Mysteries and Conundra in the Meaning and Use of Physical Dimensions

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Nondimensionalization

Navier-Stokes with Physical Parameters "f = ma"

$$\rho D_t \vec{v} = -\nabla p + \mu \nabla^2 \vec{v} + \vec{f}$$

$\vec{v}$ = velocity, p = pressure, $\vec{f}$ = body force, ρ = density, μ = dynamic viscosity

Navier-Stokes with Dimensionless Parameters

$$D_{t'} \vec{v}' = -\nabla' p' + \frac{1}{\mathrm{Re}} \nabla'^2 \vec{v}' + \vec{f}'$$

$t' \triangleq \frac{V}{L} t$, $\vec{v}' \triangleq \frac{1}{V} \vec{v}$, $p' \triangleq \frac{1}{\rho V^2} p$, $\vec{f}' \triangleq \frac{L}{\rho V^2} \vec{f}$, $\nabla' \triangleq L \nabla$, $\mathrm{Re} \triangleq \frac{\rho L V}{\mu}$
V = mean velocity, L = characteristic length

Dimensionless Parameter Reynolds Number Re Characterizes the Dynamics

Re = $\frac{\rho L^2 v^2}{\mu v L}$ = ratio of inertial forces to viscous forces
Determines laminar and turbulent flow regimes

Assuming PDE Is a Complete and Correct Description of the Dynamics

Two flows with the same Re have solutions that are identical except for scaling

Other Dimensionless Parameters

Fluid Dynamics

Reynolds number, Strouhal number, Schmidt number, Froude number
Drag and lift coefficients C_D and C_L
Pitch moment derivative C_{m_α}

Dynamics

Damping ratio $\zeta = \frac{c}{2\sqrt{km}}$
Amplification factor $Q = 1/\zeta$

Optics: Index of refraction, Fresnel number, f-number

Geometry: Radian

Key Question

Can we determine the “correct” set of dimensionless parameters when we do NOT know the underlying equations?

Scale Invariance

Dynamic Similarity

Two phenomena are *dynamically similar* if all of their dimensionless parameters are identical and completely characterize the dynamics

Dynamically similar systems "behave in the same way"

Applications

Scale models—aircraft testing in wind tunnels

Note: Scaling may be "incorrect" if crucial effects (such as surface roughness or material strength) are ignored

On Being the Right Size—J. B. S. Haldane, 1928

An insect doesn't fear gravity but may be threatened by surface tension of water

Heat loss is proportional to surface area $\Longrightarrow$ Few small animals in cold regions

Weight increases as the cube of size $\Longrightarrow$ Huge monster bugs are unlikely

These are examples of *incorrect* scaling

Historical Perspective

Zeno—5th Century BC

A lagging body cannot overcome a leading body when both are in motion

Autolycus—3rd Century BC

Uniform velocity means equal distances in equal times—geometric view

Gerard of Brussels 13th Century

Studied Euclid and Archimedes
Wrote first Latin treatise on kinematics (Liber de Motu—Book of Motion)
"The proportion of the movements of a point is that of the lines described in the same time"
Motion is no longer a ratio of lengths, but rather a ratio of length to time!!

Historical Perspective

Oxford Calculators—14th Century

Tractatus de Proportionibus Velocitatum (Thomas Bradwardine and 3 others)
Experimental result found by rolling objects down inclined planes:
"A body moving with constant velocity travels the same distance as an accelerated body in the same time if its velocity is half the final speed of the accelerated body"

Using Newton's Calculus

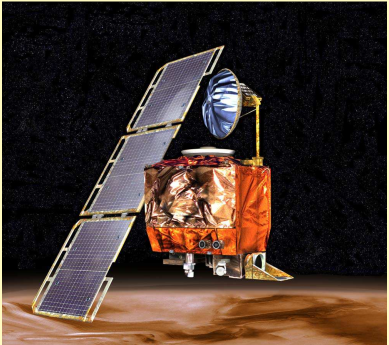
Start from rest with constant acceleration
$\dot{v}(t) = a \implies v(t) = at$
$\dot{x}(t) = v(t) \implies x(t) = \int_0^t v(s)\mathrm{d}s = \frac{1}{2}at^2$
Hence $x(t_f) = \frac{1}{2}at_f^2$
Restart from rest with constant velocity $\bar{v} = \frac{1}{2}v(t_f) = \frac{1}{2}at_f$
$\implies \bar{v}t_f = \frac{1}{2}at_f^2 = x(t_f)$!!

