of each subsequent order. This is not only not required, but actually guaranteed to be false as long as the orders grow at all; see 18. Thus, even if, for instance, both antecedents are true and  $\min(\gamma', \alpha + 1) < \gamma + 1$ , we will eventually see models of both sorts, and we will be restricted to models of the first sort only until we have begun constructing order  $\gamma + 1$ .

## 4.5 The original Appendix B paradox

In the original version of the Appendix B paradox,  $\llbracket m \rrbracket(\mathbf{x})$  is the proposition *that every proposition of which  $\mathbf{x}$  is true is true*, i.e., the proposition  $\llbracket \forall x[yx \rightarrow x] \rrbracket^{y/\mathbf{x}}$ . With this interpretation, the problematical assumption is not (13) but

$$\forall x^{(p,p)} \forall y^{(p,p)} [\forall z[xz \rightarrow z] = \forall z[yz \rightarrow z] \rightarrow x = y]. \quad (16)$$

Since each  $\llbracket m \rrbracket(\mathbf{x})$  now involves propositional quantification, traditional ramification provides a different resolution of the paradox. This comes from Church's first comprehension schema (Chu76, p. 750): for every  $\mathbf{P}$  of type  $p$  we are guaranteed to have the following true, so long as  $\mathbf{x}$  (does not appear in  $\mathbf{P}$  and) is (i) of higher order than every bound variable in  $\mathbf{P}$  and (ii) of at least as high an order as every free variable and constant in  $\mathbf{P}$ .

$$\exists \mathbf{x}[\mathbf{x} \leftrightarrow \mathbf{P}] \quad (17)$$

This comprehension schema ensures that  $\llbracket \forall z[wz \rightarrow z] \rrbracket$  is of at least as high an order as  $\llbracket w \rrbracket$ . But then, since the order of  $\llbracket w \rrbracket$  is the sum of the orders of its inputs and outputs,  $\llbracket \forall z[wz \rightarrow z] \rrbracket$  cannot be an argument to  $\llbracket w \rrbracket$ . (Again, Church's lowest propositional order is 1.) It is thus wrong to even ask whether  $\llbracket w(\forall z[wz \rightarrow z]) \rrbracket$  is true or false.

Compressed ramification does away with restrictions on the arguments to functions, so it cannot provide this resolution. I think, however, that this is not a great shortcoming. In the general case, when  $\llbracket mw \rrbracket$  does not involve quantification, even uncompressed ramification must fall back on restrictions on quantification over propositional functions; compressed ramification merely extends that reliance to the original Appendix B paradox.

### 4.5.1 Sets, properties, pluralities, etc.

A greater shortcoming of compressed ramification is that it has no easy answer to paradoxes that arise if we extend the logic to cover, for example, sets, properties, or pluralities.

- (i) For every set  $\mathbf{x}$ , there seems to be the proposition  $\mathbf{m}(\mathbf{x})$  *that  $\mathbf{x}$  is my favorite set*. But what about the set  $w$  of all and only propositions of the form  $\mathbf{m}(\mathbf{x})$  such that  $\mathbf{m}(\mathbf{x}) \notin \mathbf{x}$ —do we have  $\mathbf{m}(w) \in w$ ?

- (ii) For every property  $x$ , there seems to be the proposition  $m(x)$  *that  $x$  is my favorite property*. But what about the property  $w$  of being a proposition of the form  $m(x)$  that does not have the property  $x$ —does  $m(w)$  have  $w$ ?
- (iii) For any propositions  $xx$ , there is the proposition  $m(xx)$  *that  $xx$  are my favorites*. But what about those propositions  $ww$  that are of the form  $m(xx)$  but not one of  $xx$ —is  $m(ww)$  one of  $ww$ ?<sup>31</sup>

None of these paradoxes can be constructed in the system as it stands, but we can imagine introducing, say, a type  $s$  of sets of propositions and a constant  $\in^{(p,(s,p))}$  in order to capture the paradox in (i).

It would be natural for traditional ramification to insist that each set be capped at some order or other: this parallels the requirement that propositional functions take input of only a certain order, and this it blocks the paradox. But compressed ramification does away with the requirement that such an insistence parallels—functions in a theory of compressed ramified types can take inputs of any order—and so it cannot block the paradox in the same manner. All compressed ramification allows is ordering functions by the orders of their outputs. The parallel of this is ordering a set  $x$  by the orders of propositions of the form  $[[\mathbf{P} \in x]]^{x/x}$ —the orders of propositions about membership in  $x$ . This strikes me as a truly bizarre method of assigning orders to sets.

