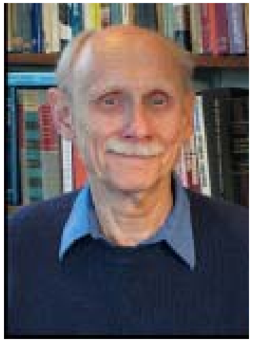
On the Advantages of Being Periodic

***r**Evolving Horizons in Systems and Control*

In Honor and Appreciation of

Elmer Gilbert



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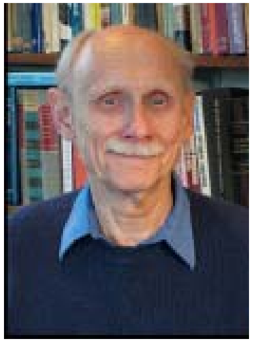
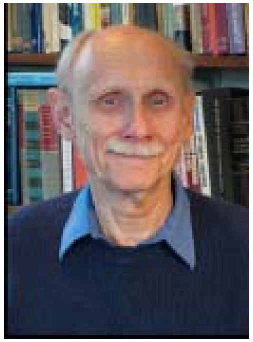
Goal

- 33 years of perspective on what my PhD research was about!



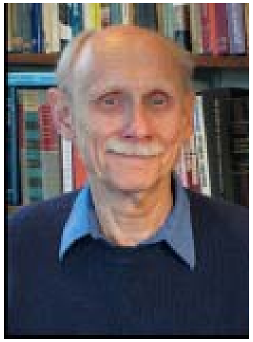
Periodic History

- Marquez—*One Hundred Years of Solitude*
  - “Just like Aureliano,” Ursula exclaimed. “It’s as if the world were repeating itself.”



Periodic = Monotonous?

- Ecclesiastes (Koheleth)
  - The sun rises and the sun sets,
    and hurries back to where it rises.
  - The wind blows to the south
    and turns to the north;
    round and round it goes,
    ever returning on its course.
  - All streams flow into the sea,
    yet the sea is never full.
    To the place the streams come from,
    there they return again.
  - What has been will be again,
    what has been done will be done again;
    there is nothing new under the sun.



Periodic Control

- What is it?
- Why do it?
- Why need it?

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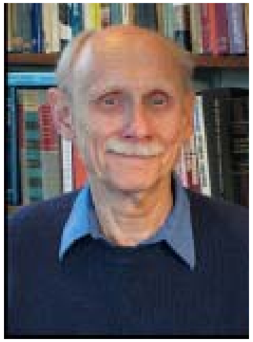
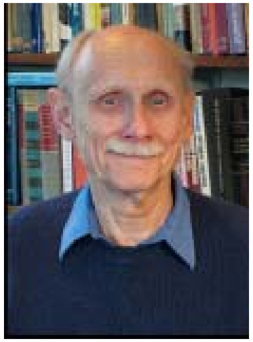


Control for the Long Term

MichiganEngineering

- How to best control a system for long-term operation?
  - Ignore transients/startup
    - E.g., ascent, descent
  - Operate sustainably
    - E.g., Cruise
    - Maximize endurance
    - Minimize fuel usage rate
- Constant operation---obvious approach
- Periodic operation---why?



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Optimal Periodic Control

SIAM J. CONTROL AND OPTIMIZATION
Vol. 15, No. 5, August 1977

OPTIMAL PERIODIC CONTROL: A GENERAL THEORY OF NECESSARY CONDITIONS*

ELMER G. GILBERT†

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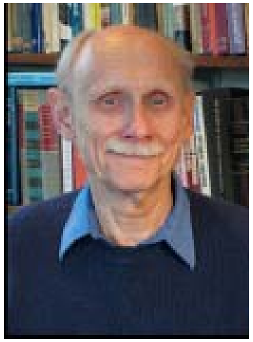


Optimal Periodic Control

Optimal periodic control problem (OPC). Find $u(\cdot)$, $x(\cdot)$ and τ which minimize J subject to

$$J = g_0(y, x(0)), \tag{2.1-1}$$

$$g_i(y, x(0)) \leqq 0, \qquad i = -j, \cdots, -1, \tag{2.1-2}$$

$$g_i(y, x(0)) = 0, \qquad i = 1, \cdots, k, \tag{2.1-3}$$

$$y = \frac{1}{\tau}\int_0^\tau \tilde{f}(x(t), u(t))\, dt \in Y, \tag{2.1-4}$$

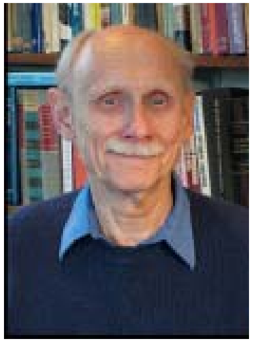
$$\dot{x}(t) = f(x(t), u(t)) \quad \text{almost all } t \in [0, T], \quad x(0) = x(\tau), \tag{2.1-5}$$

$$u(\cdot) \in \mathscr{U} = \{u(\cdot): u(\cdot) \text{ measurable and essentially bounded on } [0, T],\ u(t) \in U \text{ for all } t \in [0, T]\}, \tag{2.1-6}$$

$$x(\cdot) \in \mathscr{X} = \{x(\cdot): x(\cdot) \text{ absolutely continuous on } [0, T],\ x(t) \in X \text{ for all } t \in [0, T]\}, \tag{2.1-7}$$

$$\tau \in (0, T]. \tag{2.1-8}$$

Gilbert, SICOPT, 1977



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Optimal Steady-State Control

Optimal steady-state problem (OSS). Find u and x which minimize J subject to

(2.2-1) $$J = g_0(y, x),$$

(2.2-2) $$g_i(y, x) \leqq 0, \qquad i = -j, \cdots, -1,$$

(2.2-3) $$g_i(y, x) = 0, \qquad i = 1, \cdots, k,$$

(2.2-4) $$y = \tilde{f}(x, u) \in Y,$$

(2.2-5) $$f(x, u) = 0,$$

(2.2-6) $$u \in U,$$

(2.2-7) $$x \in X.$$

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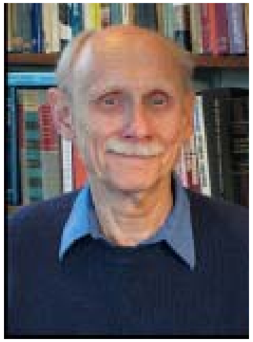


Solution Sets

$$\mathscr{S}(\mathrm{OPC} = \{(u(\cdot), x(\cdot), \tau): (u(\cdot), x(\cdot), \tau \text{ solves OPC}\}, \tag{2.3}$$

$$\mathscr{S}(\mathrm{SS}) = \{(u(\cdot), x(\cdot), \tau): \text{(2.1-2)–(2,1-8) are satisfied and } u(\cdot) \text{ and } x(\cdot) \text{ are constant}\}, \tag{2.4}$$

$$\mathscr{S}(\mathrm{SSOPC}) = \mathscr{S}(\mathrm{OPC}) \cap \mathscr{S}(\mathrm{SS}), \tag{2.5}$$

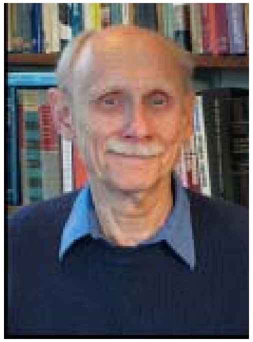
$$\mathscr{S}(\mathrm{OSS}) = \{(u(\cdot), x(\cdot), \tau): (u(\cdot), x(\cdot), \tau) \in \mathscr{S}(\mathrm{SS}) \text{ and } (u(0), x(0)) \text{ solves OSS}\}. \tag{2.6}$$

Of course, $\mathscr{S} = \varnothing$, the null set, is possible in any of the four cases. The particular circumstance $\mathscr{S}(\mathrm{SSOPC}) = \varnothing$, $\mathscr{S}(\mathrm{OSS}) \neq \varnothing$ implies that there exist time-dependent controls which do better than the best steady-state controls. If $\mathscr{S}(\mathrm{SSOPC}) \neq \varnothing$ any $\psi \in \mathscr{S}(\mathrm{SSOPC})$ is also in $\mathscr{S}(\mathrm{OSS})$ since ψ is optimum with respect to choices in $\mathscr{U} \times \mathscr{X} \times (0, T]$ and $\mathscr{S}(\mathrm{SS}) \subset \mathscr{U} \times \mathscr{X} \times (0, T]$. Also, it is clear that all elements of $\mathscr{S}(\mathrm{OSS})$ and $\mathscr{S}(\mathrm{SSOPC})$ yield identical costs J. This leads to the following.