Units

Numbers and Voltages

- Matlab works only with numbers
- Most measurements are voltages
- Effort is needed to correctly relate numbers and voltages to units

Mars Climate Orbiter: Lost in 1999 due to a mixup between Newtons and lbf (1 lbf = 4.45 N)



Why Care About Dimensions?

Dimensions Give Meaning to Numbers

Units relate numbers to dimensioned quantities.

Dimensions Facilitate Correct Modeling

Working with dimensions can suggest the correct model and help find errors.

Dimensions Connect Physics to Math

- Physics is what the real world does
- Math is a method for modeling the real world
- Dimensions and units connect the real world to mathematical models
- Bridging this gap is our main intellectual challenge

Outline

How Do We Find Dimensionless Parameters?

Buckingham Pi Theorem and why it works

Dimensions and Matrices?

What kinds of dimensions can matrices have? Eigenvalues? Eigenvectors?

Dimensionless Parameters

Can we just ignore the "dimensions" of dimensionless parameters?
Are dimensionless parameters really "dimensionless"?

Questions Arising

Can we take the log of 3 meters?
What is a radian? What is an angle?
Does $\jmath \triangleq \sqrt{-1}$ have dimensions?
Does $V(x, \dot{x}) = x^2 + \dot{x}^2$ make sense?
Is 1 m + 5 sec "legal"?

Dimensions

Mechanical Dimensions

Basic: Length, mass, time

Composite: Velocity, acceleration, momentum, force, moment, pressure, energy, power, angular velocity, angular acceleration, frequency, kinematic viscosity, dynamic viscosity, torsional rigidity, acoustic impedance

Electrical Dimensions

Basic: Charge

Composite: Voltage, current, electric field strength, electric flux, resistance, capacitance, inductance, magnetic field strength, magnetic vector potential, magnetic flux

Other Scientific Dimensions

Basic: Temperature, radiation, illumination

Note: Temperature can be absolute or relative

Units

Mechanical Units

Basic: meter (m), kilogram (kg), second (sec)
Velocity: m/sec [Zeno (Z)]
Acceleration: m/sec^2 [Galileo (G)] (1 g = 9.8 m/sec^2)
Force: 1 $kg\text{-}m/sec^2$ [Newton (N)]
Energy: $kg\text{-}m^2/sec^2$ [Joule (J)]
Power: $kg\text{-}m^2/sec^3$ [Watt (W)]
Frequency: Hertz (Hz = cycle/sec), Steinmetz (Sz = rad/sec)
Other units named for Pascal ($Pa=N/m^2$), Poise(uille), Rayl(eigh), Stokes (St = cm^2/sec)

Electrical Units

Basic: coulomb (C)
Voltage: J/C [Volt(a) (V)]
Current: C/sec [Amp(ere) (A)]
Magnetic flux: $kg\text{-}m/(sec^2\text{-}C)$ [Weber (Wb)]
Other units named for Gauss, Ohm, Henry, Oersted, Maxwell, Faraday, Tesla

Joseph Henry 1797–1878

More Dimensions and Units

Other Scientific Units

Temperature: Kelvin, Celsius, Fahrenheit, Rankine
Radiation: Curie, Sievert, Becquerel
Illumination: candela, lumen, lux

Non-Scientific Dimensions

Money, people, apples, bits, flops, samples, moles

Non-Scientific Units

Man-hour, Gigaflop, byte, dollar, Euro, credit hour
Platoon, company, battalion, brigade, division, corps, army

Dimensioned Quantities

Example

$\ell =$ length of a football field
Dimension: length
Dimensioned representation: $\ell =$ 100 yards = 300 feet
ℓ has two parts: 1) numerical part "$\mathrm{num}(\ell)$", 2) units part "$[\ell]$"
$\mathrm{num}(\ell)$ and $[\ell]$ are not unique; but we view them as unique by specific a set of units, such as kg, m, sec
It is not necessary to assign units until numbers are needed

Multiplicative Relationship

Numerical part and units part are multiplicative: $\ell =$ 100 yards = 100 (3 feet) = 300 feet
Multiplicative Law for Units: $[xy] = [x][y]$

Exception

Exception: Frequency $\omega = 2\pi\nu = \frac{2\pi}{T}$
These symbols are universally associated with specific units:
$[\omega] =$rad/sec, $[\nu] =$ cycles/sec, $[T] =$ sec

Dimensional Analysis for Word Problems

Example

You wish to drive $d = 75$ miles. Your speed will be $v = 30$ miles/hour. How long will the trip be?

