Backlash, Bifurcation, and Buckling, and the *Mysterious* Origin of Hysteresis

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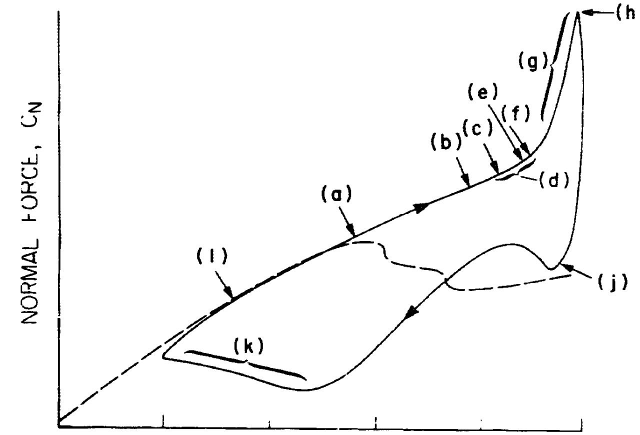
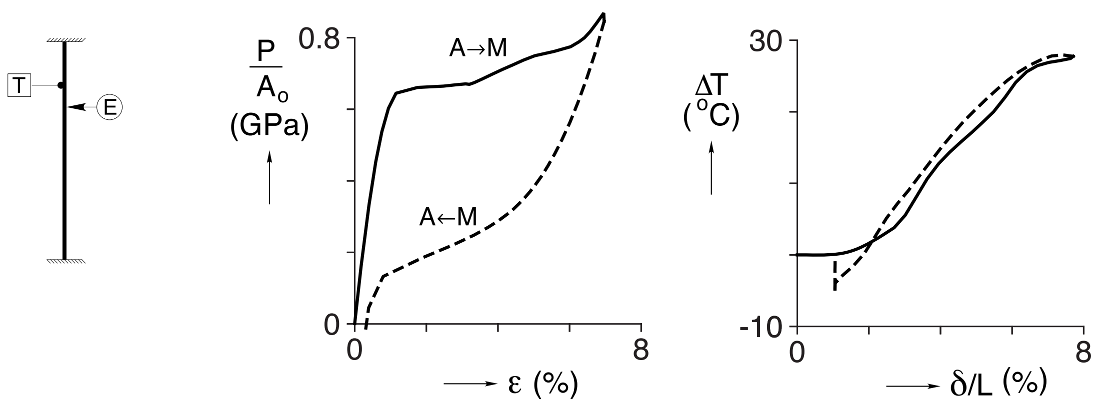
Hysteresis Is Everywhere!

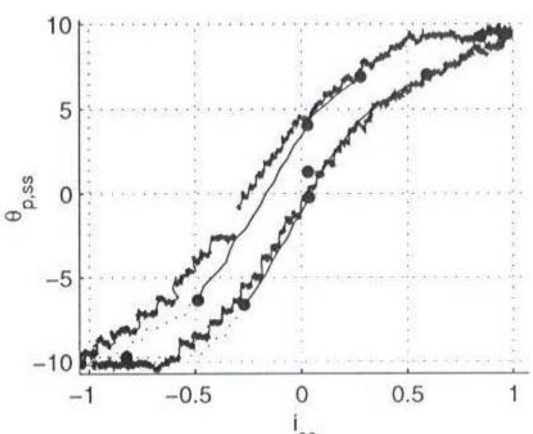
- **Structural Mechanics**
- **Ferromagnetics**
- **Smart Materials – PZTs, SMAs, electro- (or magneto-) rheological fluid**
- **Aerodynamics – dynamic stall**
- **Mechanics – backlash, friction**
- **Biological system**
- **Nonlinear systems with a continuum of semistable equilibria**



Hysteresis loops shown in the dynamic stall on NACA0012 airfoil
(image from Carr, et. al, 1977)



Nickel titanium stress/strain
(courtesy John Shaw)



Hysteresis loops shown in the motor/linkage experiment

What causes this butterfly hysteresis??

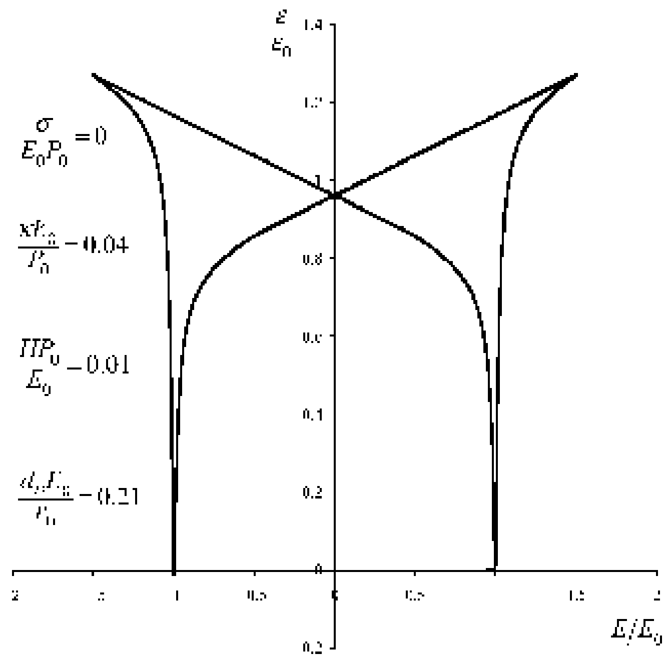
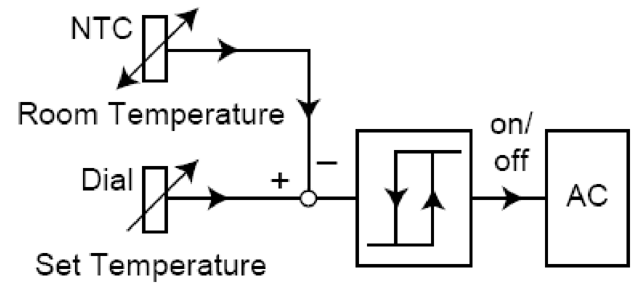


Fig. 2. Butterfly loop corresponding to the hysteresis loop in Fig. 1.

chosen such that $d_{33}E_0/\varepsilon_0 = 0.21$, selected to produce a shape comparable to experimental loops [1,30]. The origin is the starting point since the material is initially unpolarized. This also means that the material is not initially piezoelectric, as can be seen from Eq. (43). Thus no strain develops at first when the electric field is increased. However, polarization commences at $E = E_0$ and therefore the remanent strain grows quadratically with the polarization and piezoelectric strain appears and grows as well. The curve rises steeply at first because there is a rapid increase in polarization strain. However, lock up sets in and the slope of the curve diminishes. After the maximum field is reached and the then reduced, switching ceases and the response is at first purely linearly piezoelectric with the strain given by Eq. (43) with both ε^r and P^r non-zero and fixed. Therefore, the response to small electric fields at this stage is piezoelectric with a positive slope. As the field continues to be reduced, switching recommences at $E = -0.5E_0$ and since the remanent polarization is now diminishing, the remanent strain falls. Simultaneously, the piezoelectric effect is degraded and when the electric field reaches $-E_0$ and the remanent polarization is zero, the strain has reached zero as well. However, as the field is reduced below $-E_0$ a negative remanent polarization develops and the remanent and piezoelectric strains are rebuilt. Lock up now occurs as the electric field is brought towards $-1.5E_0$. After this value is reached, the electric field is

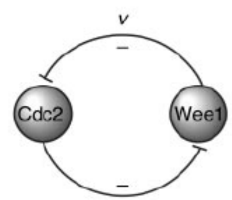
Positive and Negative Aspects

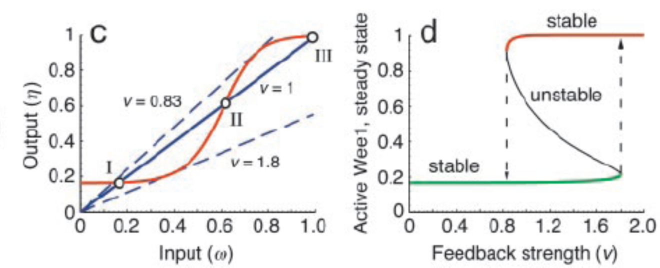
- Positive Effects
  - Used as controller element for relay systems (logic hysteresis)
  - Can model input frequency-independent energy dissipation of structural damping
  - Gives analysis tool for biosystems (cell signaling)



Logic hysteresis in thermostat
(image from von Altrock, 1996)

- Negative effects
  - Degrades precision motion control
  - May drive system to limit-cycle instability



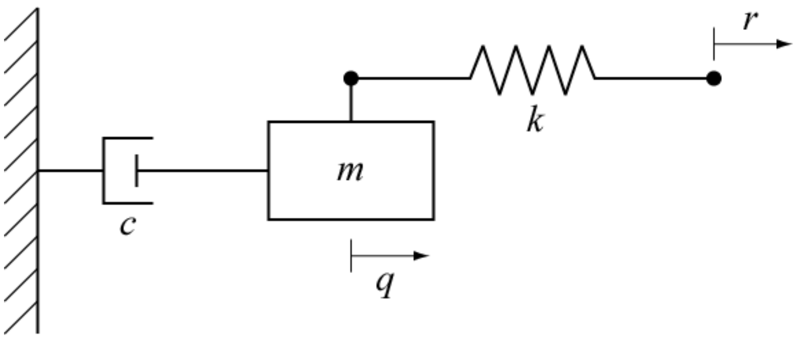


Cell signaling model and hysteresis analysis map
(image from Angeli, et. al, 2004)

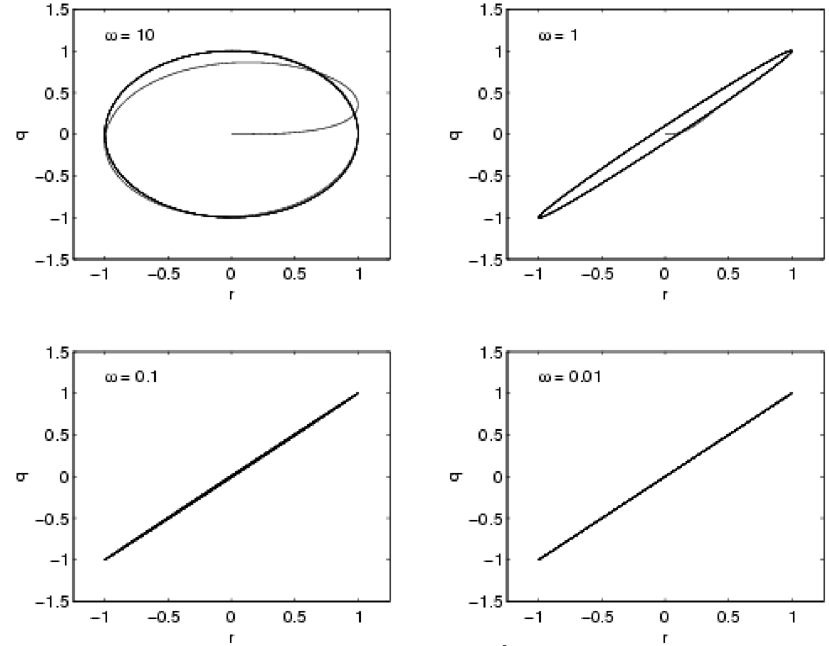
Some Basic Questions

- What is “hysteresis”?
  - Textbooks: “a system with memory” (vague)
  - “Peculiar property” (not helpful)
- What causes hysteresis?
  - Some hints in the literature (such as bifurcation)
- We need some examples…

Example 1: Linear System



$$m\ddot{q}(t) + \dot{c}(t) + k(q(t) - r(t)) = 0$$



Input-output maps w/ $r(t) = \sin \omega t$

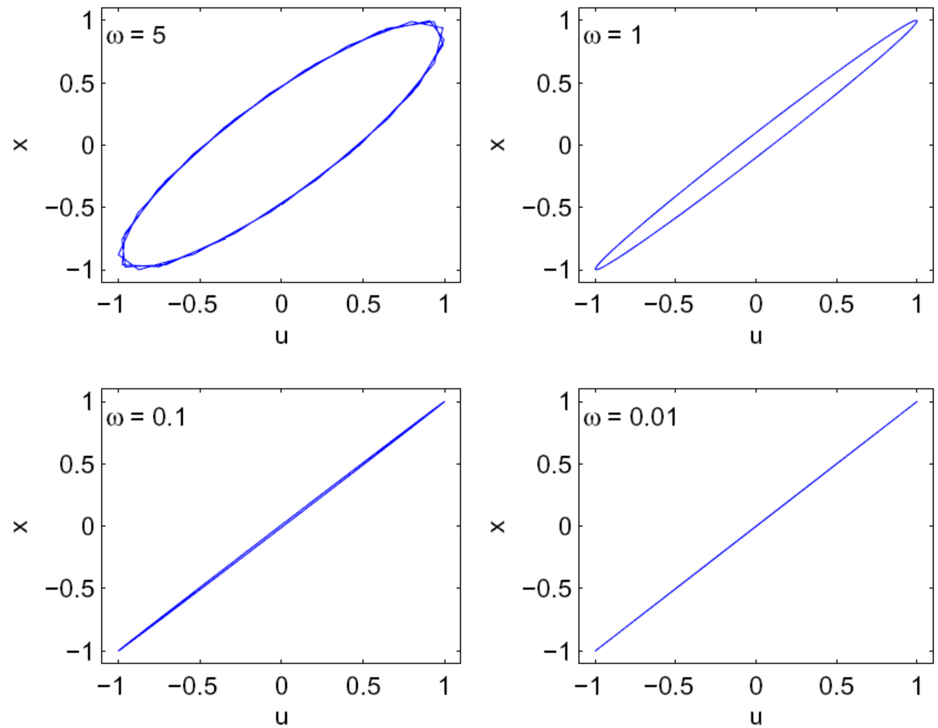
- A loop appears in the I/O map
  - The loop is indicative of **dynamics**—a Lissajous figure
- But the loop vanishes at low frequency

Example 2: Time Delay

- Hysteresis ($\upsilon\sigma\tau\epsilon\rho\acute{\epsilon}\sigma\upsilon\sigma$): “lag in arrival”
- Consider static delay

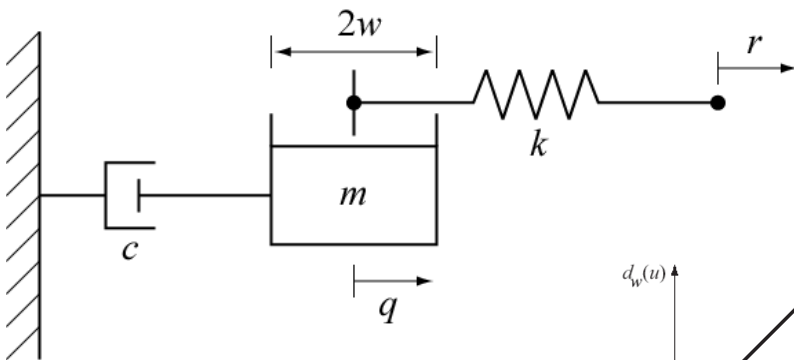
$$x(t) = u(t - \tau)$$

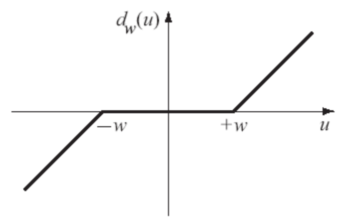
- The loop vanishes at low frequency



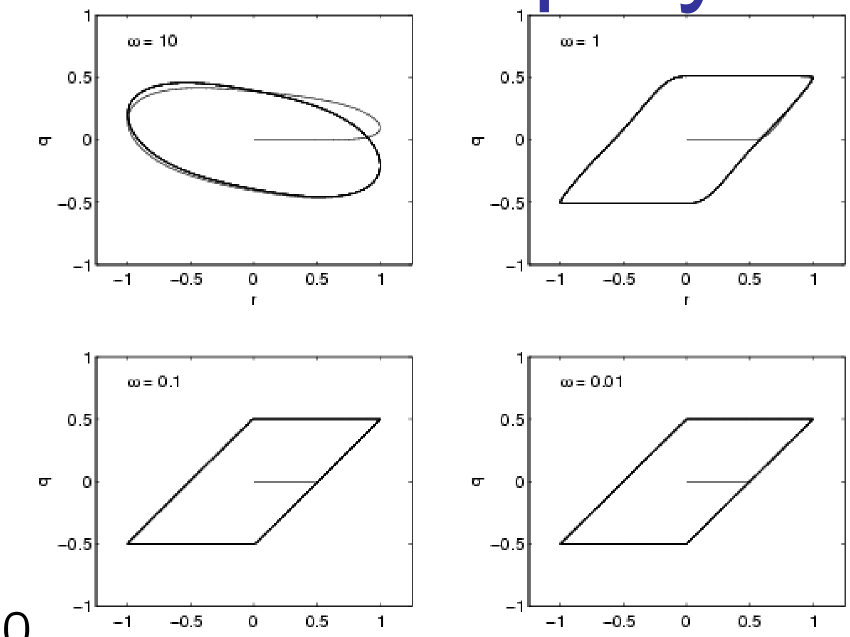
Input-output maps with $u(t) = \sin(\omega t)$