Similar contortions are required in cases (ii) and (iii), and I think the story in the case of plural quantification is even less plausible than it is in the cases of sets and properties. These paradoxes, then, show that while it might make some sense to determine the order of a function by the orders of its outputs, this does not naturally extend to some of the purposes to which functions are often put. For some purposes, for instance, we can use functions in place of sets and a membership relation, but extending the orders from Section 4.2 to sets in order to block the paradox in (i) seems hopelessly ad hoc.

## 4.6 A new Ramseyan division?

One response here is to insist that none of the paradoxes in this section can even get off the ground. I observed at the beginning of this section that

---

<sup>31</sup>This paradox is the subject of (MR00).

(13) cannot be true in the models we have been working with, and certainly it is plausible for a theory of functions, sets, properties, or pluralities to prohibit having a unique proposition for every function, set, property, or plurality of propositions. Such a story about the Appendix B paradox and its ilk would lose the attractive unification of the paradoxes that Russell was initially striving for, but perhaps not in an objectionably arbitrary manner: the idea would be that we have good independent reasons, coming from our theories of functions, sets, etc., to rule these paradoxes out from the start.

Of course, such a response leaves something to be desired, even if we're happy to embrace a Ramseyan division in principle: we still need an explanation of where our intuitions have gone wrong. Why must there be two distinct functions of type  $\langle p, p \rangle$   $\mathbf{x}$  and  $\mathbf{y}$  such that the proposition *that  $\mathbf{x}$  is my favorite function* is identical to the proposition *that  $\mathbf{y}$  is my favorite function*? What functions are they? Why can't anyone even *believe* that one is my favorite without thereby believing that the other is my favorite? Taking this route amounts to buying a simple theory of sets at the cost of intuitive stories about propositional identity and differentiation.

## 5 Additional paradoxes

The ultimate idea of Section 4.6 is to use a theory of propositions to explain away the Appendix B paradox and its ilk. Without attempting to answer the questions at the end of that subsection, I think the following two paradoxes provide some reason to think that this approach is on the right track, at least if one is already committed to ramification of one form or another.

### The proposition Liar

This paradox is the main concern of (BE87) and (Gro94). I call it the *proposition* Liar, rather than the much more natural *propositional* Liar, because there are other ways to introduce propositions into the Liar paradox. It involves a proposition that is identical to its own negation—a proposition  $\mathbf{p}$  such that  $\mathbf{p} = \llbracket \neg \rrbracket(\mathbf{p})$ . While such a proposition might be unusual, I have not said anything to prohibit it.

## The proposition Yablo

Yablo’s paradox from (Yab93) involves an infinite sequence of sentences, each of which says that all the later sentences are false. As with the proposition Liar, I want to focus on a purely intensional form of Yablo’s paradox—a paradox that arises from assumptions about propositional identity. It is easier to formalize this if we add a type  $o$  of ordinals, insisting that  $\mathcal{D}_o$  be non-empty and have no greatest element, and a constant  $>^{\langle o, \langle o, p \rangle \rangle}$ , with accompanying restrictions on  $\mathcal{T}$  and  $\mathcal{F}$  to parallel those for propositions of the form  $\llbracket = \rrbracket(\mathbf{x})(\mathbf{y})$ . The paradox then comes from introducing a constant  $Y^{\langle o, p \rangle}$  and requiring that for every  $x \in \mathcal{D}_o$ ,  $\llbracket Y \rrbracket(\mathbf{x}) = \llbracket \forall y[y > x \rightarrow \neg Y y] \rrbracket^{x/x}$ .<sup>32</sup> That is, we have a problem so long as we have an infinite hierarchy of propositions  $Y(\mathbf{x})$ , each of which says *that for every  $\mathbf{y}$  greater than  $\mathbf{x}$ ,  $Y(\mathbf{y})$  is false*.

Here, in effect, I have snuck in propositional quantification by quantifying over the arguments to a propositional function. But ramification has nothing to say about such tricks.