Remark 2.1. There are three mutually exclusive possibilities:

(i) $\mathscr{S}(\mathrm{SSOPC}) = \mathscr{S}(\mathrm{OSS}) \neq \varnothing$;
(ii) $\mathscr{S}(\mathrm{SSOPC}) = \varnothing$, $\mathscr{S}(\mathrm{OSS}) \neq \varnothing$;
(iii) $\mathscr{S}(\mathrm{SSOPC}) = \mathscr{S}(\mathrm{OSS}) = \varnothing$.

Possibility (iii) is not apt to occur since for well posed problems it is likely that $\mathscr{S}(\mathrm{OSS}) \neq \varnothing$. Possibility (i) implies that OPC has a steady-state solution and consequently, there is no advantage (even though OPC may also have time-dependent solutions) in using time-dependent control. Possibility (ii) implies time-dependent control can do better than steady-state control (a statement which holds true even if $\mathscr{S}(\mathrm{OPC}) = \varnothing$). Because of the importance of possibilities (i) and (ii) the following definitions are introduced.



More Solution Sets

DEFINITION 2.1. If $\mathscr{S}(\text{SSOPC}) = \mathscr{S}(\text{OSS}) \neq \varnothing$ the problem OPC is called *steady-state*.

DEFINITION 2.2. If $\mathscr{S}(\text{SSOPC}) = \varnothing$, $\mathscr{S}(\text{OSS}) \neq \varnothing$ the problem OPC is called *proper* (compare [5]).

The study of relative minima of OPC and OSS will prove to be of value, particularly in the case of steady-state minima.

DEFINITION 2.3. $(u(\cdot), x(\cdot), \tau) \in \mathscr{S}(\text{SS})$ is a *strong* $\{$*weak*$\}$ *relative minimum* of OPC if there exists an $\varepsilon > 0$ such that for all $(\hat{u}(\cdot), \hat{x}(\cdot), \hat{\tau})$ which satisfy (2.1-2)–(2.1-8) and $\|\hat{x}(t) - x(0)\| < \varepsilon$ $\{\|\hat{x}(t) - x(0)\| < \varepsilon,\ \|\hat{u}(t) - u(0)\| < \varepsilon\}$, $t \in [0, T]$, it follows that $g_0(y, x(0)) \leqq g_0(\hat{y}, \hat{x}(0))$.

DEFINITION 2.4. $(u(\cdot), x(\cdot), \tau) \in \mathscr{S}(\text{SS})$ is a *strong* $\{$*weak*$\}$ *relative minimum* of OSS if there exists an $\varepsilon > 0$ such that for all $(\hat{u}, \hat{x})$ which satisfy (2.2-2)–(2.2-7) and $\|\hat{x} - x(0)\| < \varepsilon$ $\{\|\hat{x} - x(0)\| < \varepsilon,\ \|\hat{u} - u(0)\| < \varepsilon\}$ it follows that $g_0(y, x(0)) \leqq g_0(\hat{y}, \hat{x})$.

In these definitions $\|\cdot\|$ denotes any norm on R^n or R^l and $\hat{y} = y$ for $u = \hat{u}$, $x = \hat{x}$, $\tau = \hat{\tau}$. Corresponding to each of the four types of relative minima, notations for the set of minima are adopted:

$$\mathscr{S}(\text{SRMSSOPC}), \mathscr{S}(\text{WRMSSOPC}), \mathscr{S}(\text{SRMOSS}), \mathscr{S}(\text{WRMOSS}).$$

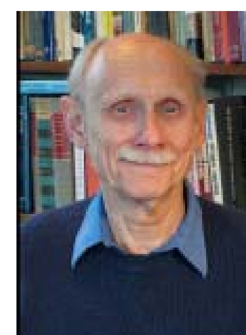
For example,

$$\mathscr{S}(\text{SRMSSOPC}) = \{(u(\cdot), x(\cdot), \tau) : (u(\cdot), x(\cdot), \tau) \in \mathscr{S}(\text{SS}) \text{ is a strong relative minimum of OPC}\}. \tag{2.7}$$

Obviously, $\mathscr{S}(\text{SSOPC}) \subset \mathscr{S}(\text{SRMSSOPC}) \subset \mathscr{S}(\text{WRMSSOPC})$ and $\mathscr{S}(\text{OSS}) \subset \mathscr{S}(\text{SRMOSS}) \subset \mathscr{S}(\text{WRMOSS})$. By using the same reasoning which led to Remark 2.1 it is easy to see that $\mathscr{S}(\text{SRMSSOPC}) \subset \mathscr{S}(\text{SRMOSS})$. However, $\mathscr{S}(\text{SRMSSOPC}) \neq \varnothing$ does not imply $\mathscr{S}(\text{SRMSSOPC}) = \mathscr{S}(\text{SRMOSS})$ because elements of $\mathscr{S}(\text{SRMSSOPC})$ do not necessarily have the same cost as elements of $\mathscr{S}(\text{SRMOSS})$. Similar reasoning applies to the case of weak relative minima. All of this is summarized in

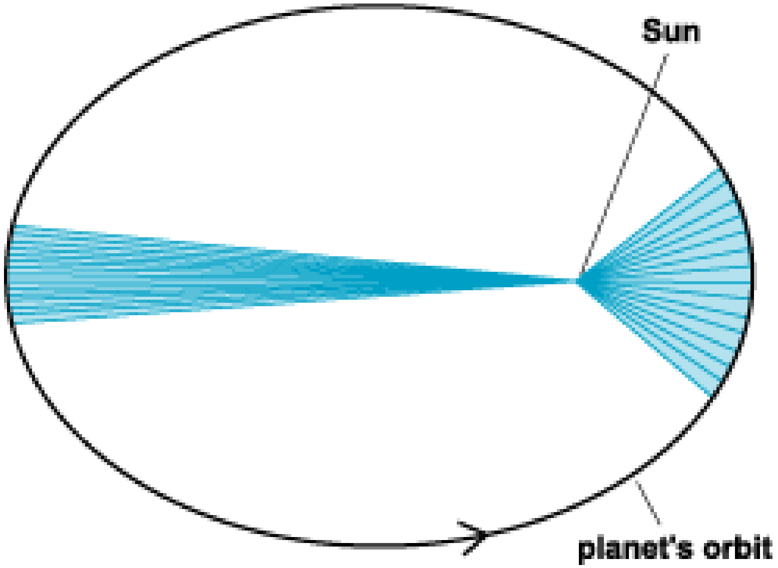
Remark 2.2. The following conclusions are valid: $\mathscr{S}(\text{SSOPC}) \subset \mathscr{S}(\text{SRMSSOPC}) \subset \mathscr{S}(\text{WRMSSOPC})$, $\mathscr{S}(\text{OSS}) \subset \mathscr{S}(\text{SRMOSS}) \subset \mathscr{S}(\text{WRMOSS})$, $\mathscr{S}(\text{SSOPC}) \subset \mathscr{S}(\text{OSS})$, $\mathscr{S}(\text{SRMSSOPC}) \subset \mathscr{S}(\text{SRMOSS})$, $\mathscr{S}(\text{WRMSSOPC}) \subset \mathscr{S}(\text{WRMOSS})$.