Time of trip $= d/v = 75\,\cancel{\text{miles}} \times \frac{1\ \text{hour}}{30\ \cancel{\text{miles}}} = 2.5$ hours

Computing with Dimensioned Quantities

Computations are performed with numbers and with units suppressed

What does $3.2 \times 2.9 = 9.28$ represent?

Carrying out computations with dimensioned quantities would add clarity and reduce the potential for errors

Dimensional Analysis for Functional Dependence

Example

A pendulum has a mass of $M = 2$ kg attached to the end of a rope whose length is $\ell = 3$ m. The acceleration due to gravity is $g = 9.8$ m/sec^2. What is the period T of oscillation?
$\left[\sqrt{\frac{l}{g}}\right] = \sqrt{\frac{[l]}{[g]}} = \sqrt{\frac{\mathrm{m}}{\frac{\mathrm{m}}{\mathrm{sec}^2}}} = \mathrm{sec} = [T] \Longrightarrow T = \Pi\sqrt{\frac{l}{g}}$ where $\Pi = T\sqrt{\frac{g}{l}}$ is a dimensionless constant

Result

We learned the functional dependence of T on g and ℓ without knowing *any* physics
We found that T does not depend on M—period is independent of mass

Dynamic Similarity

Two pendula that have the same g/l must have the same T

Think of a longer pendulum on Jupiter

Π is an *invariant* and the only dimensionless parameter in $\ddot{\theta} + \frac{g}{l}\sin\theta = 0$

Like Re in Navier-Stokes

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Pendulum Example With Matrices

Dimensioned Quantities

$T, \ell, g, M \Longrightarrow q = 4$ dimensioned quantities, $p = 3$ basic dimensions

Dimensionless Constants

$$\Pi = T^a \ell^b g^c M^d = \mathrm{sec}^a \mathrm{m}^b (\mathrm{m}/\mathrm{sec}^2)^c \mathrm{kg}^d = \mathrm{sec}^{a-2c} \mathrm{m}^{b+c} \mathrm{kg}^d$$

Π dimensionless $\Longrightarrow a - 2c = 0, b + c = 0, d = 0$

Matrix Form

$$\begin{bmatrix} 0\ \mathrm{kg}^0 \\ 0\ \mathrm{m}^0 \\ 0\ \mathrm{sec}^0 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{A} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

3 equations in 4 unknowns

A solution $x = (a, b, c, d)$ must satisfy $Ax = 0$

How many solutions does this equation have?

Matrix Analysis

Matrix Property

A solution $x = (a, b, c, d)$ must be an element of the null space of the matrix A
$\text{rank } A =$ number of linearly independent rows or columns of A
$\text{defect } A =$ number of linearly independent vectors in the null space of A
Fact: $\text{rank } A + \text{defect } A = \text{dimension of the domain of } A$

Proof: There exist nonsingular matrices R, S such that $RAS = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$

$$\begin{bmatrix} x_1 \\ 0 \end{bmatrix} = \begin{bmatrix} \overbrace{I}^{\text{rank}} & \overbrace{0}^{\text{defect}} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Apply Matrix Result

Each dimensionless parameter is given by a vector in the null space of A
Let $N =$ number of dimensionless parameters
Let $p =$ number of physical parameters
$N = p - (\text{rank } A)$
Example: $p = 5$ physical parameters depending on mass, length, and time ($D = 3$) implies at least $N = 2$ dimensionless parameter

(Edgar) Buckingham Pi Theorem (1867–1940)

Buckingham Pi Theorem (1914)

A physical meaningful law of the form

$$f(x_1, \ldots, x_p) = 0$$

is equivalent to a law of the form

$$\phi(\Pi_1, \ldots, \Pi_N) = 0$$

$\Longrightarrow$ rank A are removed, and the law is recast in terms of dimensionless parameters
"Physically meaningful" means that the function f is invariant to a change in units.
Proof: The proof is subtle and has been reworked many times