Example 3: Mechanical Freeplay





$$m\ddot{q}(t) + \dot{c}(t) + kd_{2w}\left(q(t) - r(t)\right) = 0$$



Input-output maps w/ $r(t) = \sin \omega t$

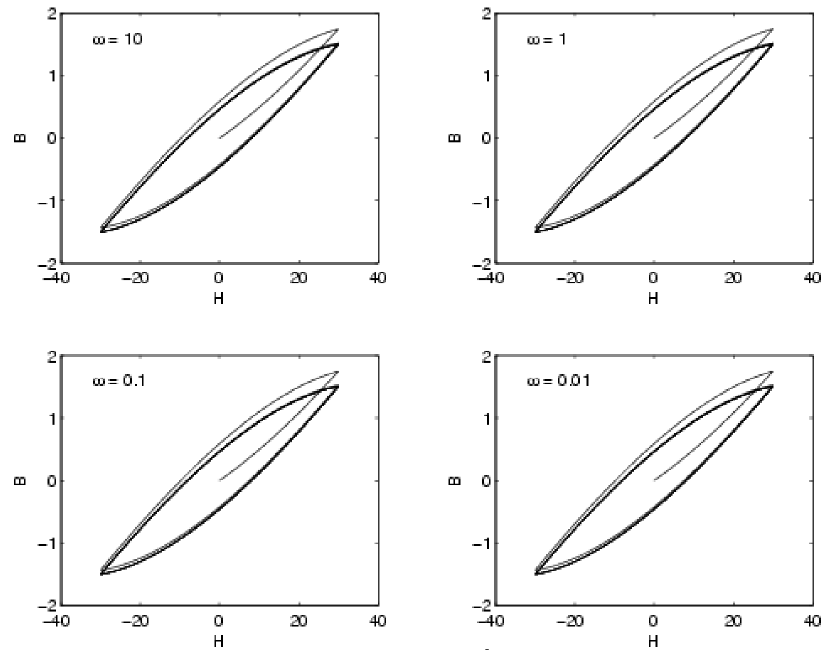
- A loop appears in the I/O map at all frequencies
- Nonvanishing I/O loop at low frequency
- The I/O loop depends on the input frequency
- Limiting loop is classical backlash

Example 4: Ferromagnetic Model

$$\dot{B}(t) = \alpha |\dot{H}(t)| \, [bH(t) - B(t)] + c\dot{H}(t)$$

where $B(t)$: magnetic flux

$H(t)$: magnetic field



Input-output maps w/ $H(t) = 30 \sin \omega t$

- A loop appears in the I/O map
- The I/O loop does NOT depend on input frequency
- Nonvanishing I/O loop at low frequency $\Rightarrow$ Hysteretic

So, What Might Hysteresis Be?

- In *all* linear dynamical systems:
  - Dynamics cause input-output loop when input frequency ω is nonzero
  - But I/O loop vanishes as $\omega \to 0$
- In *some* nonlinear systems:
  - I/O loop persists as $\omega \to 0$ $\Rightarrow$ quasi-DC I/O loop
- Hysteresis: Nonvanishing input-output loop at asymptotically low frequency
  $\Rightarrow$ Inherently nonlinear effect!

Intuitively:

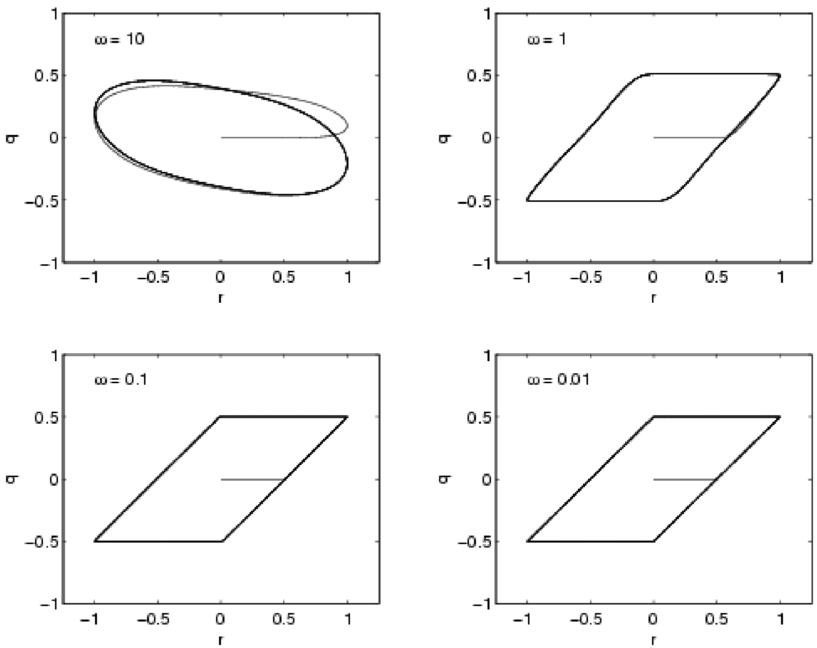
- Hysteresis is NOT dynamics
- Hysteresis is NOT statics
- Hysteresis is the *ghostly* image of the input-output map as the frequency of excitation goes to zero
  - The nontrivial "**static limit**" of the dynamic I/O map
  - Under periodic motion, which is assumed to exist

Why the Confusion?

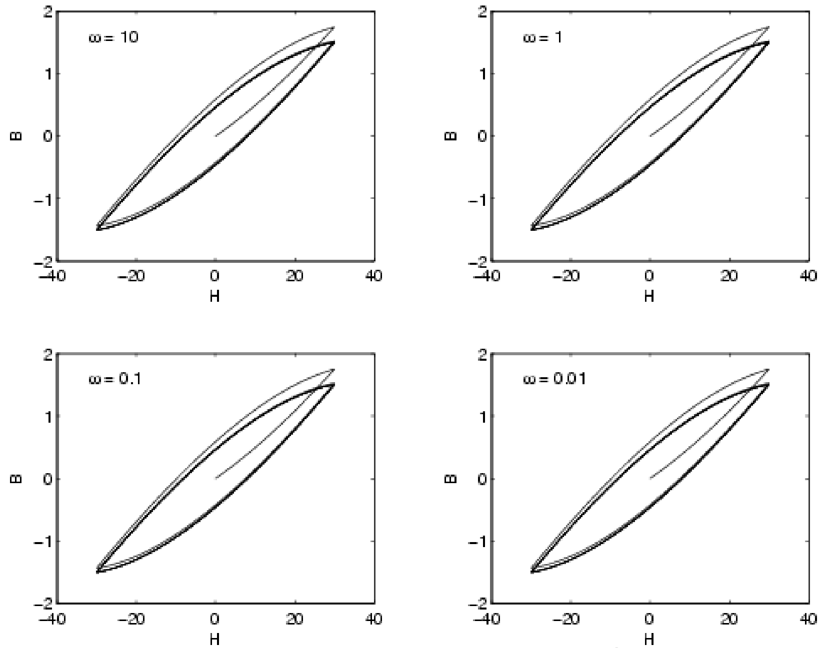
- If the input-output map is the same loop at every frequency, then the static limit is indistinguishable from the dynamics
  - The system has "**rate-independent hysteresis**"
  - Another misnomer

- Suppose the input-output map is *different* at every frequency and the static limit has a loop
  - The system has "**rate-dependent hysteresis**"
  - Another misnomer

Rate Dependence/Independence



Input-output maps of
mechanical freeplay



Input-output maps of
ferromagnetic material

Rate-Dependent Hysteresis

Rate-Independent Hysteresis

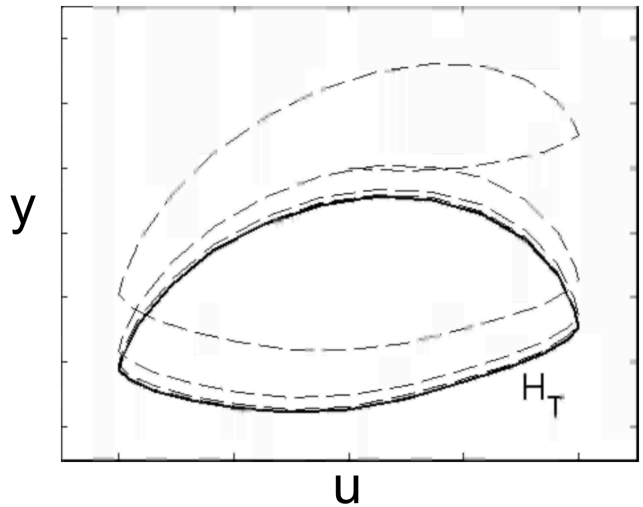
Precise Definition of Hysteresis

– $u(t)$: continuous, periodic with period α

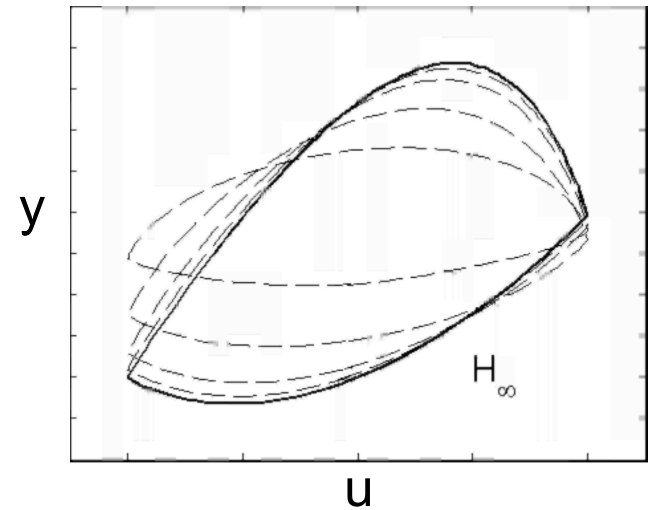
– $u_T(t) = u(\alpha t/T)$: periodic with period T. $y_T(t)$: output with $u_T(t)$

– $u_T(t)$, $y_T(t) \rightarrow \mathcal{H}_T(u)$ as $t \rightarrow \infty$, $\mathcal{H}_T(u)$: periodic I/O map

– $\mathcal{H}_T(u) \rightarrow \mathcal{H}_\infty(u)$ as $T \rightarrow \infty$, $\mathcal{H}_\infty(u)$: limiting periodic I/O map



Time Convergence to $\mathcal{H}_T(u)$



Quasi-DC Behavior $\mathcal{H}_T(u) \rightarrow \mathcal{H}_\infty(u)$

If $\mathcal{H}_\infty(u)$ has $(u, y_1), (u, y_2)$ such that $y_1 \neq y_2$, then $\mathcal{H}_\infty(u)$ is a HYSTERETIC MAP, and the model is HYSTERETIC

OK, What ***Causes*** Hysteresis?

- We will figure this out from some examples
- What kinds of models exhibit hysteresis?

Which Models Are Hysteretic ?

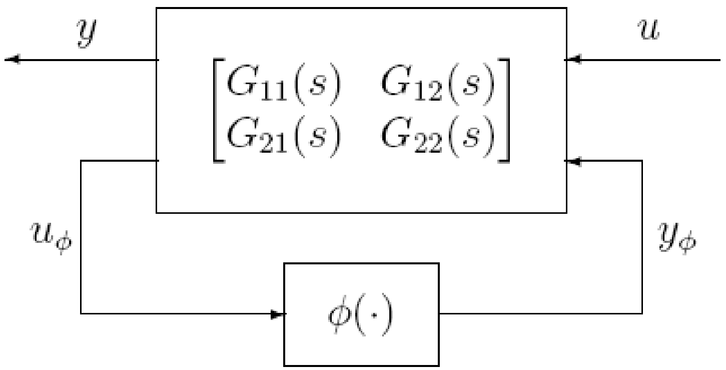
- No linear model is hysteretic
- Three specific nonlinear models

1. Nonlinear feedback models
   - Freeplay/backlash
2. Duhem models
   - Friction
3. Preisach models
   - Smart materials

Nonlinear Feedback Models

- $$\begin{aligned} \dot{x}(t) &= Ax(t) + D_1 u(t) + B y_\phi(t), \\ y(t) &= Cx(t) + D_2 u(t) + D y_\phi(t), \\ u_\phi(t) &= E_1 x(t) + E_0 u(t) + E_2 y_\phi(t), \\ y_\phi(t) &= \phi\big(u_\phi(t)\big) \end{aligned}$$

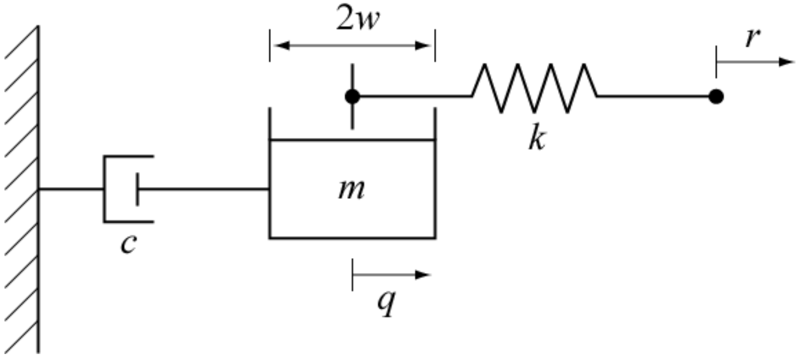
  where ϕ is memoryless nonlinearity
- The model is an LFT b/w MIMO system and ϕ

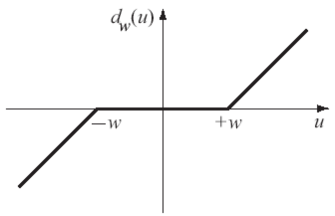


$$\begin{aligned} G_{11}(s) &\triangleq C(sI_n - A)^{-1}D_1 + D_2, \\ G_{12}(s) &\triangleq C(sI_n - A)^{-1}B + D \\ G_{21}(s) &\triangleq E_1(sI_n - A)^{-1}D_1 + E_0 \\ G_{22}(s) &\triangleq E_1(sI_n - A)^{-1}B + E_2 \end{aligned}$$