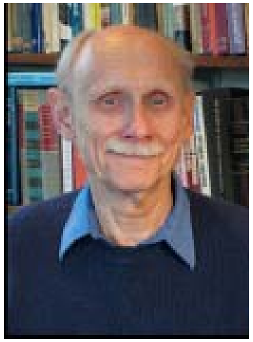
Evolving



Astrodynamics and Periodicity

- Kepler's laws and elliptical orbits





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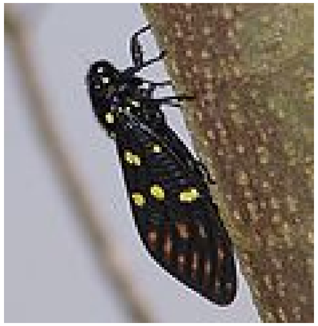
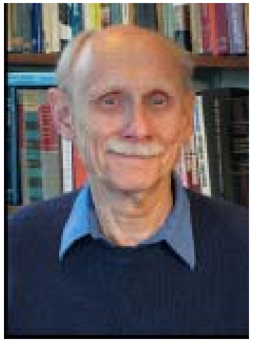
Prime Periodicity

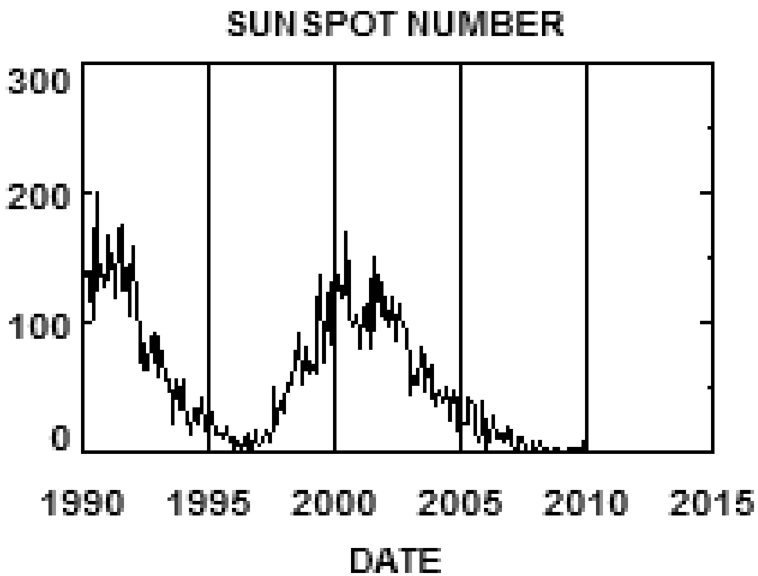
- Cicada
  - 13 and 17 year cycles
  - Predator resistance

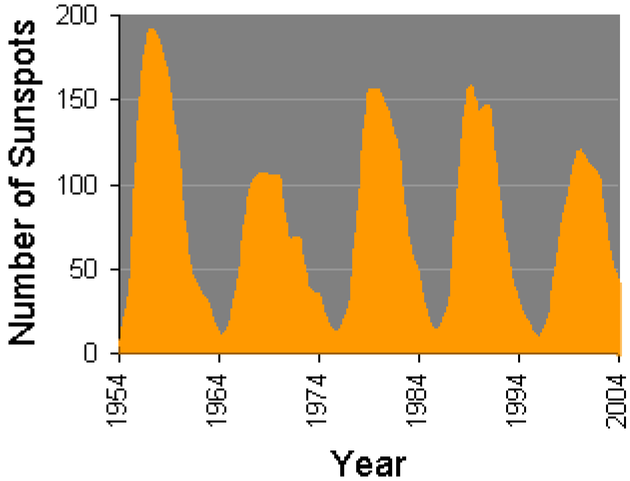


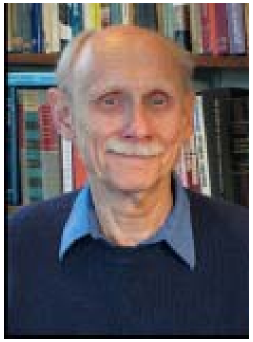


Sunspot Periodicity

- 9-14 years, 11 years on average







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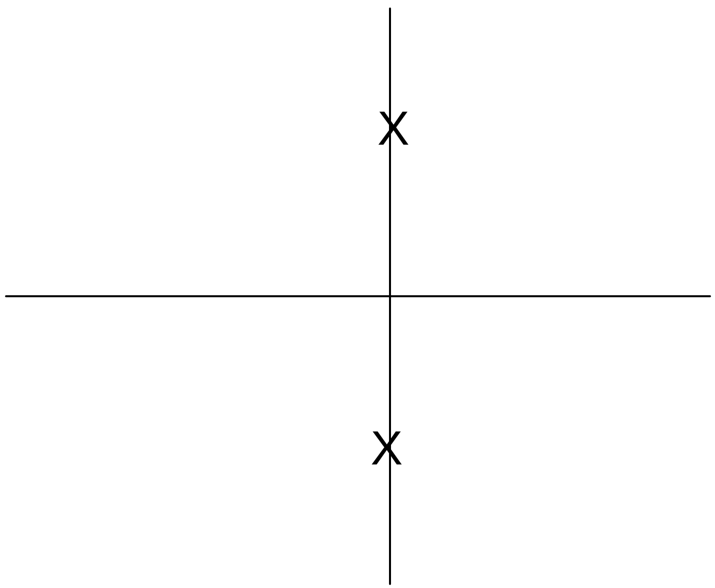
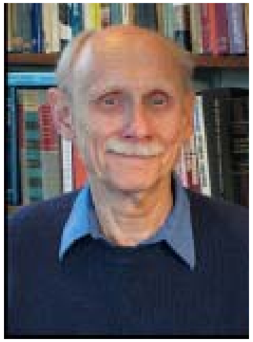
Linear Systems Periodicity

- Imaginary poles give periodic response

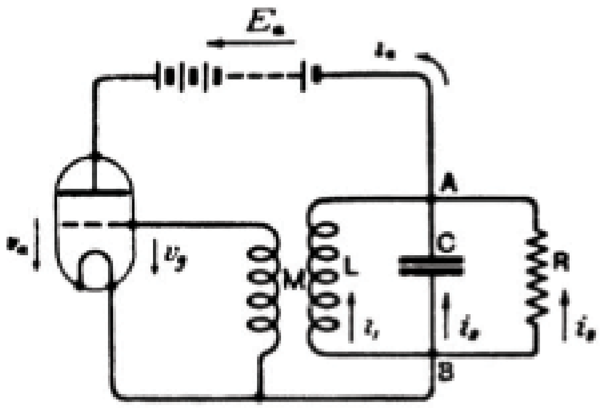


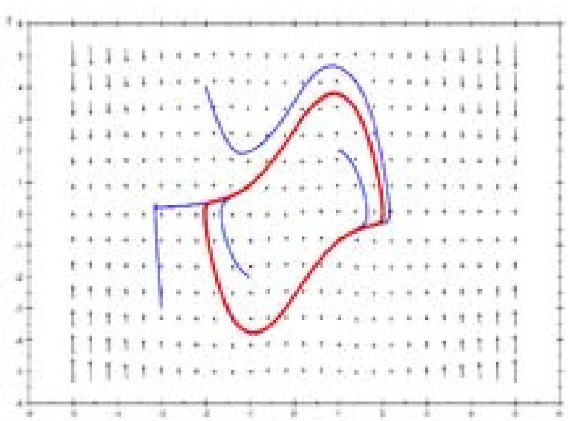


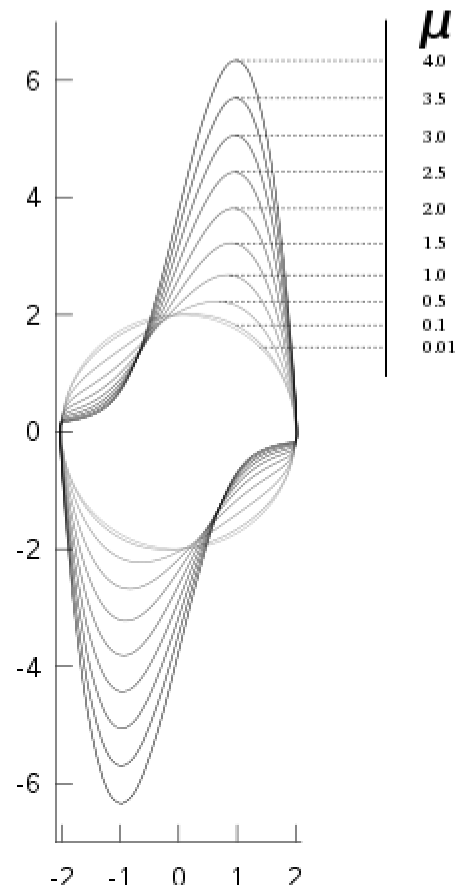
Nonlinear System Periodicity

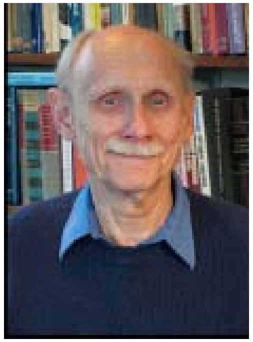
(Van der Pol 1920)