If we were to begin with models in which the identities required for these two paradoxes held, our truth-value gaps would ensure that  $\mathbf{p}$  and all the  $\llbracket Y \rrbracket(\mathbf{x})$  always lack truth values, and thus they would never appear in an order. But that does not help ramification—if we are satisfied with truth-value gaps, we can stop after Section 3.5. These are both cases where I think a proponent of ramification is best off falling back on a theory of propositions in order to rule the paradoxical circumstances out from the beginning. Certainly Russell would not have been pleased with an inherently circular proposition like  $\mathbf{p}$ , and most contemporary theories of propositions, structured or otherwise, also prohibit it. The Yablo propositions are less straightforward, but  $\llbracket Y \rrbracket$  itself is still circular, and plausibly viciously so.<sup>33</sup>

I am not convinced that this is the right approach to take to these paradoxes, which is part of why I did not set out in this paper to argue that ramification is the One True Resolution. But I want to continue to be silent on the nature of propositions; my purpose in this section is merely to point out (i) that we cannot be so silent forever, if we want to embrace ramification, and (ii) that if we are already going to rely on our theory of propositions to

---

<sup>32</sup>If we follow the suggestion in 15, we must have a unique name  $\mathbf{n}$  for every element of  $\mathcal{D}_o$  to ensure that we have enough propositions.

<sup>33</sup>Not all circularity is vicious, however—see, e.g., (Ant00)—and I do not pretend to have a good test.

explain why these paradoxes really do start from inconsistent assumptions, even though paradoxical assumptions like (1)–(4) are consistent, then extending that response to the paradoxes in Section 4 (or at least Section 4.5.1) is perhaps not as outlandish as one might have thought.

## 6 Other resolutions

I think that we can be at least somewhat pleased with the orders constructed in Section 3.6.2. They are compressed, as I argued we ought to want, but continue to respect the symmetry between (7') and (8'). There is an important sense in which  $\llbracket(7')\rrbracket$  and  $\llbracket(8')\rrbracket$  (contingently) depend on each other, and this dependence is, I think, what the strong Kleene scheme is picking up on. In the asymmetrical cases, there is a similar dependence, but in only one direction; again, I think the strong Kleene scheme picks up on this. It should be no surprise, then, that using the truth-value gaps generated by that scheme to construct the orders gives us satisfactory results.

Despite its faults, the original ramified theory of types is one of very few extant resolutions of the paradoxes with which I have been concerned. This has led some authors to treat it as a last but necessary resort, to which we must retreat when we fully confront the contradictions that our intuitions about propositions and propositional attitudes lead to. I hope that I have at least provided a more palatable option, which allows us to retain more of those intuitions while necessitating fewer commitments about propositions and attitudes themselves.

Nevertheless, Section 4.4 highlights potential shortcomings of compressed ramification, and Section 5 highlights potential shortcomings of ramification of every sort. If one wishes to look elsewhere, the basic intensional logic and truth-value gaps I have developed, using the strong Kleene scheme to identify problematic propositions, of course provides (the formal beginnings of) a truth-value gap resolution, and can also be adapted to other gapless approaches to the paradoxes. Prior (Pri61), for instance, argues that these paradoxes show that propositional attitudes sometimes fail. He relies on a first-come, first-served principle to determine which attitudes are blocked, and this gets him into trouble,<sup>34</sup> but we can get more plausible results using

---

<sup>34</sup>In the case of the asymmetrical paradox from Section 2, for instance, he is committed to saying that as of 2007, nobody other than me can bear a propositional attitude towards anything.

the truth-value gaps instead of that principle.

We can likewise take a different approach to quantifier domain restriction. Imagine starting again with a maximal, non-ramified model with truth-value gaps. Above, we cut the domain of quantification down to those propositions that had truth values and then went from there. But another option is to restrict only the quantifiers in the propositions that lack truth values. This gives us just two domains of quantification: the gappy propositions quantify over only the propositions with truth values, while the propositions with truth values quantify over everything. We can then fill in the gaps unproblematically, giving us a resolution of the paradoxes that still uses restricted domains of quantification, but which differs from anything that compressed ramification provides.<sup>35</sup>

## References

- [AL09] Joseph Almog and Paolo Leonardi, editors. *The Philosophy of David Kaplan*. Oxford University Press, Oxford, 2009.
- [And80] C. Anthony Anderson. Some new axioms for the logic of sense and denotation: Alternative (0). *Noûs*, 14(2):217–234, 1980.
- [Ant00] G. Aldo Antonelli. Virtuous circles: From fixed points to revision rules. In Anil Gupta and André Chapuis, editors, *Circularity, Definition, and Truth*, pages 1–27. Indian Council of Philosophical Research, New Dehli, 2000.
- [BE87] Jon Barwise and John Etchemendy. *The Liar*. Oxford University Press, Oxford, 1987.
- [Bea82] George Bealer. *Quality and Concept*. Oxford University Press, Oxford, 1982.
- [Chu74] Alonzo Church. Russellian simple type theory. *Proceedings of the American Philosophical Association*, 47:21–33, 1974.