Nonlinear Feedback Freeplay Model

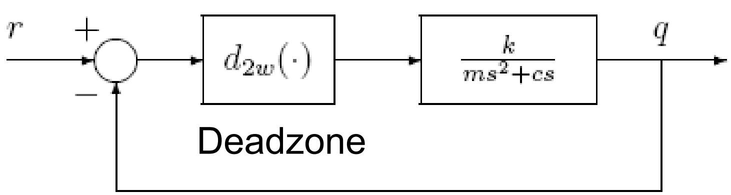
- A linear system with a memoryless nonlinearity in the feedback loop





$$\dot{x}(t) = Ax(t) + B\Big(u(t) + \phi(y(t))\Big),$$

- **Freeplay** becomes a **deadzone** in feedback



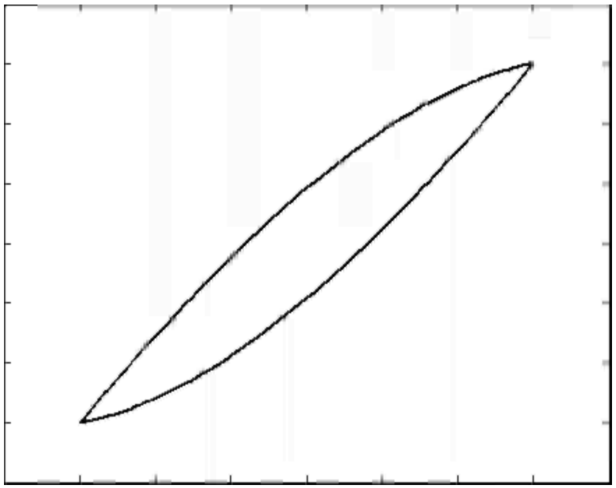
Block diagram of freeplay model

Generalized Duhem Model

- Based on input direction-dependent switching dynamical system where g(0) = 0

$$\dot{x}(t) = f(x(t), u(t))g(\dot{u}(t)), \qquad x(0) = x_0, \ t \geq 0$$
$$y(t) = h(x(t), u(t))$$

- The output changes its character when the input changes its direction
- Nonlinear ODE model
  $\Rightarrow$ Finite-Dimensional
- Can model rate-independent and rate-dependent hysteresis
- Special cases: Ferromagnetic model, Bouc-Wen model, Madelung model, Dahl friction model



Vector field analysis of the Duhem model of ferromagnetic material

Duhem-Based Friction Models

Friction Model		Duhem Type	Rate Dependence	Continuity
Coulomb		static	rate independent	discontinuous
Dahl	$\gamma = 0$	generalized	rate independent	discontinuous
	$0 < \gamma < 1$	generalized	rate independent	continuous but not Lipschitz
	$\gamma = 1$	semilinear	rate independent	Lipschitz
	$\gamma > 1$	generalized	rate independent	Lipschitz
LuGre		generalized	rate dependent	Lipschitz
Maxwell-slip		generalized	rate independent	discontinuous

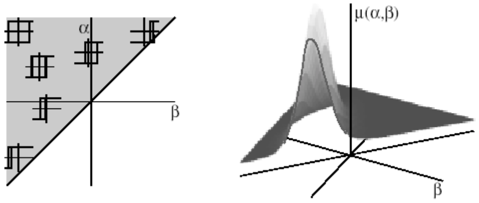
- Philip R. Dahl worked with The Aerospace Corp.
- LuGre = Lund and Grenoble
- Based on the slip model proposed by Maxwell for no-slip boundary conditions in fluid mechanics
- Non-Lipschitzian property is necessary for finite-time convergence (Sanjay Bhat’s research)

Superposition Models (Preisach Model)

- Based on rate-independent hysteretic kernels
  - Preisach model: logic hysteresis

$$y(t) = \iint_{\alpha \geq \beta} \mu(\alpha, \beta)\gamma_{\alpha\beta}[u(t)]\mathrm{d}\alpha\mathrm{d}\beta,$$

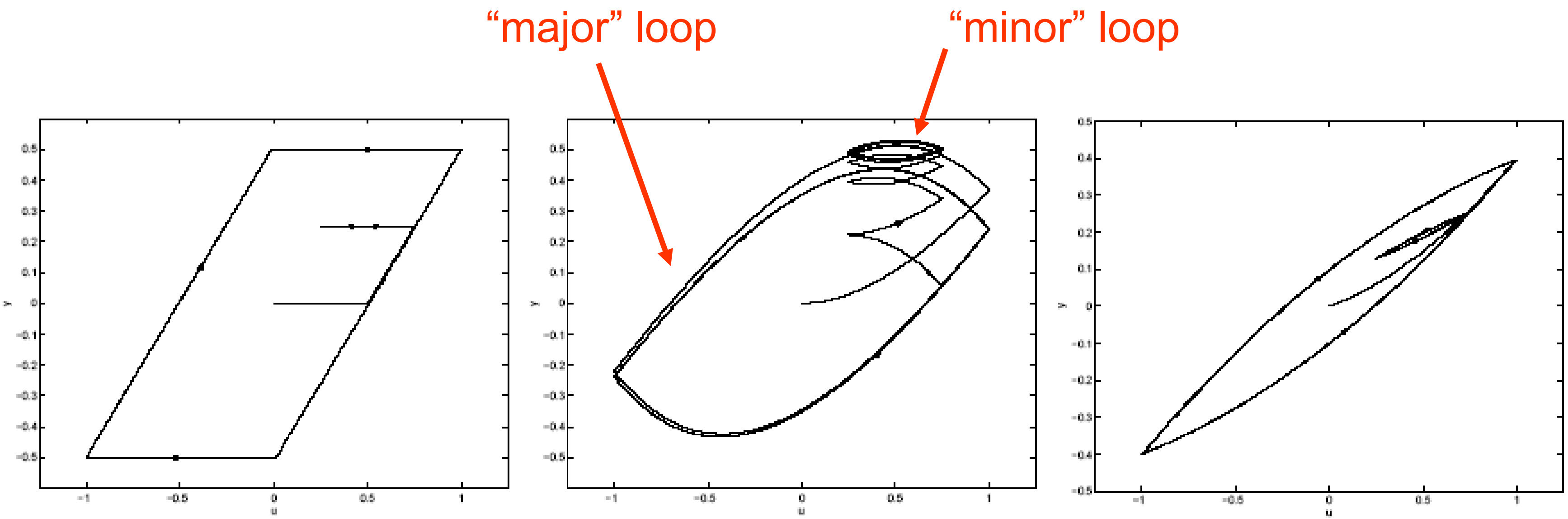
  - Kransnosel'skii-Pokorovskii model: linear stop operator (LSO)



Preisach Model and its weighting function
(image from Gorbet, 1997)

- Uses hysterons
- Can capture complicated reversal behaviors
- Usually rate-independent
- Usually infinite dimensional
- Models piezo materials

Reversal Behavior



(a) Nonlinear Feedback model

(b) Duhem model:

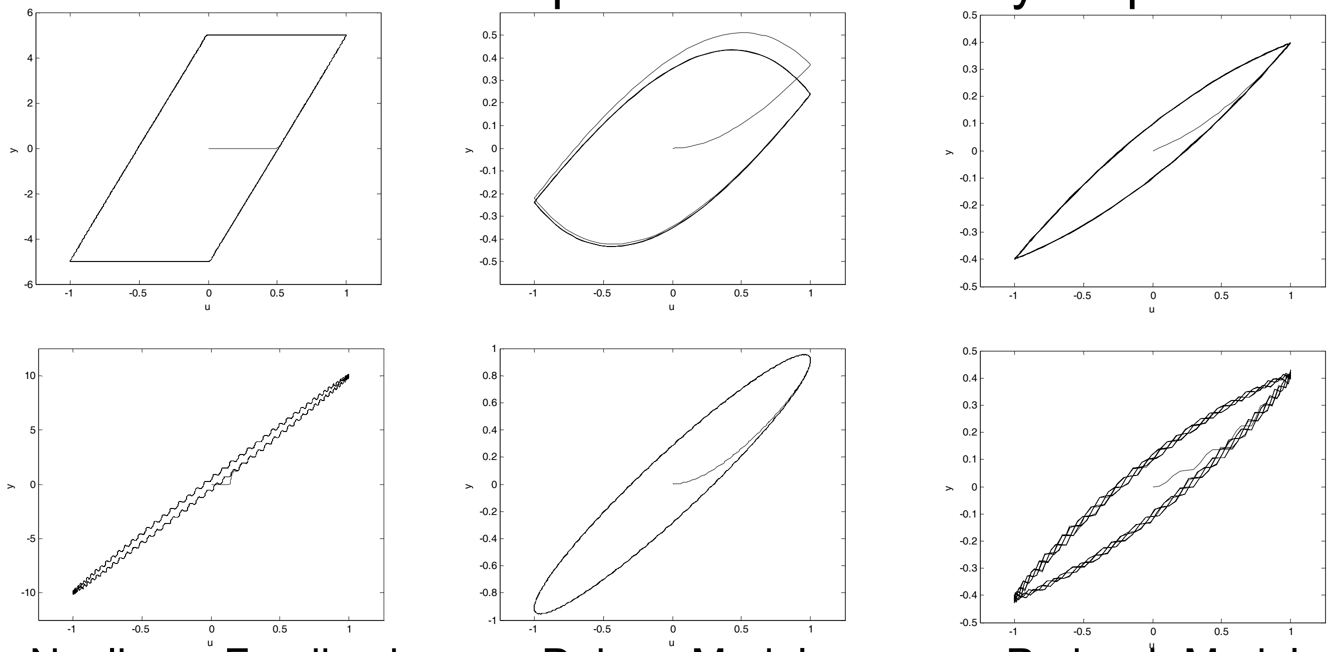
Local memory

(c) Preisach model:

Nonlocal memory

Response to Superimposed Dither

- Nonlinear feedback model: averaged to memoryless nonlinearity by *large* amplitude dither
  - Deadzone requires **large-amplitude** dither
- Duhem model: averaged to LTI system by high frequency dither
  - Dither amplitude can be **infinitesimal**
- Preisach model: does not respond to dither of any amplitude



Original Hysteresis

Hysteresis with dither

Nonlinear Feedback | Duhem Model | Preisach Model

Analysis of Hysteresis

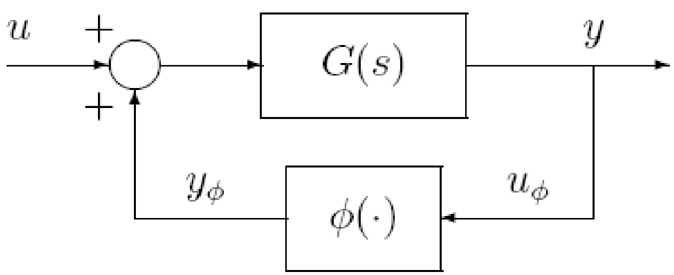
Nonlinear Feedback

Generalized Duhem

Nonlinear Feedback Model

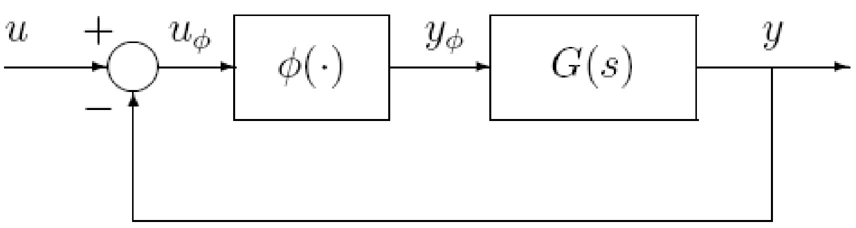
- Special Cases

$$\dot{x}(t) = Ax(t) + B\Big(u(t) + \phi(y(t))\Big),$$
$$y(t) = Cx(t)$$



$$\begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} = \begin{bmatrix} G(s) & G(s) \\ G(s) & G(s) \end{bmatrix}$$

$$\dot{x}(t) = Ax(t) + B\phi\Big(u(t) - y(t)\Big),$$
$$y(t) = Cx(t)$$



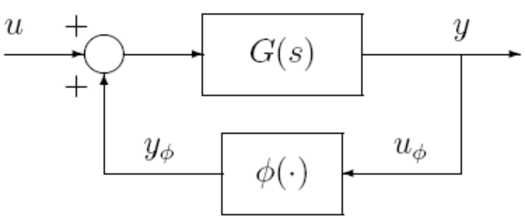
$$\begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} = \begin{bmatrix} 0 & G(s) \\ 1 & -G(s) \end{bmatrix}$$

Equilibria Map

- The equilibria map is the set $\mathcal{E}$ of points $(\bar{u}, C\bar{x}) \in \mathbb{R}^2$ where $\bar{x}$ is an equilibrium of the NF model with constant $\bar{u}$
- Let (A, B, C) be given in controllable canonical form

$$A = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} c_0 & c_1 & \cdots c_{n-1} \end{bmatrix}$$

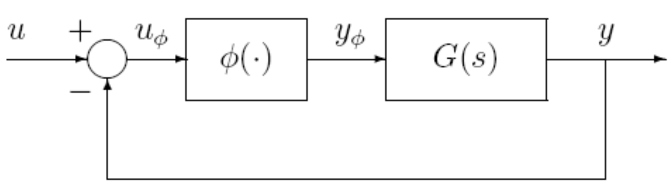
- Then $\mathcal{E}$ is given by



$$\mathcal{E} = \{(\bar{u}, c_0\bar{x}) \in \mathbb{R}^2 : a_0\bar{x} = \phi(\bar{u} - c_0\bar{x})\}$$

or

$$\mathcal{E} = \{(\bar{u}, c_0\bar{x}) \in \mathbb{R}^2 : a_0\bar{x} = \phi(c_0\bar{x}) + \bar{u}\}$$

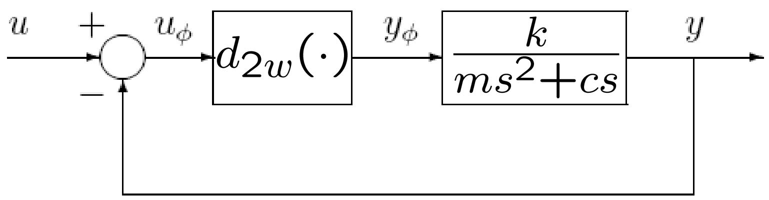


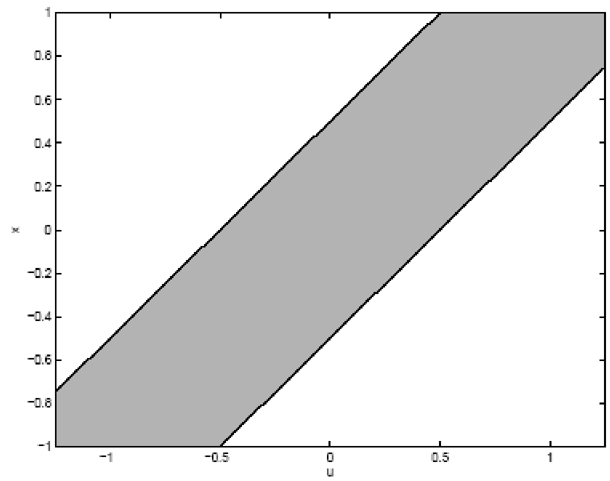
$\Rightarrow$ depends only by a_0 & c_0

Deadzone Equilibria Map

- Mechanical Freeplay Model

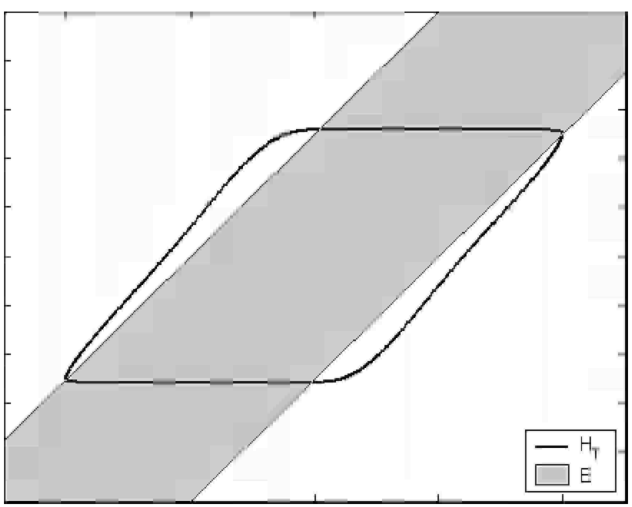
$$m\ddot{x}(t) + c\dot{x}(t) + kd_{2w}\Big(x(t) - u(t)\Big) = 0$$





Is Hysteresis Map Contained in $\mathcal{E}$?