$$\ddot{x} - \mu(1 - x^2)\dot{x} + x = 0$$







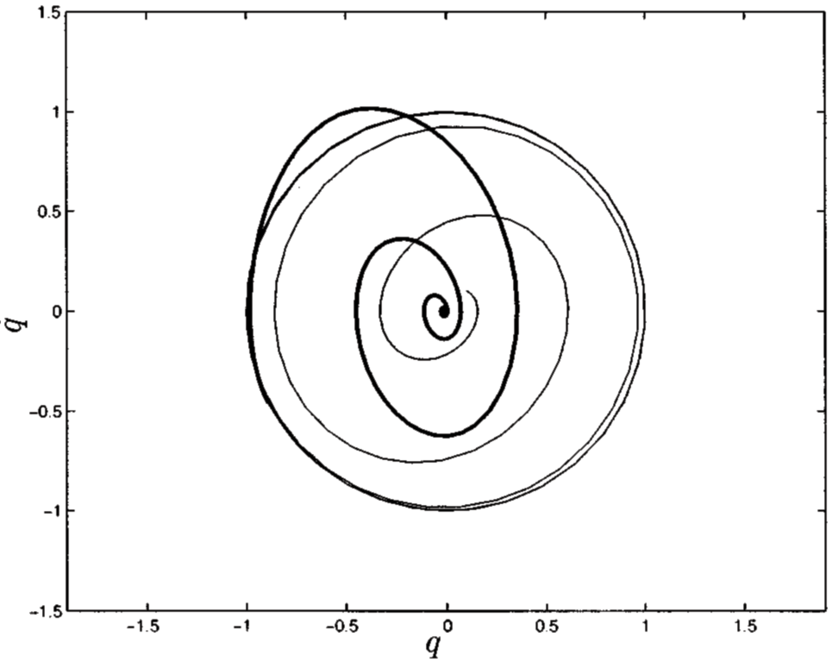


Evolving Horizons in Systems and Control

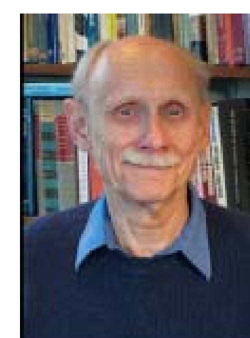
Roup Oscillator

2001

- To obtain a circular/sinusoidal limit cycle with amplitude *a* and frequency ω_n
- Speed of convergence to limit cycle determined by λ



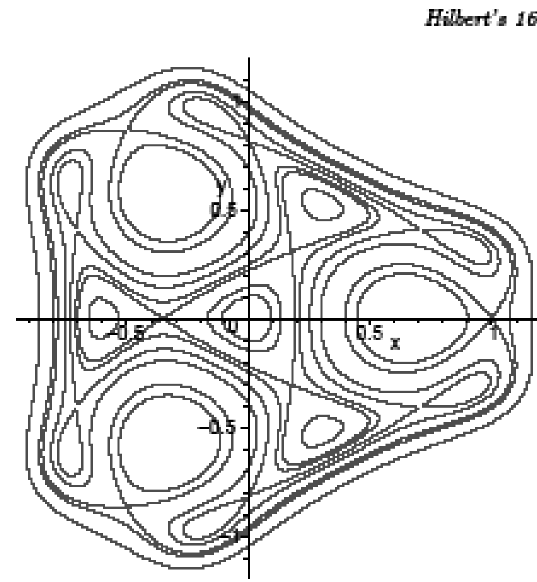
$$\ddot{q} + \lambda(q^2 + \omega_n^{-2}\dot{q}^2 - a^2) + \omega_n^2 q = 0$$

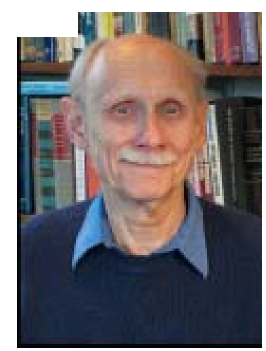


Hilbert's 16th Problem (1900)

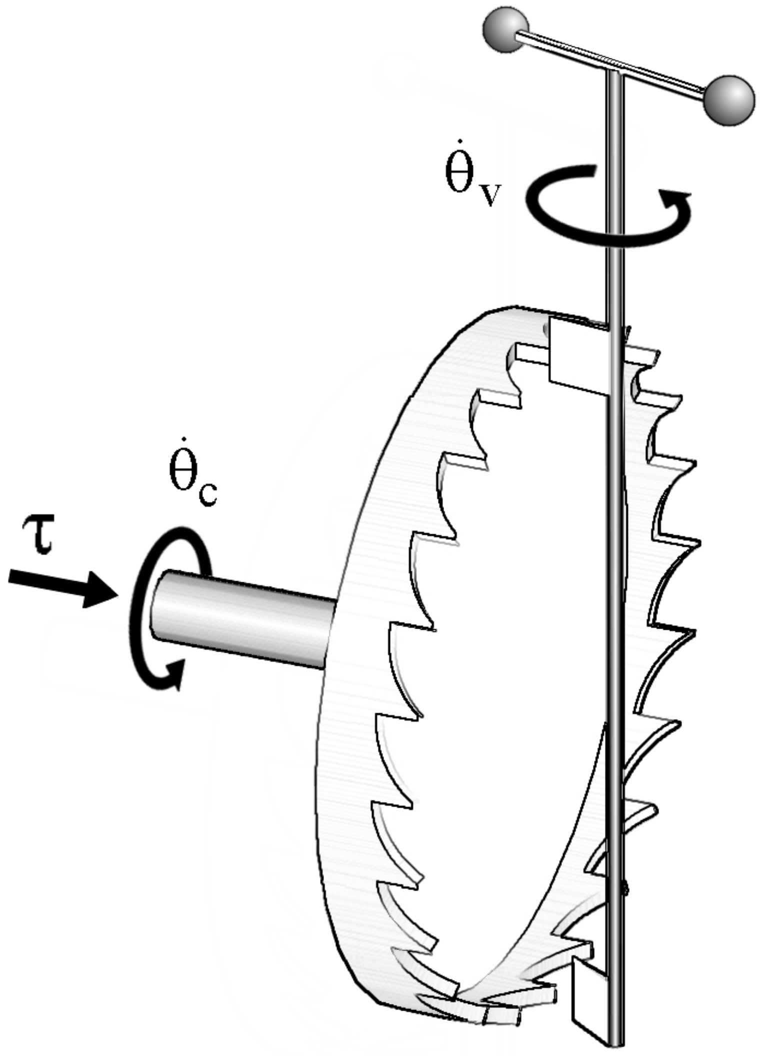
$$\dot{x} = P(x,y)$$
$$\dot{y} = Q(x,y)$$