Example

d = distance traveled in time T due to acceleration g
$\Longrightarrow d = \frac{1}{2}gT^2 \Longrightarrow f(d, g, T) = d - \frac{1}{2}gT^2 = 0$
$\Pi_1 = \frac{gT^2}{d} \Longrightarrow 1 - \frac{1}{2}\Pi_1 = 0 \Longleftrightarrow \phi(\Pi_1) = 2 - \Pi_1 = 0$

Pendulum Solution with Matrices

Pendulum

$$A = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Longrightarrow \text{rank } A = 3 \Longrightarrow \text{defect } A = 4 - 3 = 1$$

Solution $(a, b, c, d) = (2, -1, 1, 0)$

Note: We seek solutions in integers or rational numbers

Π_1

$\Longrightarrow \Pi_1 = T^2 \ell^{-1} g^1 M^0 = \frac{T^2 g}{\ell}$ dimensionless

$$\Longrightarrow T^2 = \frac{\Pi_1 \ell}{g} \Longrightarrow T = \Pi_1' \sqrt{\frac{\ell}{g}}$$

How Can We Find Solutions (Dimensionless Parameters) Systematically?

Propeller Example

Problem

What is the force generated by a propeller on an aircraft?

6 Physical Parameters

Propeller force f, diameter d of the propeller, velocity v of the airplane, density ρ of the air, rotation speed N of the propeller, kinematic viscosity ν (m^2/sec) of the air

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & -3 & 0 & 2 \\ -2 & 0 & -1 & 0 & -1 & -1 \end{bmatrix} \Longrightarrow \operatorname{rank} A = 3 \Longrightarrow \operatorname{defect} A = 6 - 3 = 3$$

How to find solutions?

Basis Method for Finding Solutions

Choose 3 Physical Parameters that Span the Fundamental Dimensions

Choose d, v, f

kg $= [\frac{fd}{v^2}]$

m $= [d]$

sec $= [\frac{d}{v}]$

Express the Remaining Physical parameters in terms of d, v, f

$$[\nu] = \left[\frac{d^2}{d/v}\right] = [dv]$$

$$[N] = \left[\frac{1}{d/v}\right] = [\frac{v}{d}]$$

$$[\rho] = \left[\frac{fd/v^2}{d^3}\right] = \left[\frac{f}{d^2v^2}\right]$$

3 Dimensionless Parameters

$\Pi_1 = \frac{dv}{\nu}$ Reynolds number

$\Pi_2 = \frac{dN}{v}$ top-speed ratio

$\Pi_3 = \frac{f}{d^2v^2\rho}$ dynamic-force ratio

Matrix Powers

Dimensioned Entries of a Matrix

$$m\ddot{q} + kq = 0 \Longrightarrow \underbrace{\begin{bmatrix} \dot{q} \\ \ddot{q} \end{bmatrix}}_{\dot{x}} \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} q \\ \dot{q} \end{bmatrix}}_{x} \quad [x] = \begin{bmatrix} \mathrm{m} \\ \frac{\mathrm{m}}{\mathrm{sec}} \end{bmatrix} \Longrightarrow [A] = \begin{bmatrix} \frac{1}{\mathrm{sec}} & [\,] \\ \frac{1}{\mathrm{sec}^2} & \frac{1}{\mathrm{sec}} \end{bmatrix}$$

Can we square A?

Note that: $[A] = \begin{bmatrix} \frac{1}{\mathrm{sec}} \\ \frac{1}{\mathrm{sec}^2} \end{bmatrix} \begin{bmatrix} 1 & \mathrm{sec} \end{bmatrix}$

$$\Longrightarrow [A]^2 = \begin{bmatrix} \frac{1}{\mathrm{sec}} \\ \frac{1}{\mathrm{sec}^2} \end{bmatrix} \underbrace{\begin{bmatrix} 1 & \mathrm{sec} \end{bmatrix} \begin{bmatrix} \frac{1}{\mathrm{sec}} \\ \frac{1}{\mathrm{sec}^2} \end{bmatrix}}_{\text{works!}} \begin{bmatrix} 1 & \mathrm{sec} \end{bmatrix} = \frac{1}{\mathrm{sec}}[A]$$