- Hysteresis map $\mathcal{H}_\infty$ is the static limit of the input-output maps
- Suggests that every point in $\mathcal{H}_\infty$ is an equilibrium and thus in $\mathcal{E}$



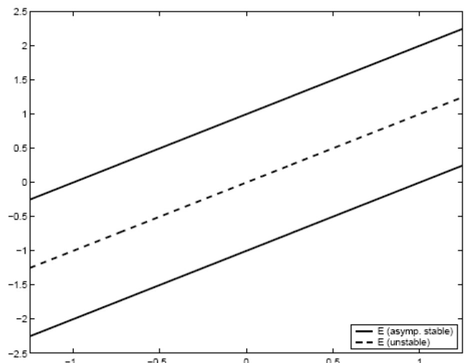
Equilibria map $\mathcal{E}$ and hysteretic map $\mathcal{H}_\infty$ for a mechanical backlash model

$$\mathcal{H}_T(u) \to \mathcal{H}_\infty(u) \subseteq \mathcal{E}$$

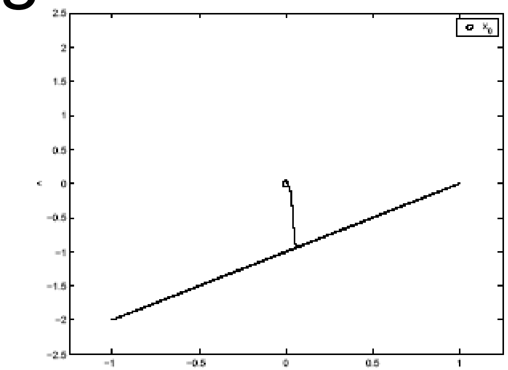
Not always!

No Hysteresis Example

- Hysteresis requires two distinct equilibria
- $(u, y_1), (u, y_2) \in \mathcal{H}_\infty \implies \mathcal{E}$ should be multi-valued map
- But not every multi-valued $\mathcal{E}$ generates hysteresis



$\mathcal{E}$ of $\dot{x}(t) = \big(u(t) - x(t)\big)^3 - \big(u(t) - x(t)\big)$



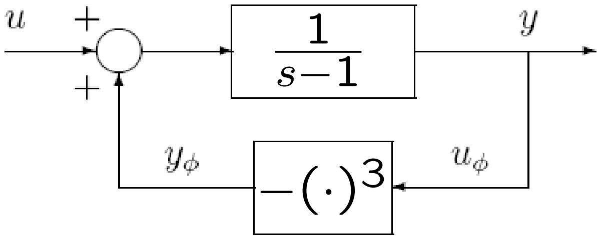
Non-hysteretic I/O map

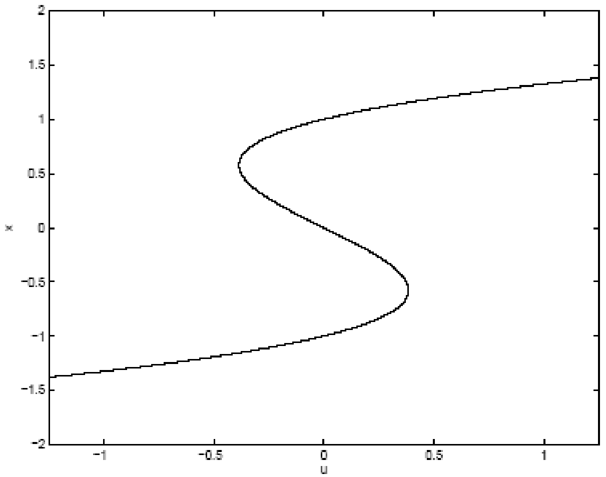
- Ingredients of hysteresis
  - Step convergence
  - Multi-valued limiting equilibria map
  - Bifurcation (vertical segments) or a continuum of equilibria

Cubic Equilibria Map Example

- Cubic Hysteresis Model

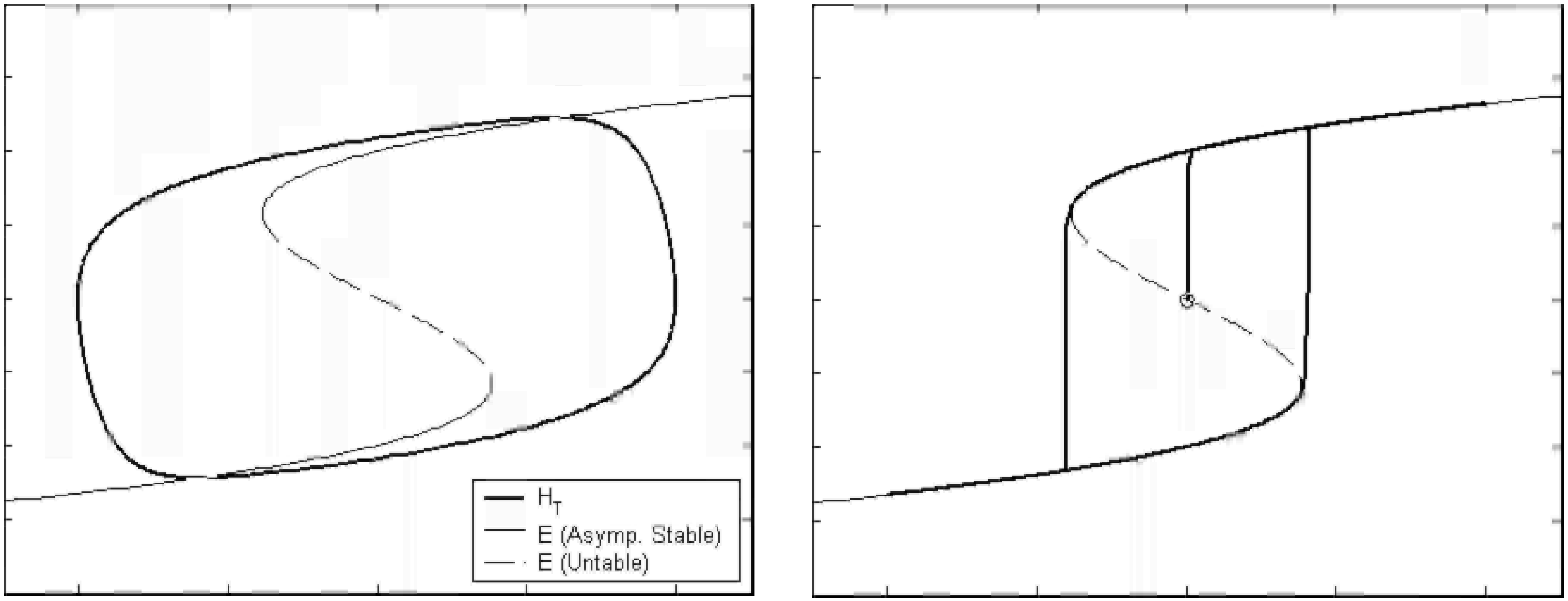
$$\dot{x}(t) = -x^3(t) + x(t) + u(t)$$





$\mathcal{E}$ and $\mathcal{H}_\infty$

- Generally $\mathcal{H}_\infty \not\subseteq \mathcal{E}$!!
- Cubic hysteresis – a bifurcation occurs

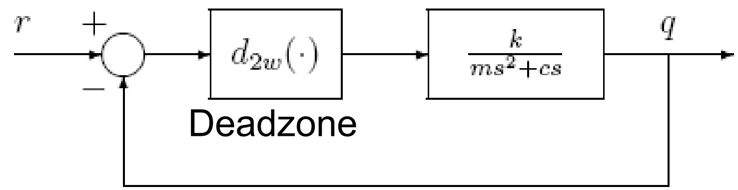


- Except for the limiting vertical transition trajectories, $\mathcal{H}_\infty \subseteq \mathcal{E}$

Bifurcation Video

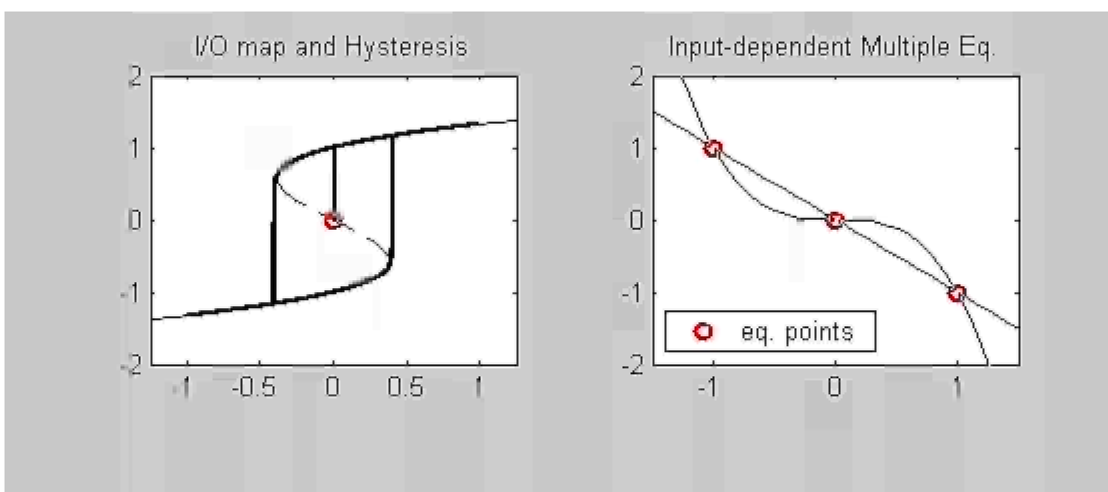
- A linear system with a memoryless nonlinearity in the feedback loop

$$\dot{x}(t) = Ax(t) + B\big(u(t) + \phi(y(t))\big),$$



Block diagram of freeplay model

- Nonlinearity introduces input-dependent multiple equilibria
- Input change causes bifurcation
  $\Rightarrow$ I/O trajectory “chases" the stable moving equilibria



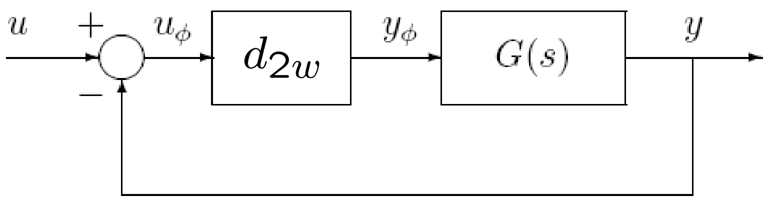
Principle of Multistability

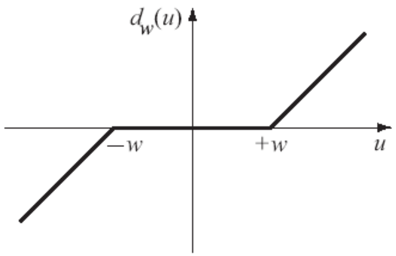
- The presence of hysteresis requires the existence of multiple, attracting equilibria
  - Finite set of attracting equilibria OR
  - Continuum of attracting equilibria
  - Existence of multiple equilibria is not sufficient

- The input-dependent structure of this set as well as the dynamics of the system determine the **presence** and **properties** of the hysteresis

Deadzone-Based Freeplay Hysteresis

$$\dot{x}(t) = Ax(t) + B d_{2w}\big(u(t) - y(t)\big),$$
$$y(t) = Cx(t)$$





- Let (A, B, C) be given by the controllable canon. form $\Rightarrow$ limiting equilibria map $\mathcal{E}$ is given by

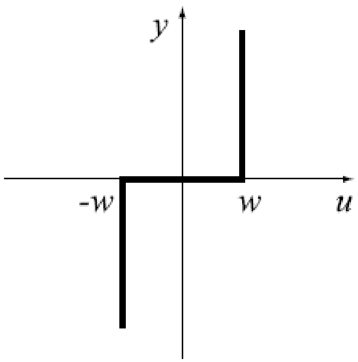
$$\mathcal{E} = \{(\bar{u}, \bar{x}) \in \mathbb{R}^2 : a_0\bar{x} + d_{2w}(c_0\bar{x} - \bar{u}) = 0\}$$

$\Rightarrow$ depends only on a_0 & c_0, where a_0 and c_0 are constant terms in numerator and denominator of $G(s)$, respectively.

$\mathcal{E}$ Set of Deadzone-Based Freeplay Hysteresis

- Can determine $\mathcal{E}$ from a_0 and c_0

• Case 1



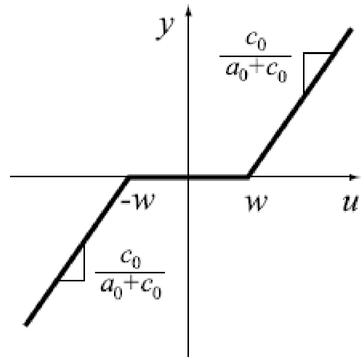
$a_0 \neq 0$ &

$a_0 + c_0 = 0$

$\mathcal{E}$ does not exist for all $u \in \mathbb{R}$

$\Rightarrow$ No Hysteresis

• Case 2



$a_0 \neq 0$ &

$a_0 c_0 \geq 0$,

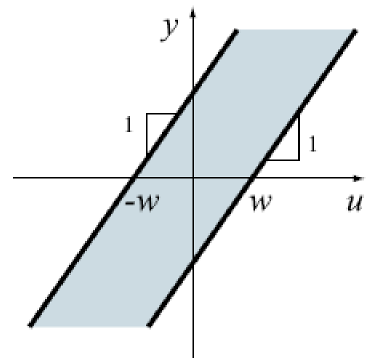
or $a_0 \neq 0$ &

$c_0(a_0 + c_0) < 0$

single valued map

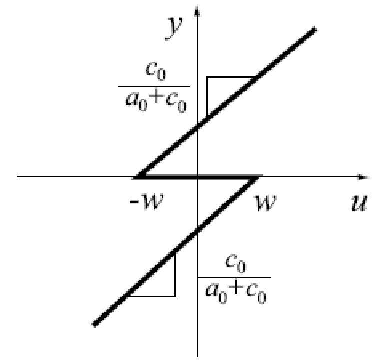
$\Rightarrow$ No Hysteresis

• Case 3



$a_0 = 0$

• Case 4



$a_0 \neq 0$ &

$c_0(a_0 + c_0) > 0$

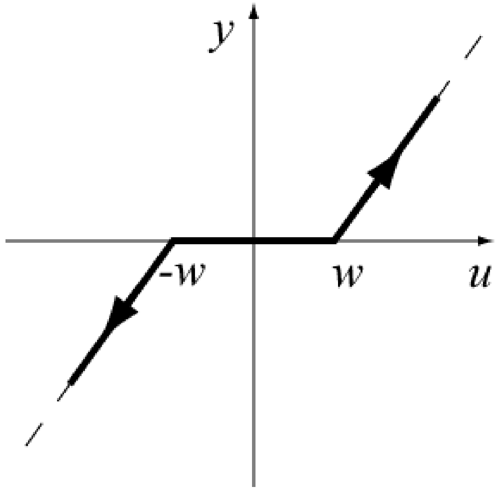
Multi-valued map

$\Rightarrow$ Hysteresis

$\mathcal{H}_\infty$ of DZ-Based Freeplay Hysteresis

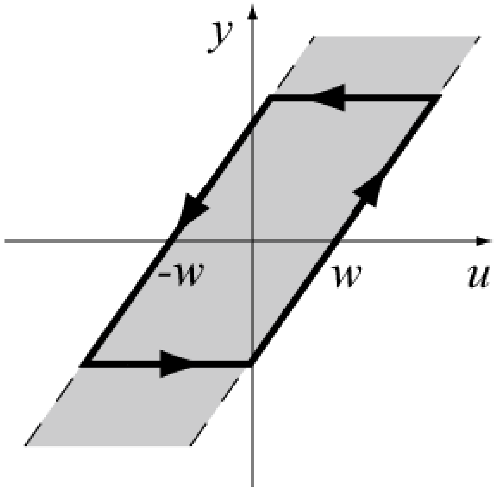
- Suppose DZ-BH is step convergent
- Limiting periodic I/O map $\mathcal{H}_\infty$ is given by