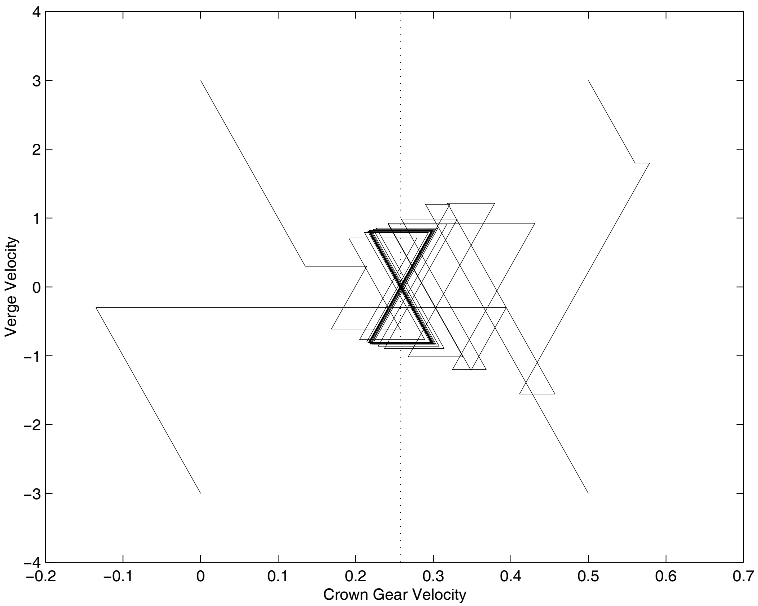
- How many limit cycles does a planar polynomial have?
  - Dulac's theorem 1923: Finite number for each system
    - Incorrect proof stood for 80 years—current status uncertain
  - For n=2, 4 are possible
  - For n=3, 11 are possible
  - For n=5, 24 are possible
  - No upper bound known for ANY n
- Besides the Riemann hypothesis, the most "elusive" of Hilbert's problems

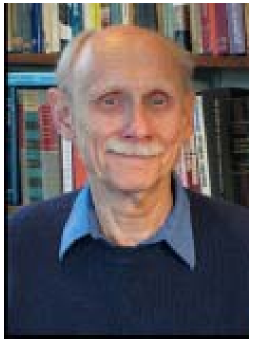




Clocks







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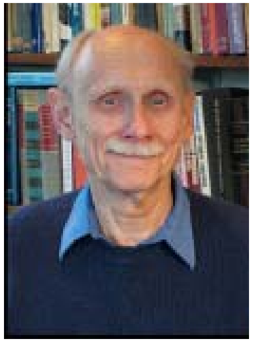
Self-Recycling Worlds

Shrimp

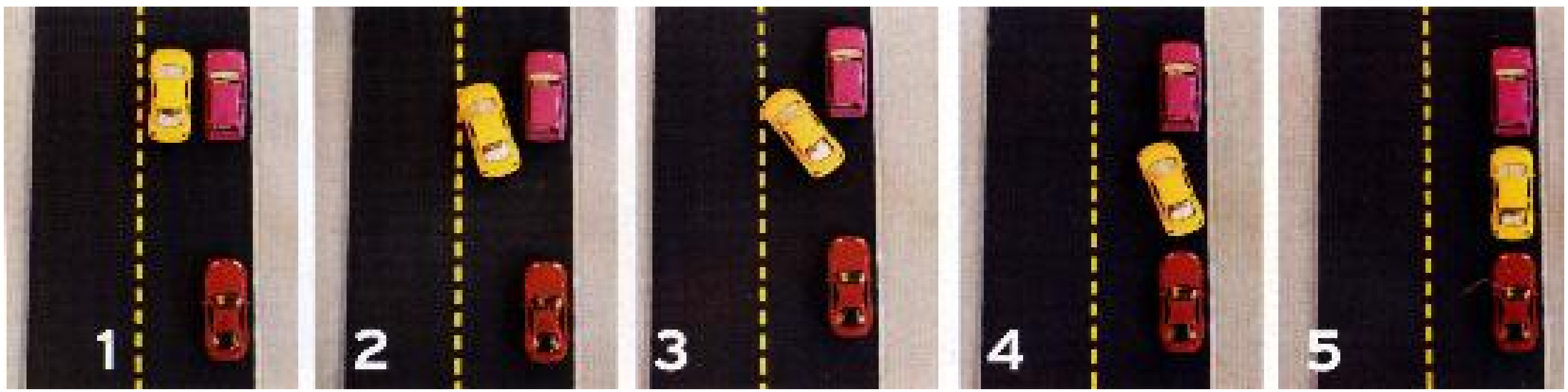
Ecospheres



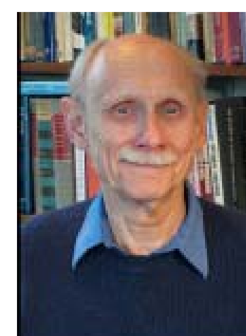
Evolving Horizons in Systems and Control

Nonholonomic Systems

- System with constraint on a velocity..
  - But not a constraint on position



Turning radius is constrained
Multiple passes may be needed
The number of required passes increases as the turning radius decreases

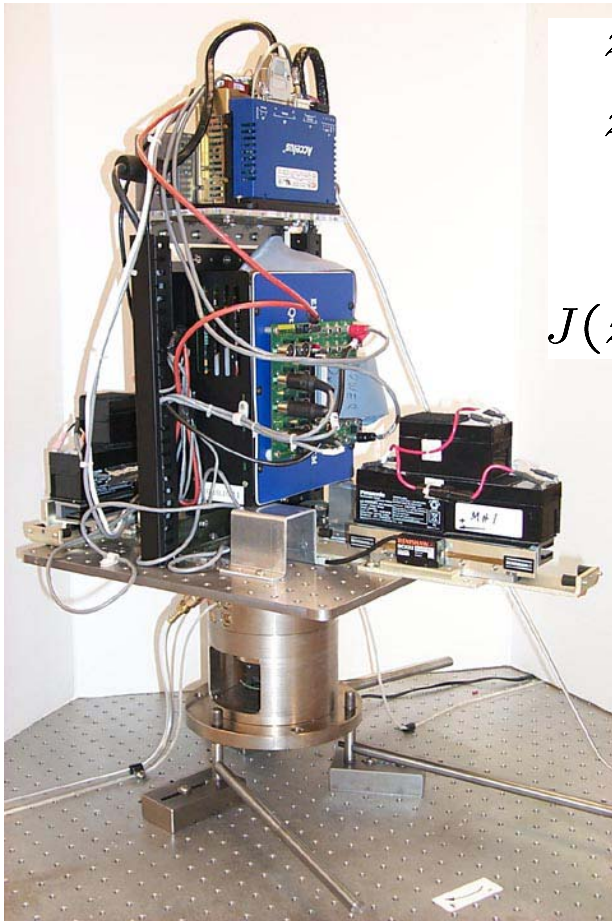


Evolving Horizons in Systems and Control

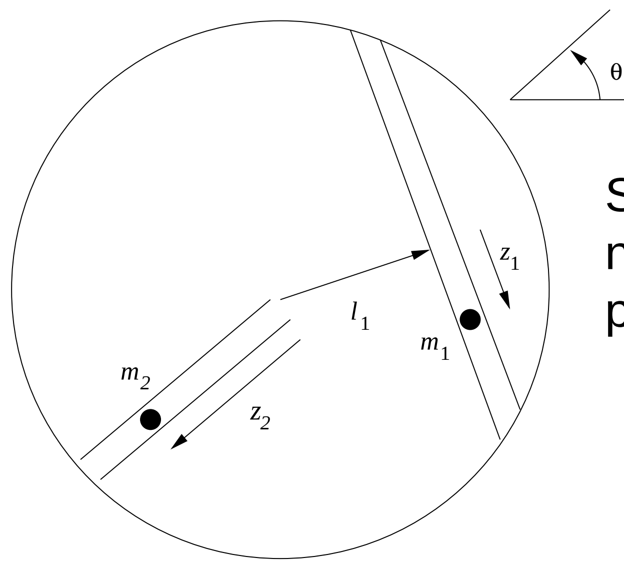
Shape Change Actuation

(Shen and McClamroch)

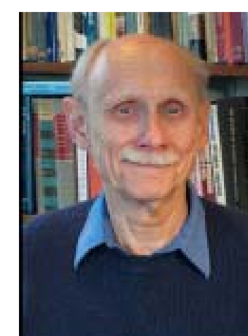
Angular momentum is conserved
…..but attitude is <u>not</u> constrained



$$
\begin{aligned}
\dot{z}_1 &= v_1 \\
\dot{z}_2 &= v_2 \\
\dot{\theta} &= \frac{m_1 l_1}{J(z)} v_1 + \frac{m_2 l_2}{J(z)} v_2 \\
J(z) &= I + m_1(l_1^2 + z_1^2) + m_2(l_2^2 + z_2^2)
\end{aligned}
$$