$[A^r]$ exists for all $r \Longrightarrow$ can compute matrix exponential e^{At}

Arbitrary Square A

$[A^2]$ exists if and only if $[A] = zw^{\mathrm{T}}$ and $w^{\mathrm{T}}z$ exists

Determinants

Mass-Spring Example

$$[A] = \begin{bmatrix} \frac{1}{\text{sec}} \\ \frac{1}{\text{sec}^2} \end{bmatrix} \begin{bmatrix} 1 & \text{sec} \end{bmatrix}$$

$\Longrightarrow \det[A] = \frac{1}{\text{sec}} \cdot \frac{1}{\text{sec}} - [\,] \cdot \frac{1}{\text{sec}^2}$ Exists

Arbitrary Square Matrix

$\det[A]$ exists if and only if $[A] = zw^{\mathrm{T}} \Longleftrightarrow$ outer product of dimension vectors

Fact: If $[A^2]$ exists, then $\det[A]$ exists

$\Longrightarrow \det[A]$ existence is a weaker condition

The converse is not true: $[A] = \begin{bmatrix} \text{m} & \text{m}^2 \\ \text{s} & \text{m-s} \end{bmatrix}$

Inverse

$A^{-1} = \frac{1}{\det A} A^{\text{adj}} \Longleftrightarrow \det[A]$ and $[A^{\text{adj}}]$ exist

Warning: Can happen that $[A][A^{-1}] \neq [A^{-1}][A]$

State Space Models

Dynamics and Output Equations

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

Dimensions of Coefficient Matrices

$$[A] = \frac{1}{\text{sec}}[x][x^{\mathrm{T}}]^{\{-1\}}$$
$$[B] = \frac{1}{\text{sec}}[x][u^{\mathrm{T}}]^{\{-1\}}$$
$$[C] = [y][x^{\mathrm{T}}]^{\{-1\}}$$
$$[D] = [y][u^{\mathrm{T}}]^{\{-1\}}$$

Transfer Functions Have Dimensions

$$H(s) = C(sI - A)^{-1}B + D$$
$$[H] = [y][u^{\mathrm{T}}]^{\{-1\}}$$

Note: $[s] = [\jmath\omega] = \frac{1}{\text{sec}}$

Eigenvectors and Eigenvalues

Eigenvalues

$\dot{x} = Ax \Longrightarrow [A_{i,i}] = \frac{1}{\text{sec}}$
Form $\lambda I - A \Longrightarrow [\lambda] = [A_{i,i}] = \frac{1}{\text{sec}}$

Eigenvectors

$Av = \lambda v \Longrightarrow [A][v] = [\lambda][v] \Longrightarrow \frac{1}{\text{sec}}[x][x^{\mathrm{T}}]^{\{-1\}}[v] = \frac{1}{\text{sec}}[v]$
$\Longrightarrow [x][x^{\mathrm{T}}]^{\{-1\}}[v] = [v]$
Can show: $\Longrightarrow [v] = (\text{arbitrary dimension}) \times [x]$
We usually take $[v] = [x]$

Arbitrary A—Not Necessarily State Space

Must require $[A_{i,i}] = [A_{j,j}]$ for all i, j

Groups

Definition

A group is a set of objects and a rule for combining those objects in pairs assuming that there is an identity element and every object has an inverse.

Numerical Examples

Real or complex numbers with addition ($x + 0 = x$) (∞ elements)
Nonzero real or complex numbers with multiplication ($1 \cdot x = x$) (∞ elements)
$\{1, -1\}$ (2 elements)
$\{1, -1, \jmath, -\jmath\}$ (4 elements)
The quaternions $q = a + b\imath + c\jmath + dk$ (∞ elements)

Geometric Examples

A set of functions or operations that are all invertible—-Especially a set of permutations
Rotating a square 90 deg CW or CCW (4 elements)
Rotating a cube 90 deg CW or CCW about 3 axes (24 elements)
Rotating Rubik's cube (43,252,003,274,489,856,000 $\approx$ 43 quintillion elements)

Klein 4-Group

Mattress Flipping Group

Flipping a mattress 180 deg about 3 axes
4 elements——-1 identity plus 3 square roots of the identity
All elements commute