• Case 2



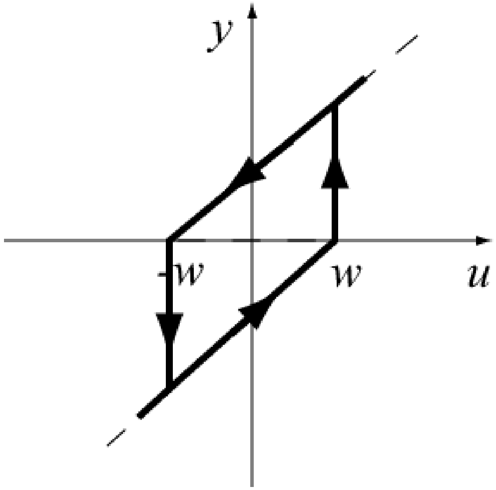
⇒ No Hysteresis

• Case 3



⇒ Backlash-type hysteresis

• Case 4

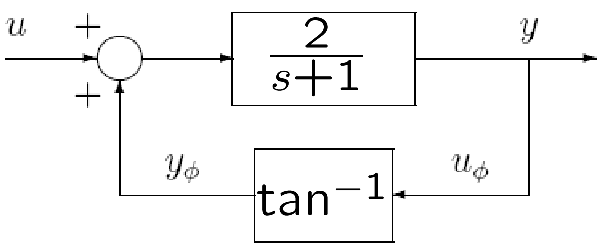


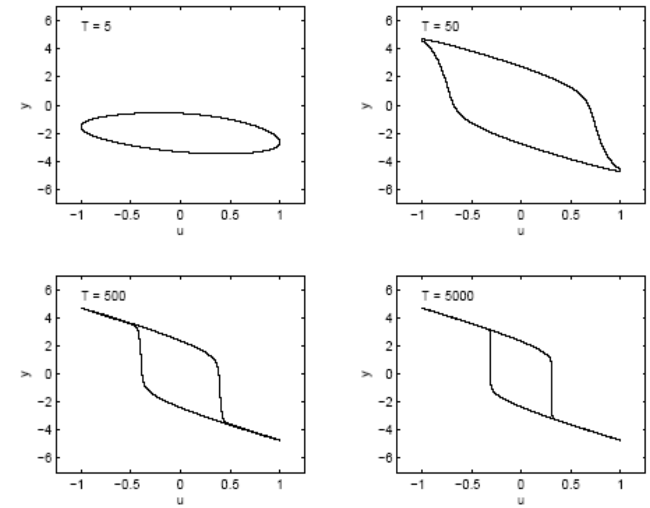
⇒ Bifurcation-type hysteresis

What Other Nonlinearities Can Cause Hysteresis?

- Arctangent model

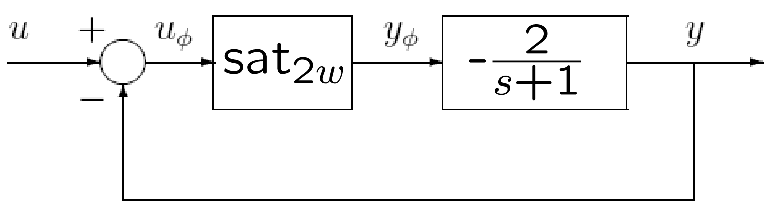
$$\dot{x}(t) = -x(t) + u(t) + \tan^{-1} 2x(t)$$

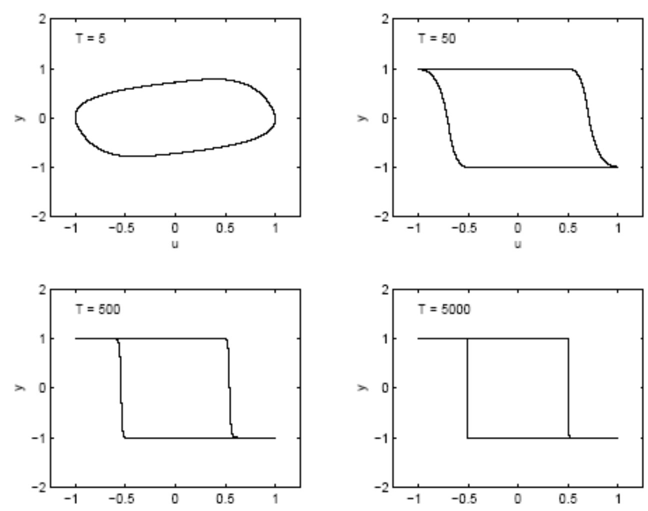




- Saturation model

$$\dot{x}(t) = -x(t) + \mathrm{sat}_w\big(u(t) + 2x(t)\big)$$





Saturation can cause hysteresis !

Generalized Duhem Model

- Nonlinear feedback models are rate dependent
- Generalized Duhem models can rate dependent or rate Independent
  - Rate and shape dependent
- Dither properties
- Friction modeling

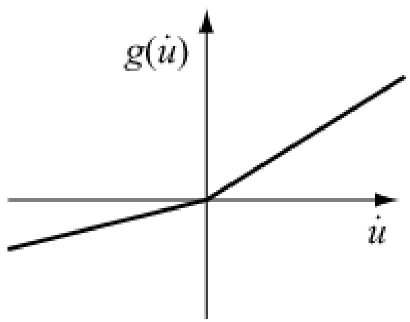
Generalized Duhem Model (GDM)

- GDM: $\dot{x}(t) = f(x(t), u(t))g(\dot{u}(t)), \quad x(0) = x_0,\ t \geq 0$
  $y(t) = h(x(t), u(t))$

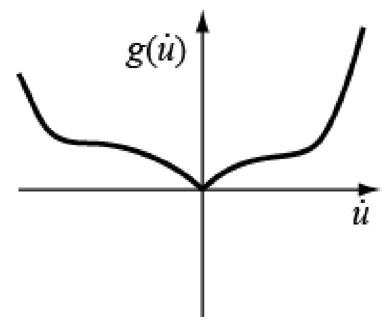
  where $x \in \mathbb{R}^n,\ u, y \in \mathbb{R},\ f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^{n \times r},\ g : \mathbb{R} \to \mathbb{R}^r$ and $g(0) = 0$
- Every constant *u* gives equilibria
- Rate dependent: Hysteresis map depends on the rate of the input
- Shape dependent: Hysteresis map depends on the shape of the input

When Is GDM Rate-Independent?

- FACT: GDM is rate independent if g is positively homogeneous, that is, $g(\alpha v) = \alpha g(v)$ for all $\alpha \geq 0$



positively homogeneous



not positively homogeneous

- FACT: if $\mathcal{H}_T$ exists for rate-independent GDM,

  $\mathcal{H}_T = \mathcal{H}_\infty$ for all T

$\Rightarrow$ Rate-Independent Hysteresis

u-parameterized Model

- Fact: g pos. homogeneous $\Rightarrow g(v) = \begin{cases} h_+ v, & v \geq 0 \\ h_- v, & v < 0 \end{cases}$
- Hence the RI GDM can be written as

$$\frac{\mathrm{d}x(t)}{\cancel{\mathrm{d}t}} = \begin{cases} f\left(x(t), u(t)\right) h_+ \frac{\mathrm{d}u(t)}{\cancel{\mathrm{d}t}}, & \dot{u}(t) \geq 0, \\ f\left(x(t), u(t)\right) h_- \frac{\mathrm{d}u(t)}{\cancel{\mathrm{d}t}}, & \dot{u}(t) < 0, \end{cases} \quad x(0) = x_0 \qquad (1)$$

$$y(t) = h(x(t), u(t)) \qquad (2)$$

- u -parameterized model

$$\frac{\mathrm{d}\hat{x}(u)}{\mathrm{d}u} = \begin{cases} f(\hat{x}(u), u) h_+, & \text{when } u \text{ increases,} \\ f(\hat{x}(u), u) h_-, & \text{when } u \text{ decreases,} \end{cases} \quad \hat{x}(u_0) = x_0$$

$$\hat{y}(u) = h(\hat{x}(u), u)$$

- Time-varying dynamical system with NON-MONOTONIC “time” u!

Rate-Independent Semilinear Duhem Model

$$
\begin{aligned}
\dot{x}(t) &= \begin{bmatrix} \dot{u}_+(t)I_n & \dot{u}_-(t)I_n \end{bmatrix} \left(\begin{bmatrix} A_+ \\ A_- \end{bmatrix} x(t) + \begin{bmatrix} B_+ \\ B_- \end{bmatrix} u(t) + \begin{bmatrix} E_+ \\ E_- \end{bmatrix} \right) \\
y(t) &= Cx(t) + Du(t), \qquad x(0) = x_0,\ t \geq 0
\end{aligned}
$$

where

$$
\dot{u}_+(t) \triangleq \max\{0, \dot{u}(t)\}, \quad \dot{u}_-(t) \triangleq \min\{0, \dot{u}(t)\}
$$

- u-parameterized

$$
\begin{aligned}
\frac{\mathrm{d}\hat{x}(u)}{\mathrm{d}u} &= \begin{cases} A_+\hat{x}(u) + B_+u + E_+, & u \uparrow \\ A_-\hat{x}(u) + B_-u + E_-, & u \downarrow \end{cases} \\
\hat{y}(u) &= C\hat{x}(u) + Du,\ \hat{x}(u_0) = x_0
\end{aligned}
$$

- "ramp+step" response for "time" u

When Does I/O Map Converge?

- Let $u(t) \in [u_{\min}, u_{\max}]$ be periodic
- If $\rho\left(e^{(u_{\max}-u_{\min})A_+} e^{-(u_{\max}-u_{\min})A_-}\right) < 1$

then (u, y) converges to

$$\mathcal{H}_\infty = \left\{\left(u, y_+(u)\right) : u \in [u_{\min}, u_{\max}]\right\} \cup \left\{\left(u, y_-(u)\right) : u \in [u_{\min}, u_{\max}]\right\}$$

$$\begin{aligned}
\hat{y}_+(u) &= Ce^{A_+(u-u_{\min})}\hat{x}_+ - CA_+^{\mathsf{D}}\left[uI - u_{\min}e^{A_+(u-u_{\min})}\right]B_+ \\
&\quad -CA_+^{2\mathsf{D}}\left[I - e^{A_+(u-u_{\min})}\right]B_+ - CA_+^{\mathsf{D}}\left(I - e^{A_+(u-u_{\min})}\right)E_+ \\
&\quad +C\mathcal{X}_+(u, u_{\min}) + C\mathcal{Y}_+(u - u_{\min}) + Du, \\
\hat{y}_-(u) &= Ce^{A_-(u-u_{\max})}\hat{x}_- - CA_-^{\mathsf{D}}\left[uI - u_{\max}e^{A_-(u-u_{\max})}\right]B_- \\
&\quad -CA_-^{2\mathsf{D}}\left[I - e^{A_-(u-u_{\max})}\right]B_- - CA_-^{\mathsf{D}}\left(I - e^{A_-(u-u_{\max})}\right)E_- \\
&\quad +C\mathcal{X}_-(u, u_{\max}) + C\mathcal{Y}_-(u - u_{\max}) + Du
\end{aligned}$$

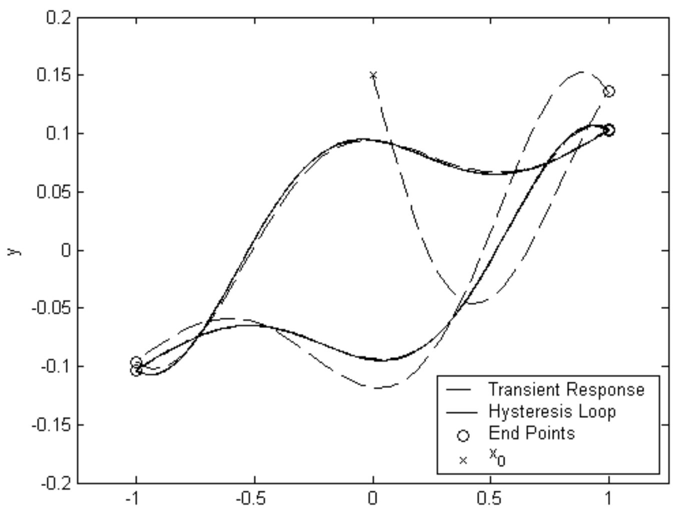
A^{D} = Drazin generalized inverse

Convergence of Rate-Independent SDM

$$\dot{x}(t) = \begin{bmatrix} \dot{u}_+(t)I_n & \dot{u}_-(t)I_n \end{bmatrix} \left(\begin{bmatrix} h_+A \\ h_-A \end{bmatrix} x(t) + \begin{bmatrix} h_+B \\ h_-B \end{bmatrix} u(t) + \begin{bmatrix} h_+E \\ h_-E \end{bmatrix} \right)$$

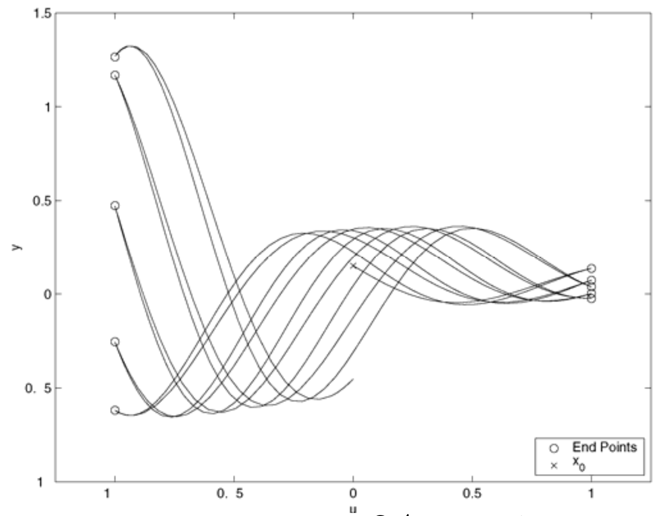
$$y(t) = Cx(t), \quad x(0) = x_0, \ t \geq 0$$

with $A = \begin{bmatrix} -1 & 4 \\ -4 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}$



$h_+ = 1, h_- = -1, \ \rho(e^{2A_+}e^{-2A_-}) = 0.018$

Converges



$h_+ = 1, h_- = 1.1, \ \rho(e^{2A_+}e^{-2A_-}) = 1.221$

Does not converge

Rate-Dependent Semilinear Duhem Model

- $\dot{x}(t) = \left[\begin{array}{cc} g_+\big(\dot{u}(t)\big)I_n & g_-\big(\dot{u}(t)\big)I_n \end{array}\right] \left(\left[\begin{array}{c} A_+ \\ A_- \end{array}\right] x(t) + \left[\begin{array}{c} B_+ \\ B_- \end{array}\right] u(t) + \left[\begin{array}{c} E_+ \\ E_- \end{array}\right] \right)$

  $y(t) = Cx(t) + Du(t), \qquad x(0) = x_0,\ t \geq 0$

  where $g_+(\dot{u}) \triangleq \max\{0, g(\dot{u})\}, \quad g_-(\dot{u}) \triangleq \min\{0, g(\dot{u})\}$,

  and g is NON-positively homogeneous and $g(0) = 0$

- FACT: If $\rho\left(e^{(u_{\max}-u_{\min})g'_+(0)A_+} e^{-(u_{\max}-u_{\min})g'_-(0)A_-}\right) < 1$

  then rate-dependent SDM I/O map converges to a closed curve $\mathcal{H}_\infty$ as $T \to \infty$