Stroke constraints necessitate multiple passes



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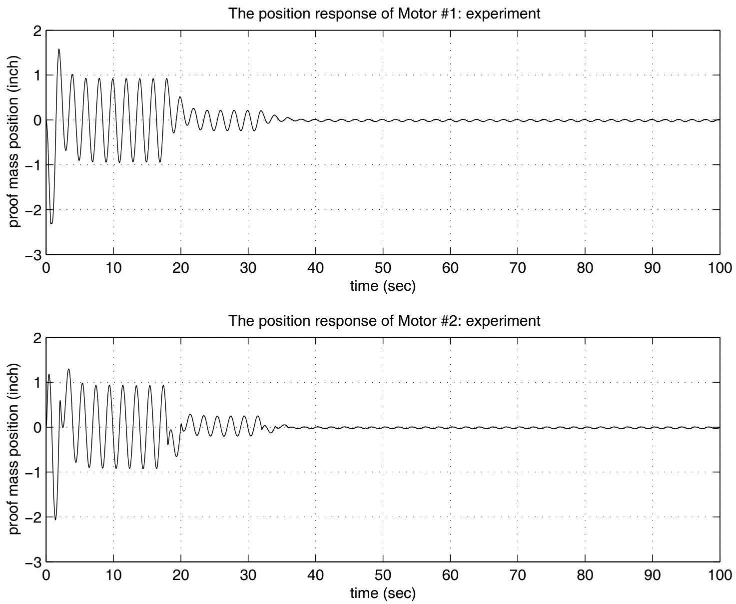
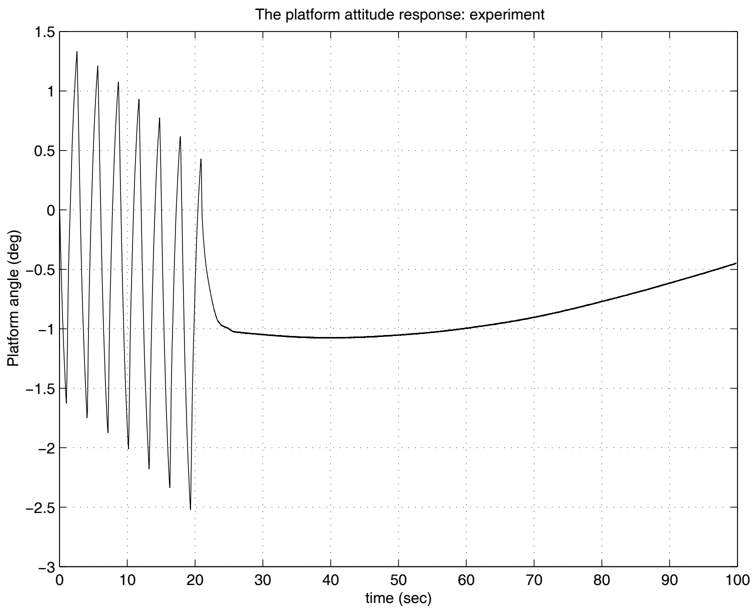
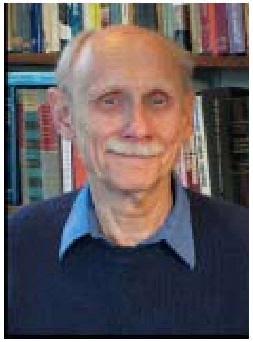
Platform Reorientation





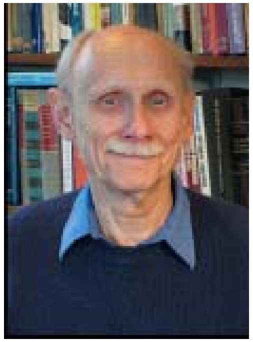


Evolving Horizons in Systems and Control

Periodic Flight

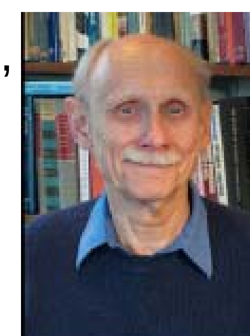
- How do birds stay aloft and cover huge distances with minimal energy?

- How can we keep an aircraft aloft indefinitely?

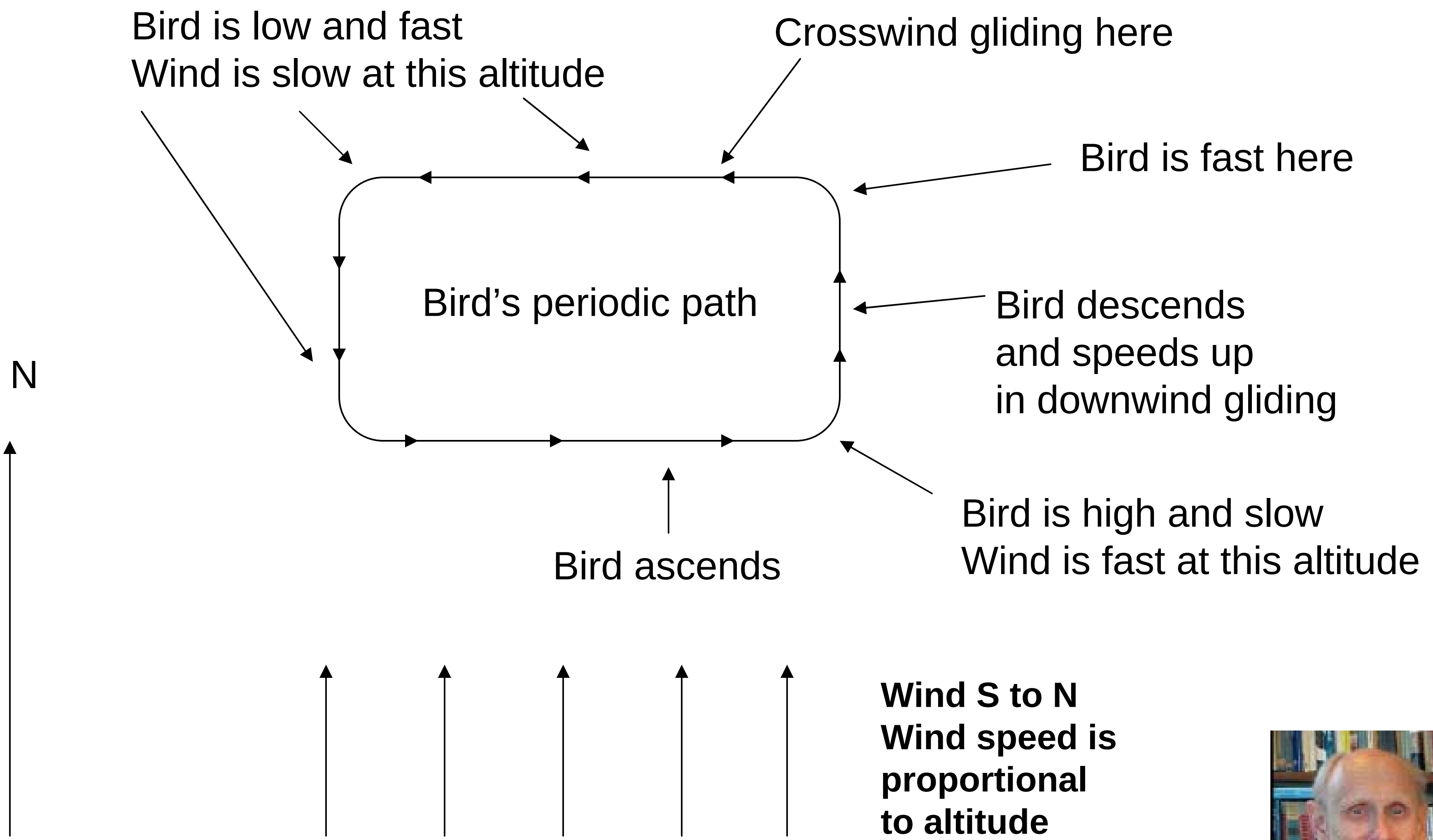


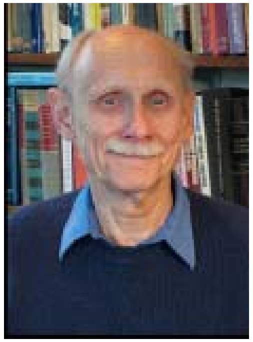
Dynamic Soaring

- Harvesting energy from wind gradients—from special terrain
  - **Not** from vertical components/requires special conditions
  - **Conjectured by Lord Rayleigh 1883 for albatrosses**
  - **Accomplished in 1974 by glider pilot:**
    - **"By repeating this manoeuvre he successfully maintained his height for around 20 minutes without the existence of ascending air…"**
  - 392 mph RC glider record from 45 mph winds in 2009
  - UAV strategy to reduce fuel consumption
    - Zhao/Qi 2004
    - "All problem formulations are subject to UAV equations of motion, UAV operational constraints, proper initial conditions, and terminal conditions that enforce a periodic flight."