---

<sup>35</sup>I pursue both this and the revision of Prior’s resolution in (Tuc11).

- [Chu76] Alonzo Church. Comparison of Russell's resolution of the semantical antinomies with that of Tarski. *Journal of Symbolic Logic*, 41:747–760, 1976.
- [Chu93] Alonzo Church. A revised formulation of the logic of sense and denotation. Alternative (1). *Noûs*, 27:141–157, 1993.
- [Fit64] Frederic B. Fitch. Universal metalanguages for philosophy. *Review of Metaphysics*, 17(3):396–402, 1964.
- [GCB74] Dorothy L. Grover, Joseph L. Camp, Jr., and Nuel D. Belnap, Jr. A prosentential theory of truth. *Philosophical Studies*, 27:73–125, 1974.
- [Gla04] Michael Glanzberg. A context-hierarchical approach to truth and the liar paradox. *Journal of Philosophical Logic*, 33(1):27–88, 2004.
- [Gro94] Willem Groeneveld. Dynamic semantics and circular propositions. *Journal of Philosophical Logic*, 23(3):267–306, 1994.
- [Kap95] David Kaplan. A problem in possible-world semantics. In Walter Sinnott-Armstrong, Diana Raffman, and Nicholas Asher, editors, *Modality, Morality, and Belief: Essays in Honor of Ruth Barcan Marcus*, pages 41–52. Cambridge University Press, Cambridge, England, 1995.
- [Kne72] William Kneale. Propositions and truth in natural languages. *Mind, New Series*, 81(322):225–243, 1972.
- [Kri75] Saul Kripke. Outline of a theory of truth. *Journal of Philosophy*, 72:690–715, 1975.
- [Lin03a] Sten Lindström. Frege's paradise and the paradoxes. In Krister Segerberg and Risiek Sliwinski, editors, *A Philosophical Smorgasbord: Essays on Action, Truth, and Other Things in Honor of Frederick Stoutland*, volume 52 of *Uppsala Philosophical Studies*. Uppsala University, Uppsala, 2003.
- [Lin03b] Sten Lindström. Possible-worlds semantics and the liar: Reflections on a problem posed by Kaplan. In Artur Rojszxxzak, Jacek Cacro, and Gabriel Kurczewski, editors, *Philosophical Dimenions of Logic*

*and Science: Selected Contributed Papers from the 11th International Congress of Logic, Methodology, and Philosophy of Science*, volume 320 of *Synthese Library*, pages 297–314. Kluwer Academic Publishers, Dordrecht, 2003. Reprinted in (AL09, pp. 93–108).

- [MR00] Vann McGee and Agustín Rayo. A puzzle about *de rebus* belief. *Analysis*, 60(4):297–299, October 2000.
- [Par74] Charles Parsons. The liar paradox. *Journal of Philosophical Logic*, 3:381–412, 1974.
- [Pri61] Arthur N. Prior. On a family of paradoxes. *Notre Dame Journal of Formal Logic*, 2:16–32, 1961.
- [RU06] Agustín Rayo and Gabriel Uzquiano, editors. *Absolute Generality*. Oxford University Press, Oxford, 2006.
- [Rus03] Bertrand Russell. *The Principles of Mathematics*. Cambridge University Press, Cambridge, 1903.
- [Tho80] Richmond H. Thomason. A model theory for propositional attitudes. *Linguistics and Philosophy*, 4:47–70, 1980.
- [Tho88] Richmond H. Thomason. Motivating ramified type theory. In Gennaro Chierchia, Barbara Partee, and Raymond Turner, editors, *Properties, Types and Meaning, Vol. 1*, pages 47–62. Kluwer Academic Publishers, Dordrecht, 1988.
- [Tuc10] Dustin Tucker. Intensionality and paradoxes in Ramsey’s ‘The Foundations of Mathematics’. *Review of Symbolic Logic*, 3(1):1–25, 2010.
- [Tuc11] Dustin Tucker. *Propositions and Paradoxes*. Ph.D. dissertation, Department of Philosophy, University of Michigan, Ann Arbor, Michigan, 2011.
- [Yab93] Stephen Yablo. Paradox without self-reference. *Analysis*, 53:251–252, 1993.