  $\Rightarrow$ Consequence of rate-independent SDM

Friction

- How to model it?
  - Duhem models

Friction Model		Duhem Type	Rate Dependence	Continuity
Coulomb		static	rate independent	discontinuous
Dahl	$\gamma = 0$	generalized	rate independent	discontinuous
	$0 < \gamma < 1$	generalized	rate independent	continuous but not Lipschitz
	$\gamma = 1$	semilinear	rate independent	Lipschitz
	$\gamma > 1$	generalized	rate independent	Lipschitz
LuGre		generalized	rate dependent	Lipschitz
Maxwell-slip		generalized	rate independent	discontinuous

- Why is it hysteretic??
  - Mathematically
  - Physically

Dahl Friction Model

- Nonlinear Model

$$\dot{F}(t) = \sigma \left|1 - \frac{F(t)}{F_C}\text{sgn}\,\dot{u}(t)\right|^{\gamma} \text{sgn}\left(1 - \frac{F(t)}{F_C}\text{sgn}\,\dot{u}(t)\right)\dot{u}(t)$$

F = Friction force

u = relative displacement

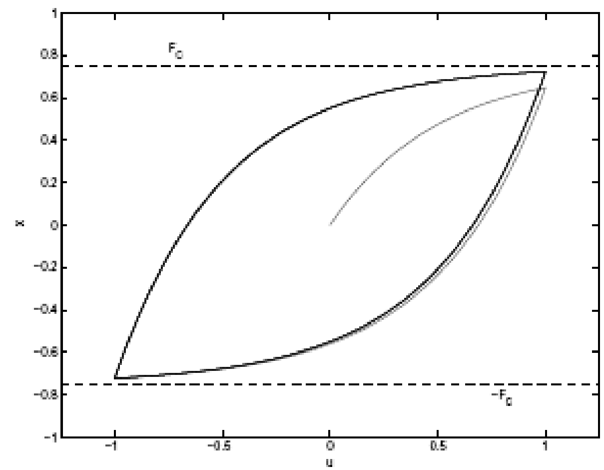
- Rewrite as

$$\dot{F}(t) = \sigma \left[\mathcal{F}_+(F(t)) \quad \mathcal{F}_-(F(t))\right] \begin{bmatrix} \dot{u}_+(t) \\ \dot{u}_-(t) \end{bmatrix},$$

where

$$\mathcal{F}_+(F(t)) \triangleq \sigma \left|1 - \frac{F(t)}{F_c}\right|^{\gamma} \text{sgn}\left(1 - \frac{F(t)}{F_c}\right),$$

$$\mathcal{F}_-(F(t)) \triangleq \sigma \left|1 + \frac{F(t)}{F_c}\right|^{\gamma} \text{sgn}\left(1 + \frac{F(t)}{F_c}\right),$$

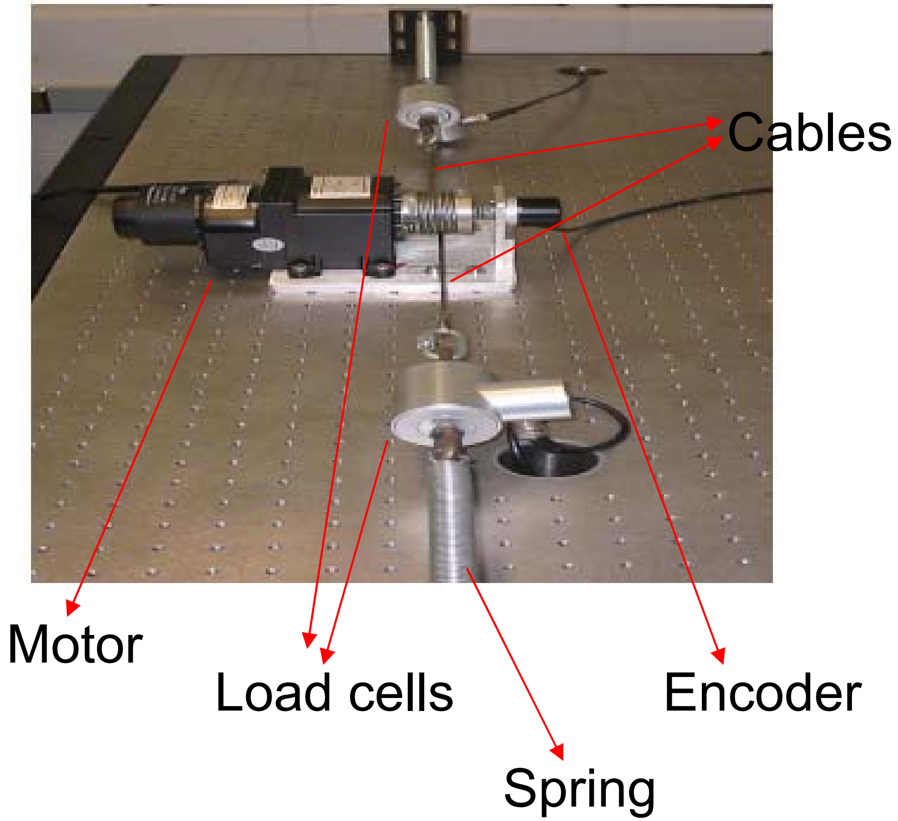


Hysteresis loop between Friction force and relative displacement

$\Rightarrow$ Rate-Independent Generalized Duhem model

DC Motor Experimental Setup

- 5 Outputs
  - 2 Load cells
  - Tachometer built in
  - Encoder for angular deflection
  - Current supplied to the motor
- 1 Input : Current supplied through a Quanser current amplifier
- Connected to a digital computer through a dSpace system



Dynamics of the DC motor Setup

- Dynamics of the setup given by

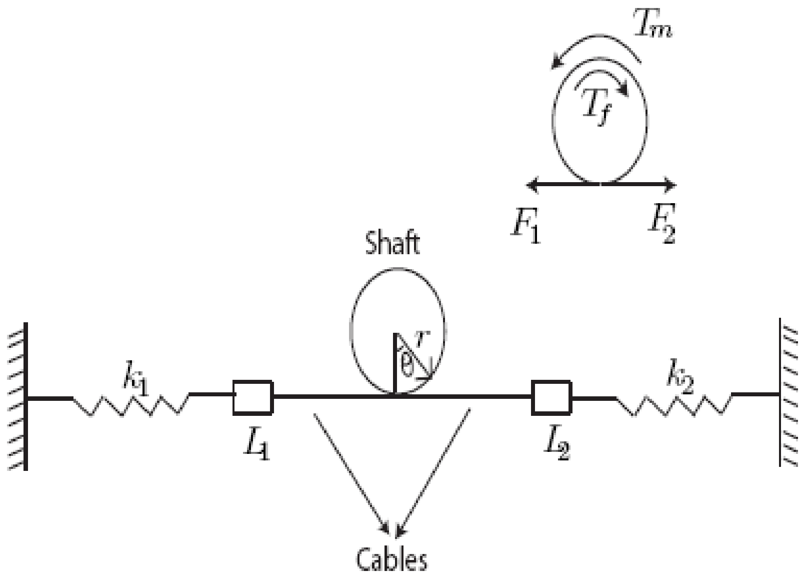
$$I\ddot{\theta} = T_m - T_f + F_2 r - F_1 r$$

- The spring forces

$$F_1 = f_1 + k_1\delta_1, \quad F_2 = f_2 + k_2\delta_2;$$

$$\delta_1 = r\theta, \quad \delta_2 = -r\theta$$

- Motor torque proportional to current $\Rightarrow$ $T_m = k_m i_m$



Hence,

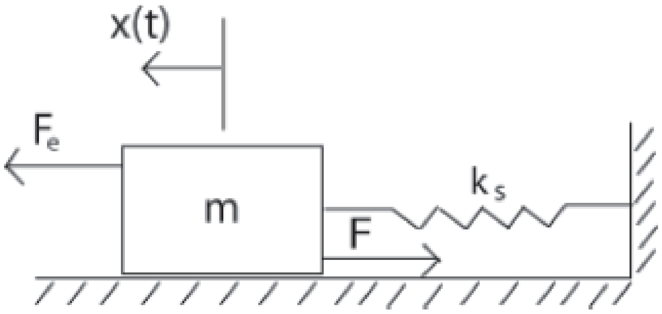
$$\ddot{\theta} + \frac{(k_1 + k_2)r^2}{I}\theta = \frac{k_m}{I} i_m - \frac{1}{I} T_f - \frac{(f_1 - f_2)r}{I}$$

Similarity to a Mass-Spring System

- The dynamics of the experiment are same as that of the mass-spring system shown below

$$\ddot{x}(t) + \frac{k_s}{m} x(t) = \frac{1}{m} F_e(t) - \frac{1}{m} F(x(t), \dot{x}(t))$$

$$\ddot{\theta}(t) + \frac{(k_1 + k_2) r^2}{I} \theta(t) = \frac{k_m}{I} i_m(t) - \frac{1}{I} T_f(\theta(t), \dot{\theta}(t))$$



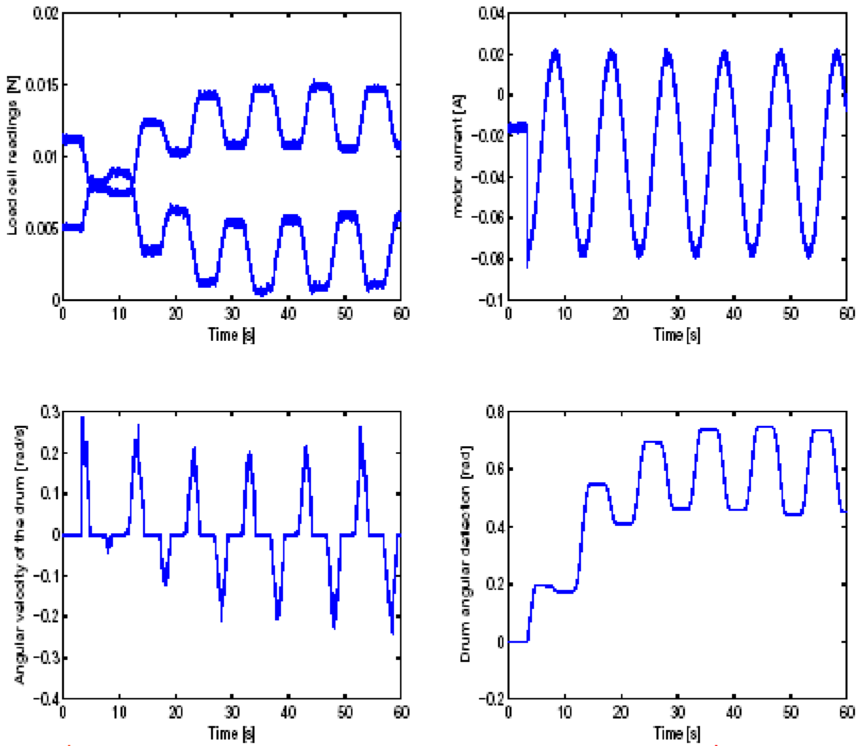
- Simulate the dynamics using different friction models for T_f and compare with the experimental results

Experimental Results

- Sinusoidal current fed to the motor

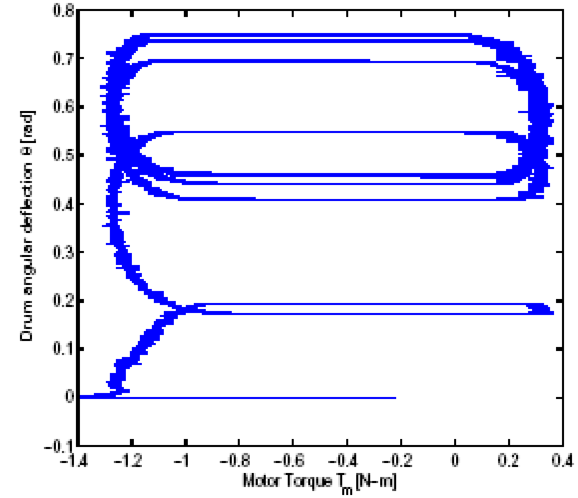
Input current

Load-cell readings



Angular velocity

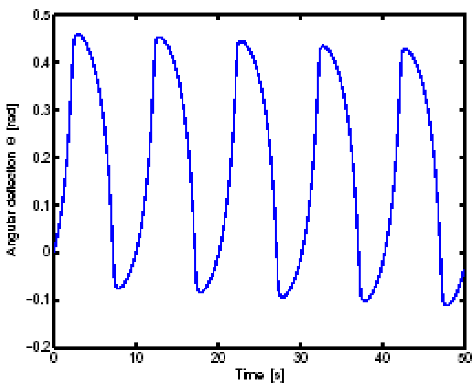
Angular deflection θ



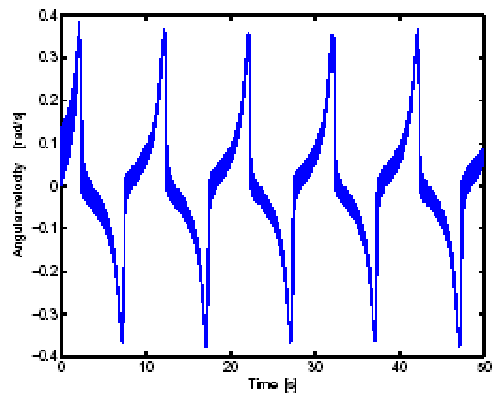
Hysteresis map from motor torque to angular deflection θ

Simulation with the Dahl Model

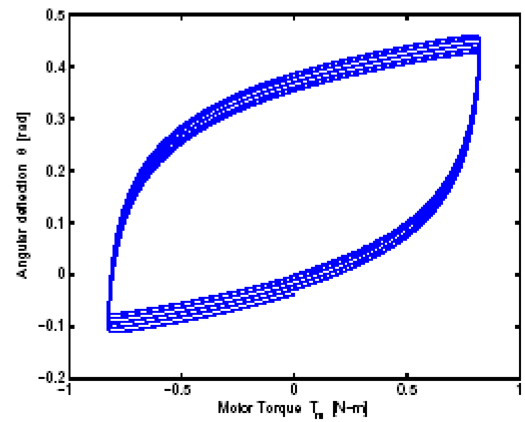
- Low frequency sinusoidal input current



Angular deflection θ



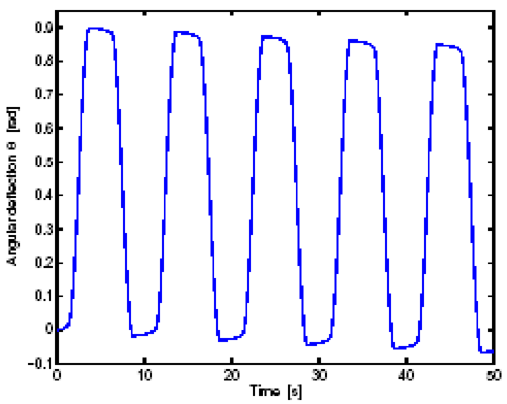
Angular velocity

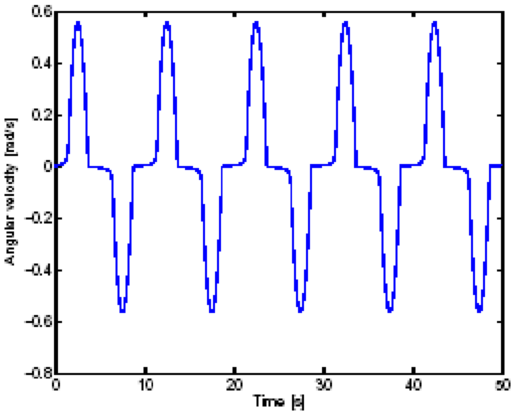


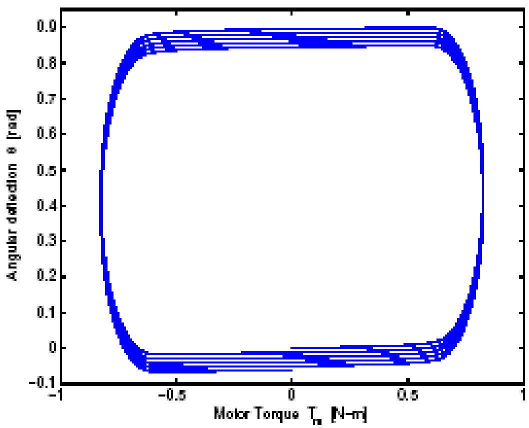
Hysteresis map from motor torque to angular deflection θ