How the Albatross Loiters





Evolving Horizons in Systems and Control

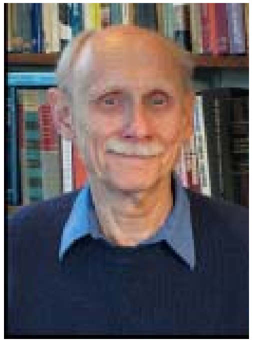
Assigning Equilibria

Can we maintain arbitrary equilibria?

$$\dot{x} = Ax + Bu \Rightarrow 0 = A\bar{x} + B\bar{u}$$

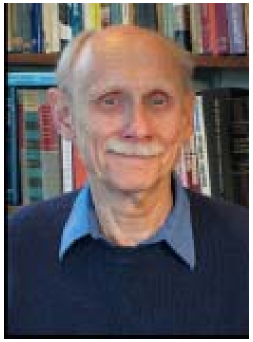
- Why not?
  - Range of B is too small---need $m \geq n$
  - <u>Unattainable</u> equilibria

- Contradicts controllability??



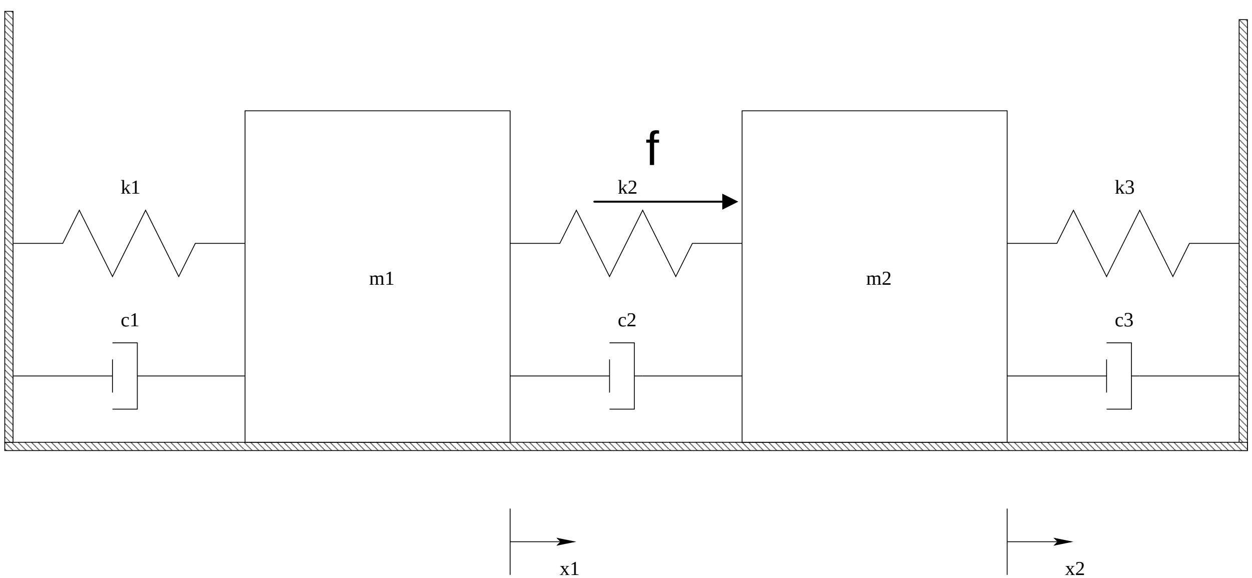
Vibration versus Shape Control

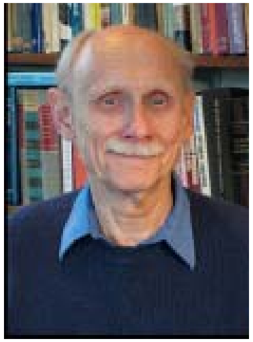
- Vibration suppression
  - Bring motion to rest (origin) and stay there
  - E.g., vibrating membrane
- Assign shape
  - Bring motion to rest at desired equilibrium
  - Vibrating membrane with desired aperture shape



Vibration versus Shape Control

- 4-state structure is controllable with one force input
  - Can bring masses to arbitrary configuration at arbitrary time
  - Cannot stay there!
  - Desired equilibrium is unattainable
- Idea:
  - Reach, leave, and return quickly to desired “equilibrium”
- Do this periodically

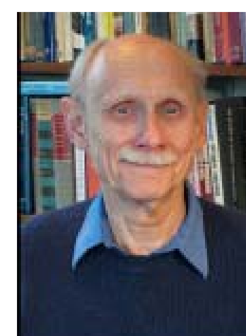




Hovering/Loitering

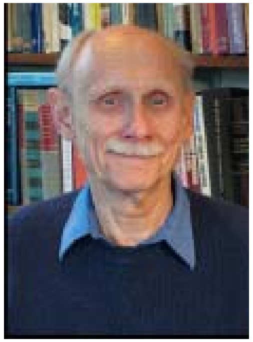
- All airplanes are controllable in position and velocity
  - Can reach desired position with zero velocity
    - Not a good idea for most airplanes
- But most airplanes cannot hover
  - Not enough actuation
  - Works for helicopters

- Hummingbirds control periodically by flapping
  - Flapping induces small periodic motion
- Research problem: What is the best way to maintain operation near an unattainable equilibrium?
  - Loitering limited by actuation constraints
  - Barabanov "Non-assignable Equilibria," Automatica, 2007



Dealing with Unattainable Equilibria

- Slow switching
  - Quasi steady state (QSS)
  - Convexify SSs to meet constraints
- Fast switching
  - Relaxed steady state (RSS)
  - Convexifies velocity set
- Slow switching between RSS's
  - Quasi-relaxed steady state (QRSS)
  - Convexify RSS's



Aircraft Cruise

The vehicle cruise problem is formulated as follows. The performance function

$$J(T(\cdot), V(\cdot), \tau) = V_{\text{avg}}(F_{\text{avg}})^{-1} = \text{specific range} \quad (1)$$

depends on the thrust $T(\cdot)$ (measurable on $[0, \tau)$), the speed $V(\cdot)$, and the period $\tau > 0$ which satisfy the following constraints:

$$\dot{V} = -D(V) + T(t), \quad V(0) = V(\tau) \geqslant 0, \quad (2)$$

$$0 \leqslant T(t) \leqslant 1, \quad a.a.\ t \in [0, \tau], \quad (3)$$

$$F_{\text{avg}} = \frac{1}{\tau}\int_0^\tau F(T(t))\,\mathrm{d}t = \text{average fuel rate}, \quad (4)$$