Groups and Dimensions

Examples of Groups of Dimensions

$\{\mathrm{m}^0, \mathrm{m}^1, \mathrm{m}^{-1}, \mathrm{m}^2, \mathrm{m}^{-2}, \ldots\}$
$\{\mathrm{kg}^0\text{-}\mathrm{m}^0, \mathrm{kg}^0\text{-}\mathrm{m}^1, \mathrm{kg}^1\text{-}\mathrm{m}^0, \mathrm{kg}^1\text{-}\mathrm{m}^1, \ldots\}$

Dimensionless Parameters Provide the Identity Element of Each Group

$\mathrm{m}^0 = \frac{\mathrm{m}}{\mathrm{m}}$ length ratio

strain $= \frac{\Delta L}{L}$

radian $= \frac{\text{arc length}}{\text{radius}}$

kg^0 mass ratio (massian)

mass ratio = $\frac{\text{relativistic mass}}{\text{rest mass}}$

N^0 force ratio (forcian)

Reynolds number Re = $\frac{\rho L^2 v^2}{\mu v L} = \frac{\text{inertial force}}{\text{viscous force}}$

Coefficient of friction $\mu = \frac{\text{friction force}}{\text{normal force}}$

$\mathrm{m}^0\text{-}\mathrm{sec}^0$ velocity ratio (velocian)

Mach number = $\frac{\text{vehicle speed}}{\text{sound speed}}$

Angles

Triangle Invariance

The ratios of the sides of triangles are invariant—-independent of size

For a right triangle:

$$\sin\theta \triangleq \frac{\text{length of opposite}}{\text{length of hypotenuse}}$$

Therefore, $[\sin\theta] = \frac{\mathrm{m}}{\mathrm{m}} = \mathrm{m}^0 = [\,] \implies \theta$ is dimensionless–an invariant of the triangle

θ is the “angle” that defines “opposite”——-But what is an “angle”?

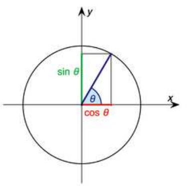
Arc Length

Angle as Arc Length on a Circle

Let's say that an angle θ is the arc length on a circle "beyond" the "opposite" scaled by the hypotenuse (radius)

The scaled arc length ranges from 0 to 2π

$\sin(\theta) =$ vertical component of the point on the circumference divided by the radius



Radians

We measure the scaled arc length in radians, where 2π radians is one circle

Radians are dimensionless—but may have units (e.g., degrees) since they have a natural scale–2π

3 Questions About Radians

1. Mass on a Spring

Consider translational motion of a mass on a spring (nothing rotating)

$m\ddot{q} + kq = 0 \implies q(t) = \sin \omega_{\mathrm{n}} t$, where $\omega_{\mathrm{n}} \triangleq \sqrt{\frac{k}{m}}$

Units: $[\omega_{\mathrm{n}}] = \frac{1}{\mathrm{sec}} \implies \frac{\mathrm{rad}}{\mathrm{sec}}$?

Need radians to use the sine function, but where's the circle?

2. [Energy] = [Moment]??

A unit of energy is the Joule, which is 1 N-m = 1 kg-$\mathrm{m}^2/\mathrm{s}^2$

A unit of moment is the N-m, which is 1 kg-$\mathrm{m}^2/\mathrm{s}^2$

We know moment $\times$ angle = energy but N-m-m^0 = N-m

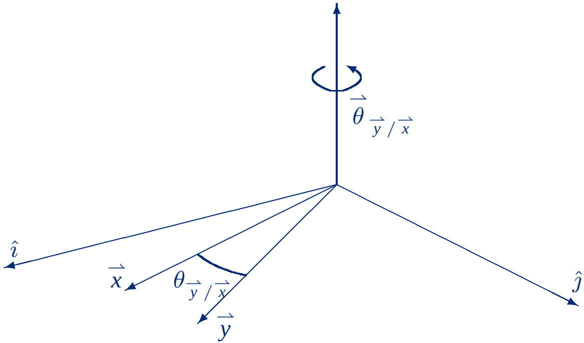
3. Is a Radian Really m/m?

[scaled arc length] = $\frac{\mathrm{m}}{\mathrm{m}} = \mathrm{m}^0 = [\ ]$

Does arc length cancel with radius?