Simulation with the LuGre Model

- Low frequency sinusoidal input current







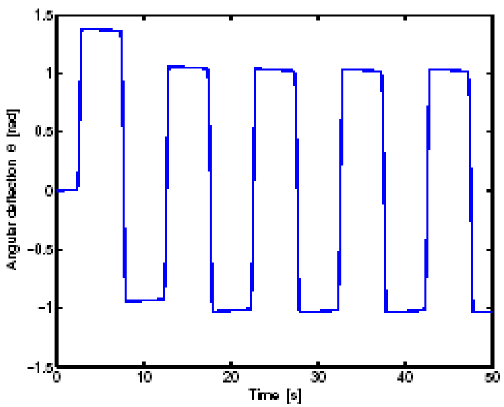
Angular deflection θ

Angular velocity

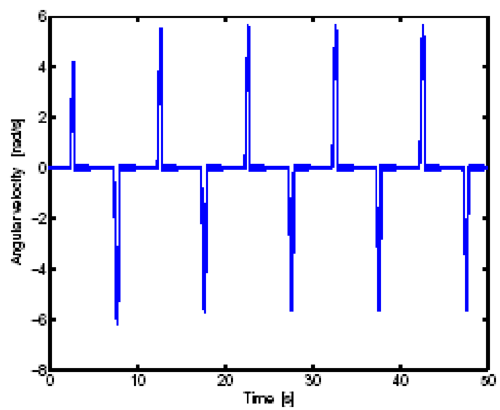
Hyteresis map from motor torque to angular deflection θ

Simulation with Maxwell-slip Model

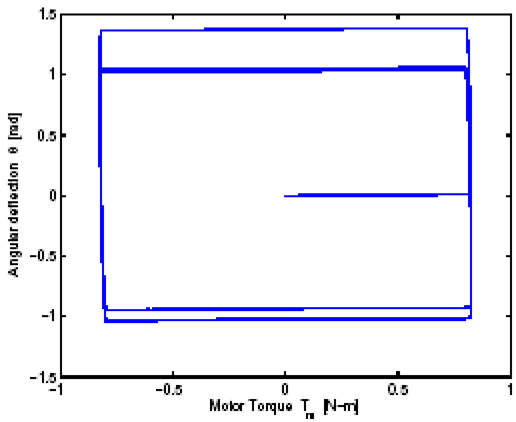
- Low frequency sinusoidal input current



Angular deflection θ



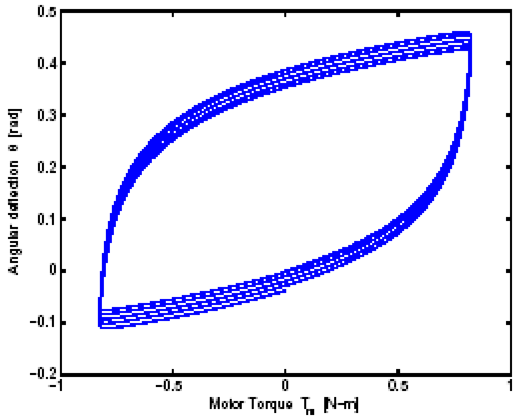
Angular velocity



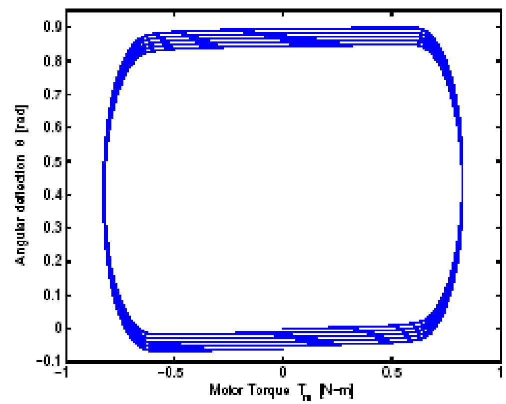
Hyteresis map from motor torque to angular deflection θ

Compare Simulation with Experimental Results

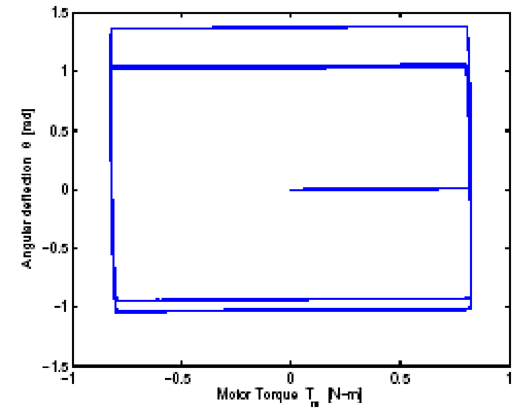
- Hysteresis loop from Motor Torque to Angular Deflection



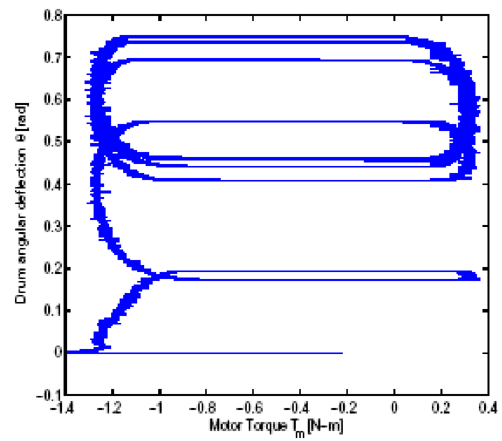
Dahl model



LuGre model



Maxwell-slip model



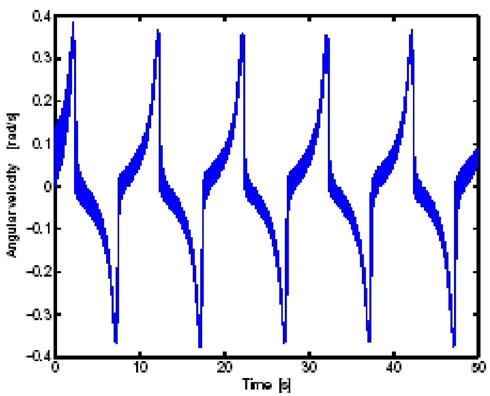
Experimental

Aside from the transients and the input current bias,

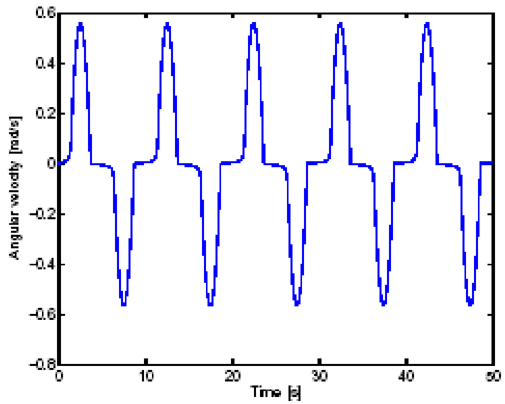
LuGre model Matches the best

Compare Simulation with Experimental Results

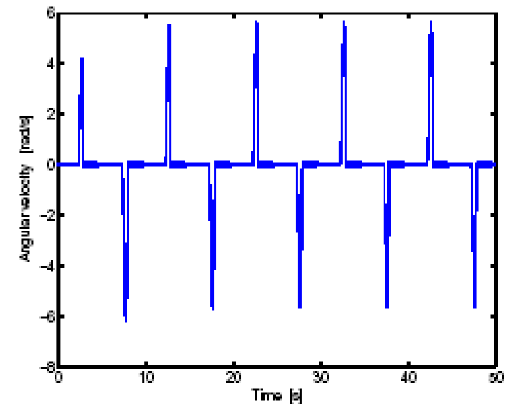
- Comparing the Angular velocity



Dahl model

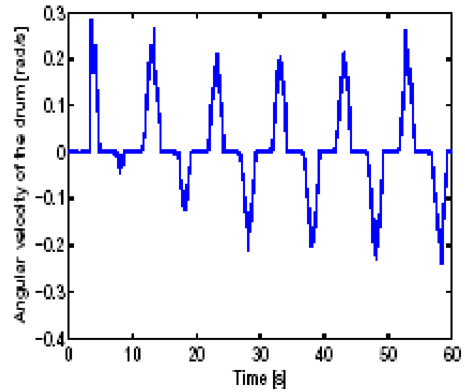


LuGre model



Maxwell-slip model

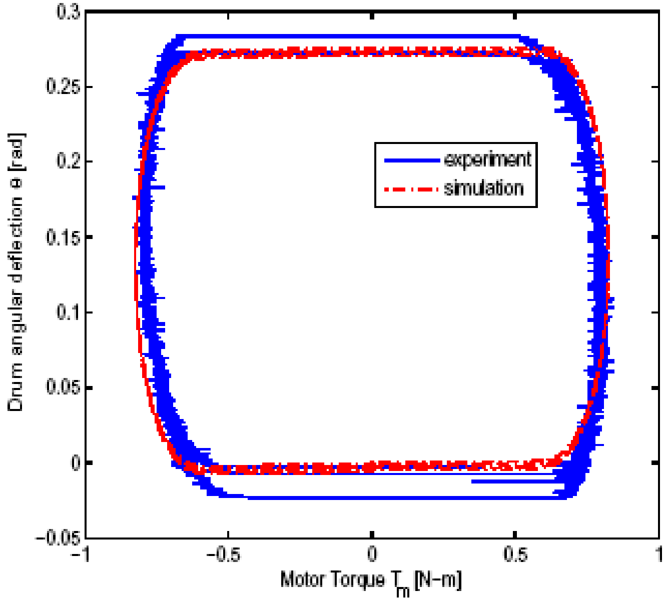
Experimental



LuGre model
Matches the best

Identification of the Gearbox Friction

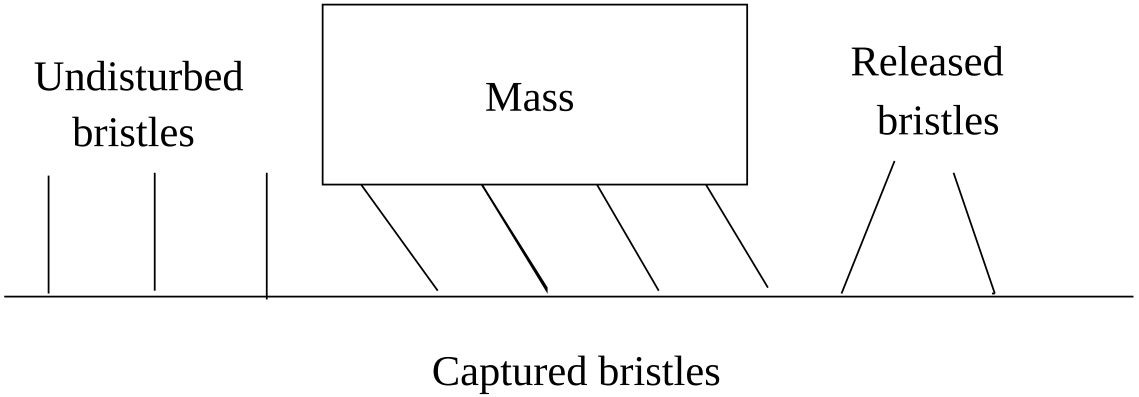
- LuGre friction parameters that model the gearbox friction were identified



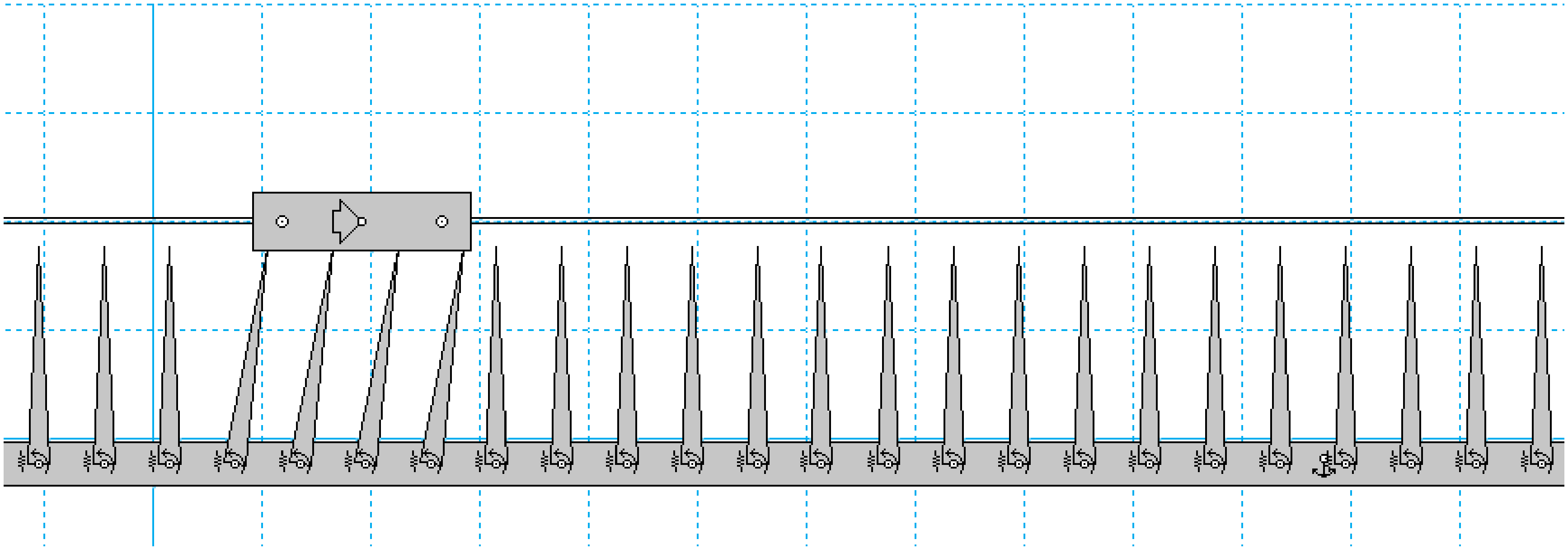
The experimental and simulated (using the identified LuGre friction model) hysteresis maps

Friction

- Hysteretic friction can perhaps be explained by sudden release
- Sudden release converts linear bristle damping into hysteretic friction



Friction



Damping

- How to model it?
- Why is it hysteretic?