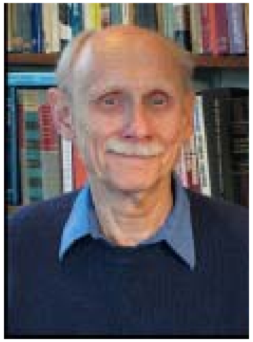
$$V_{\text{avg}} = \frac{1}{\tau}\int_0^\tau V(t)\,\mathrm{d}t = \text{average speed}, \quad (5)$$

$$V_{\text{avg}} \geqslant V_{\min} \geqslant 0. \quad (6)$$

The condition $V(0) = V(\tau)$ assures that both $V(\cdot)$ and $T(\cdot)$ are periodic when the domain of these functions is extended to $(-\infty, +\infty)$ by

$$V(t+\tau) = V(t) \quad \text{and} \quad T(t+\tau) = T(t).$$

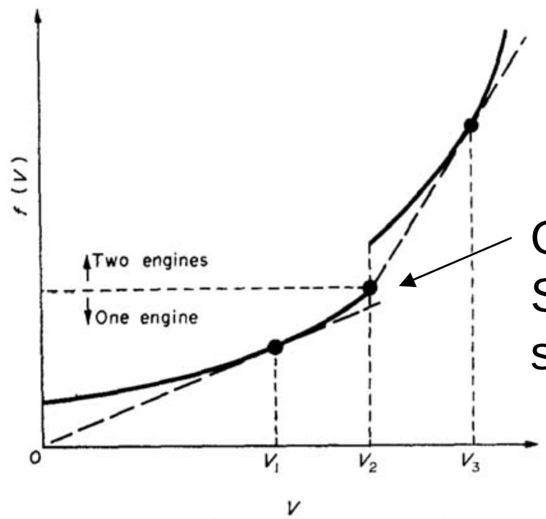
Gilbert, Automatica, 1976



Evolving Horizons in Systems and Control

SS, RSS,QSS, QRSS

- SS—equilibrium solution
- Relaxed steady state—fast switching between SS's
- QSS—slow switching between SS's
- QRSS—slow switching between RSS's



QSS
Slow
switching

FIG. 3. Steady-state fuel rate for ship.

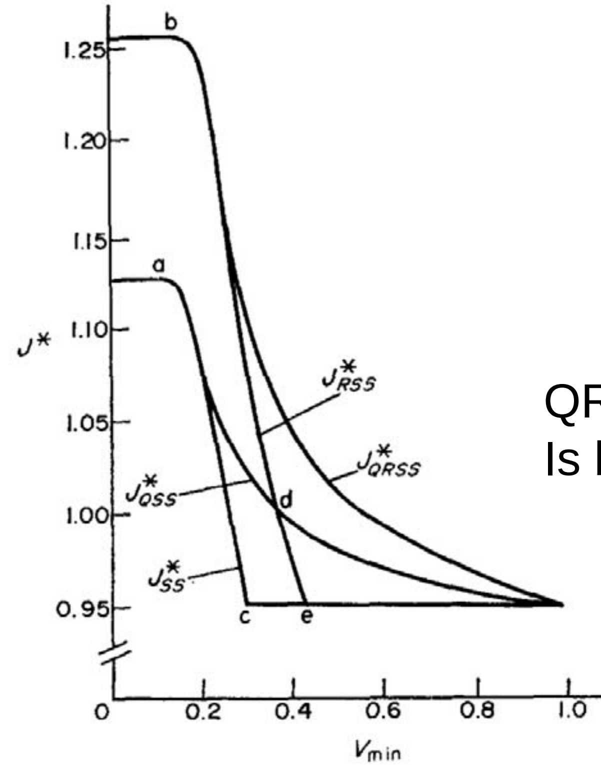
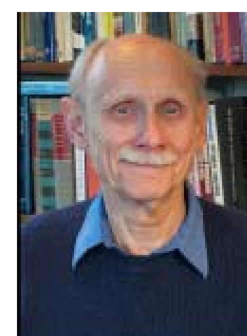


FIG. 2. Optimal cost functions for Example 1.

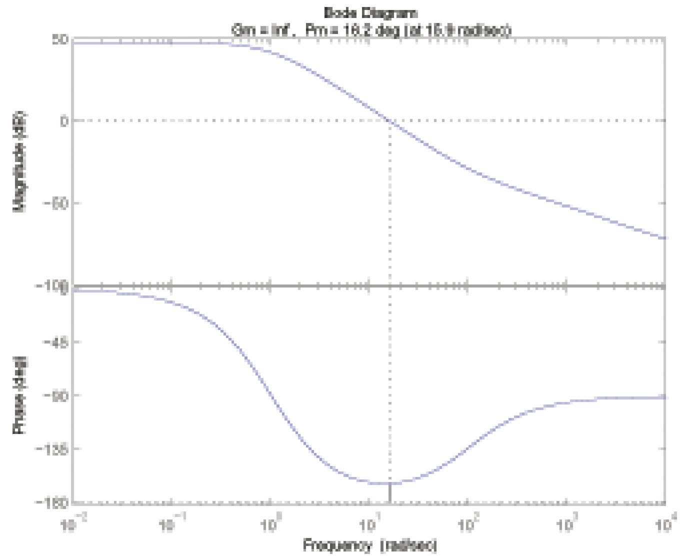
QRSS
Is best

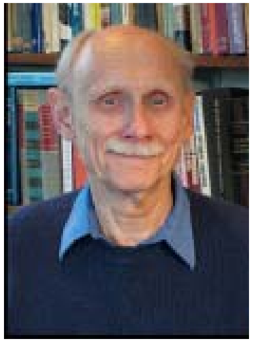


Evolving Horizons in Systems and Control

Forced Periodicity: Linear Case

- Fundamental theorem of linear systems
  - Sinusoidal forcing of an AS LTI systems is eventually periodic
  - Gain and phase captured by Bode plot



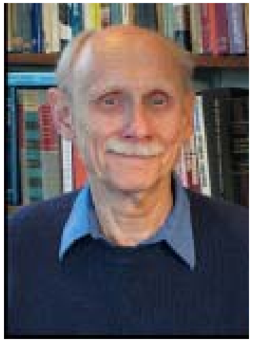


Evolving Horizons in Systems and Control

Forced Periodicity: Nonlinear Case

- Periodic forcing is a special case of periodically time-varying dynamics
- When does a nonlinear system have the property “periodic-input/eventually-periodic-output”?
- Subharmonic, superharmonic, and nonperiodic solutions may exist
- Extremely complex problem

$$\begin{aligned}\dot{x} &= f(x, u)\\ \dot{x} &= f(x, \sin \omega t)\\ \dot{x} &= f(x, t)\end{aligned}$$

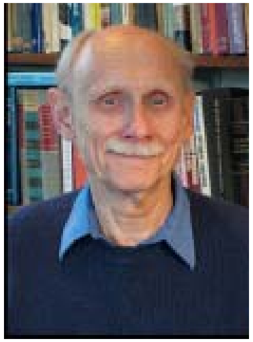


Local Improvement

- Dynamics and cost $\dot{x} = f(x,u) \quad J = \frac{1}{\tau}\int_0^\tau g(x,u)dt$
- Find optimal steady state solution $(\bar{x},\bar{u})$
- Linearize the cost and dynamics $\delta\dot{x} = A\delta x + B\delta u$
- Note $\delta J = 0$
- Evaluate $\delta^2 J$ with $\delta u = \sin\omega t$
- Find ω such that $\Pi(\omega) < 0$

$$\Pi(\omega) = G^*(\omega)L_{xx}G(\omega) + G^*(\omega)L_{xu} + L_{ux}G(\omega) + L_{uu}$$

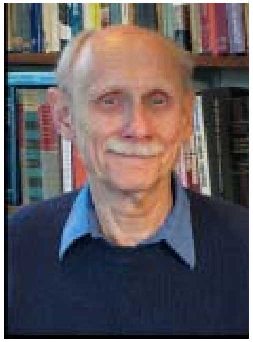
  - Pi Test----Guardabassi
- Then periodic control can locally do better than constant control



Evolving Horizons in Systems and Control

Why Periodic Control?

- Periodic control is necessitated by
  - Unassignable equilibria
  - Constraints
  - Nonconvex velocity set
- Periodic control can do dramatically better than constant control
  - Dynamic soaring
- Periodic control ensures sustainability
- Nature and humans have discovered these advantages and benefits



Evolving Horizons

- Thank you, Elmer, for your constant guidance and wisdom, and for always setting the highest example of scholarship and integrity

Evolving Horizons in Systems and Control