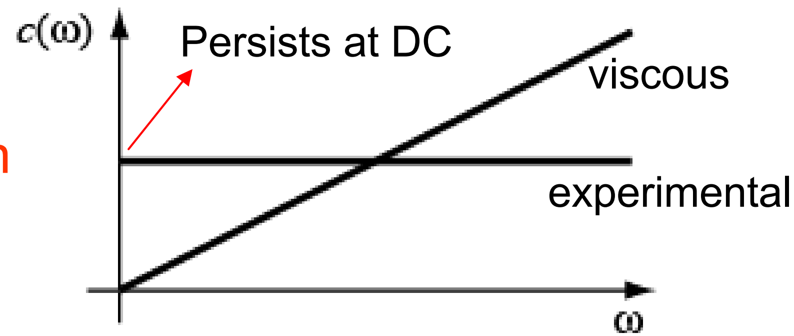
Hysteretic Energy Dissipation

- Hysteresis sometimes corresponds to energy loss
- Energy loss in static limit = area enclosed by force-to-position hysteresis loop $\mathcal{H}_\infty$
  - Line integral around loop
  - Area measured in Joules
- Dissipation need NOT be hysteretic
  - Viscous damping is not hysteretic
- Hysteresis need NOT be dissipative
  - Position to position hysteresis is usually not dissipative

Structural/Material Damping

- Conventional viscous damping predicts energy dissipation in 1 cycle as $\pi c \omega A^2 \Rightarrow$ rate-dependent
- Experiments shows the energy dissipation is rate independent
- Conventional model: frequency-domain structural damping model

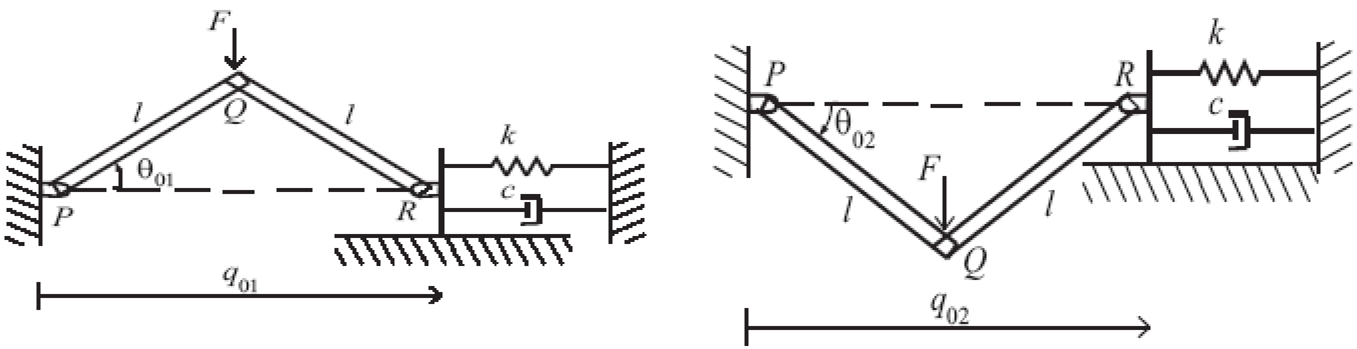
$$m\ddot{q}(t) + \jmath\eta q(t) + kq(t) = f$$



- Has complex, unstable time-domain solutions--noncausal
- Energy dissipated per cycle depends only on the signal amplitudes but not on the frequency
- Constant energy dissipation as $\omega \to 0 \Rightarrow$ Hysteretic damping
  - **Energy dissipation "at" DC!**
- Real damping has more complex frequency dependence

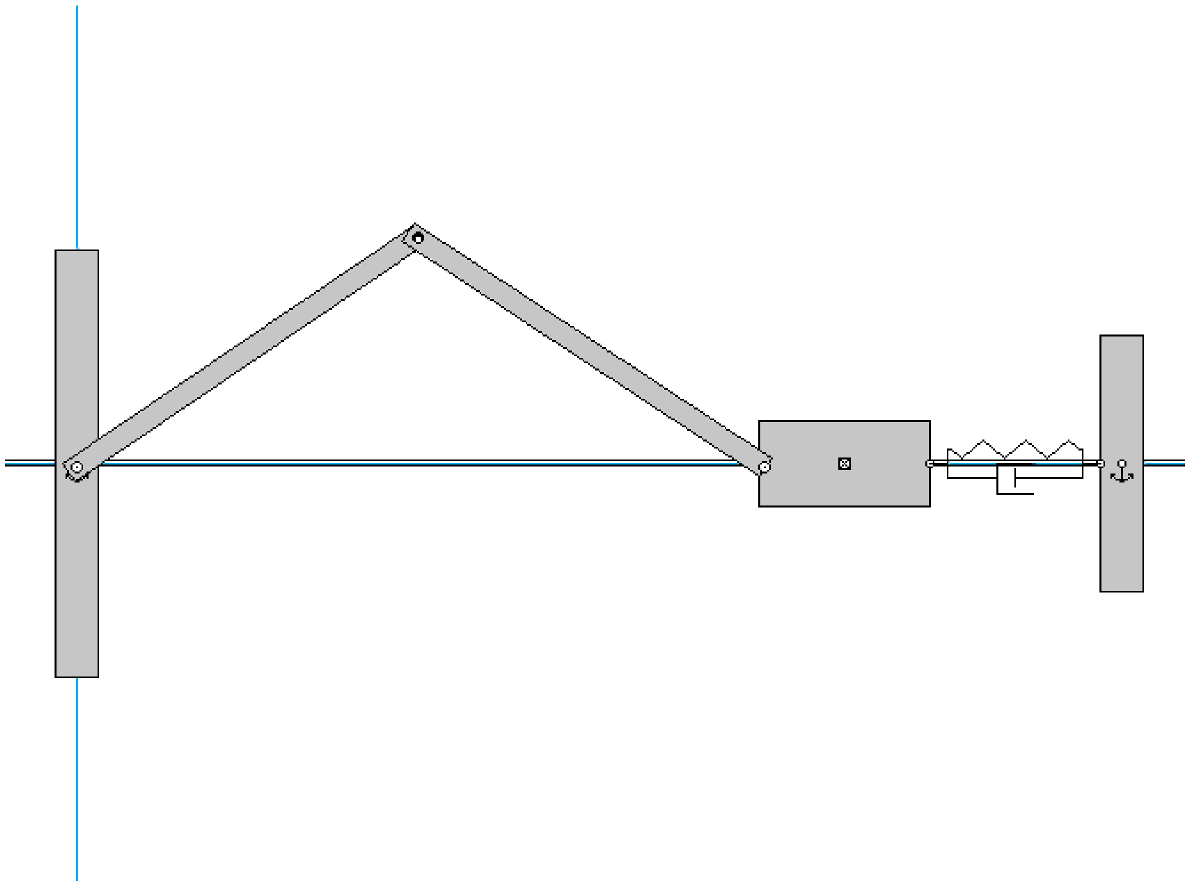
Hysteresis in a Two-bar Linkage

- Two-bar linkage has 2 stable equilibria for constant F
  - Note **linear/viscous** damping
- Exhibits snap-through buckling
- Multiple equilibria



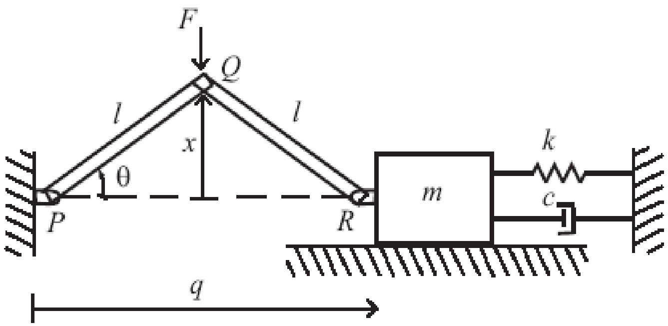
Two stable equilibria of the linkage for a constant F

Snap-Through Buckling

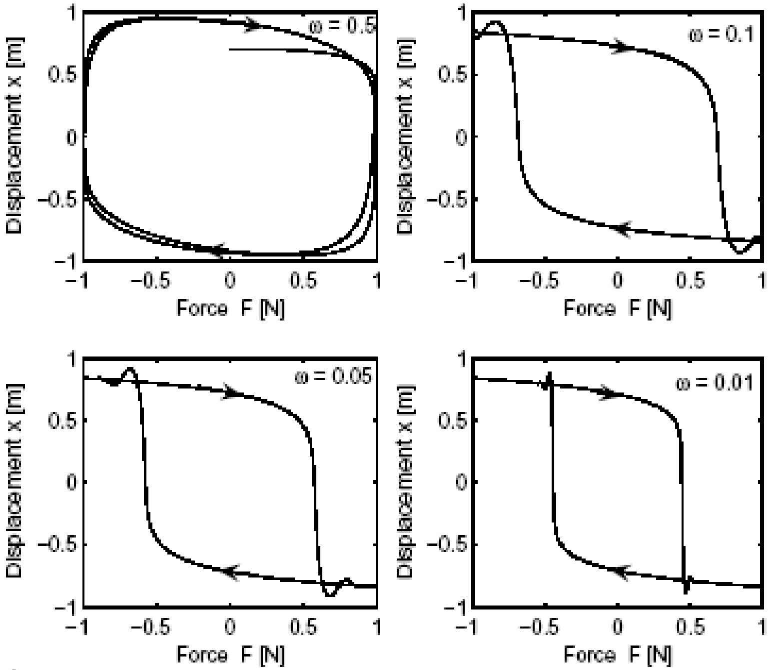


Hysteresis in a Two-bar Linkage

- Simulations show hysteresis between the vertical force F and the vertical displacement x



Two-bar linkage showing x and F



Area of each loop is the energy dissipated by the linear damper during one cycle of operation

Damping

- Multistability plus viscous damping causes hysteretic damping
- Hysteretic damping can possibly be explained by microbuckling

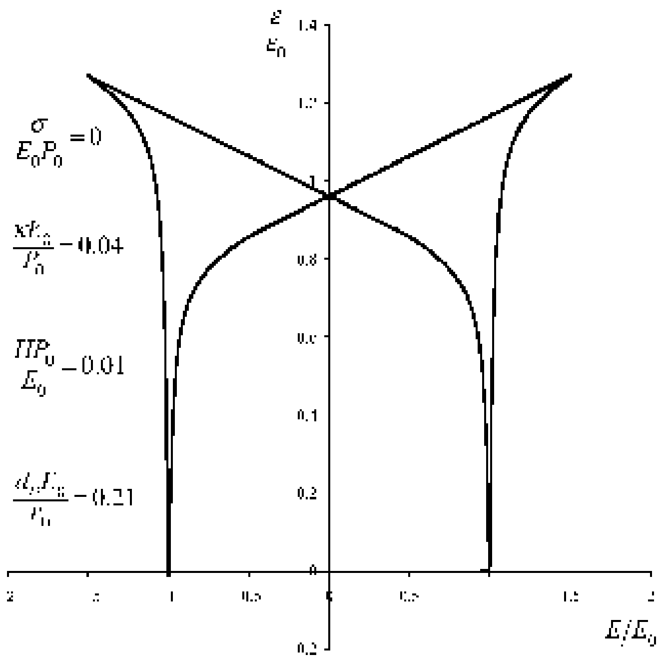


Fig. 2. Butterfly loop corresponding to the hysteresis loop in Fig. 1.

chosen such that $d_{33}E_0/\varepsilon_0 = 0.21$, selected to produce a shape comparable to experimental loops [1,30]. The origin is the starting point since the material is initially unpolarized. This also means that the material is not initially piezoelectric, as can be seen from Eq. (43). Thus no strain develops at first when the electric field is increased. However, polarization commences at $E = E_0$ and therefore the remanent strain grows quadratically with the polarization and piezoelectric strain appears and grows as well. The curve rises steeply at first because there is a rapid increase in polarization strain. However, lock up sets in and the slope of the curve diminishes. After the maximum field is reached and the then reduced, switching ceases and the response is at first purely linearly piezoelectric with the strain given by Eq. (43) with both ε^r and P^r non-zero and fixed. Therefore, the response to small electric fields at this stage is piezoelectric with a positive slope. As the field continues to be reduced, switching recommences at $E = -0.5E_0$ and since the remanent polarization is now diminishing, the remanent strain falls. Simultaneously, the piezoelectric effect is degraded and when the electric field reaches $-E_0$ and the remanent polarization is zero, the strain has reached zero as well. However, as the field is reduced below $-E_0$ a negative remanent polarization develops and the remanent and piezoelectric strains are rebuilt. Lock up now occurs as the electric field is brought towards $-1.5E_0$. After this value is reached, the electric field is

Butterfly Hysteresis in the Linkage Mechanism

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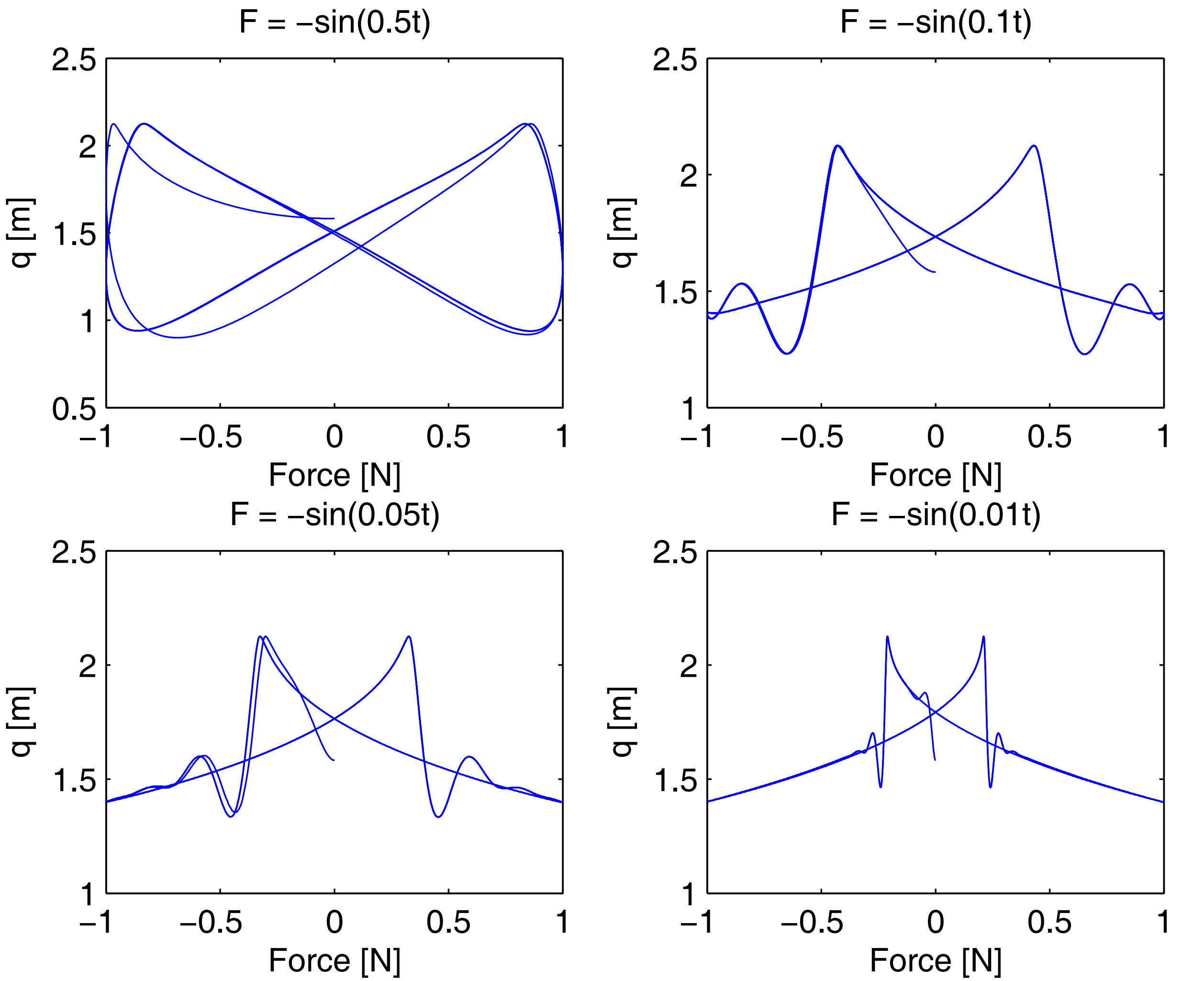

Conclusions

- Hysteresis is the static limit of the periodic dynamic response
- Hysteresis arises from the principle of multistability
- The principle of multistability suggests mechanisms that can explain hysteretic phenomena
  - Friction------sudden release of viscously damped bristles
  - Damping-----snap-through buckling