No an issue for strain $\Delta L/L$ where ΔL and L are in the same direction

Angle Vector



Work Done by a Moment Applied to a Rigid Body

Moments are represented by cross products: $\vec{M} = \vec{r} \times \vec{f}$

Work = energy transferred to the rigid body = $\vec{M} \cdot \vec{\theta}$

$\vec{\theta}$ is the angle vector about which the body rotates

The Exponential

For Real x

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

Adding x and x^2 suggests that x is dimensionless

Euler's Formula for $x = \jmath\theta$

$e^{\jmath\theta} = \cos\theta + \jmath\sin\theta$

$\cos\theta = 1 - \frac{\theta^2}{2!} + \cdots$

$\sin\theta = \theta - \frac{\theta^3}{3!} + \cdots$

$\implies [\jmath][\sin\theta] = [\cos\theta] = [\,] \implies [\jmath] = [\,]$

What Does $\jmath$ Do?

Multiplication by $\jmath$ rotates complex numbers by 90 degrees

Does not change length, just direction—like a rotation matrix

Projectile Example (Huntley, 1952)

Problem Statement

A projectile is fired horizontally from a height h, with an initial velocity v_0.
What is its range R?

Dimensioned Parameters

$[h] = \mathrm{m}$, $[v_0] = \mathrm{m/sec}$, $[R] = \mathrm{m}$, $[g] = \mathrm{m/sec}^2$

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & -2 \end{bmatrix} \Longrightarrow \mathrm{rank}\, A = 2 \Longrightarrow \mathrm{defect}\, A = 4 - 2 = 2$$

Solutions are $(-1, 0, 1, 0)$ and $(-1, 2, 0, -1)$

$\Longrightarrow \Pi_1 = \frac{h}{R}$ and $\Pi_2 = \frac{v_0^2}{hg}$

If we maintain these dimensionless constants, then we expect to maintain dynamic similarity

But, do we really need to maintain BOTH of these?

Projectile Example Second Try

Horizontal $\neq$ Vertical $\implies$ 3 Dimensions

$\mathrm{m}_x \triangleq$ "horizontal" meter (horizontal length dimension)

$\mathrm{m}_z \triangleq$ "vertical" meter (vertical length dimension)

$[h] = \mathrm{m}_z$, $[v_0] = \mathrm{m}_x/\mathrm{sec}$, $[R] = \mathrm{m}_x$, $[g] = \mathrm{m}_z/\mathrm{sec}^2$

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & -2 \end{bmatrix} \implies \text{rank } A = 2 \implies \text{defect } A = 4 - 3 = 1$$

Unique solution is $(1, 2, -2, -1)$

$\implies \Pi_3 = \dfrac{hv_0^2}{R^2 g} \implies R = \Pi_3 v_0 \sqrt{\dfrac{h}{g}}$ Only 1 dimensionless constant now!

$(1, 2, -2, -1) = 2(1, 0, -1, 0) + (-1, 2, 0, -1)$ is a linear combination of previous solutions $\implies \Pi_3 = \Pi_1^2 \Pi_2^2$

$\implies$ We used more information to find a *single less-restrictive* vector in the null space of A

Note: R does not need to scale with h

Huntley 1952: “Vector Lengths”

We Found:

By viewing all lengths as having the same dimension, we obtained two correct but unnecessarily restrictive dimensionless parameters

We mixed up vertical and horizontal directions

Dimensional analysis seems to benefit from a distinction between vertical displacement and horizontal displacement

“Vector Lengths”

1 m North + 1 m East = $\sqrt{2}$ m NE

It appears that “vector lengths” are essential

Radians as a Physical Parameter

Suppose that the projectile is launched at an angle α to the horizontal.

How can dimensionless data be used to determine dimensionless constants?

Angles Revisited

Angles as Ratios

Let views angles as $[\theta] = \dfrac{\mathrm{m}_x}{\mathrm{m}_y}$

θ is dimensionless but has orientation

What Is Orientation?

Orientation is an *attribute* of a dimension

Denote orientations by $1_0, 1_x, 1_y, 1_z$

- 1_0 means no orientation
- $1_x, 1_y, 1_z$ denote orthogonal directions

Orientation Group (Siano, 1985)

Group Multiplication Rules

$1_0 1_0 = 1_0$, $1_0 1_x = 1_x$, $1_0 1_y = 1_y$, $1_0 1_z = 1_z$ (identity element)
$1_x 1_y = 1_y 1_x$, $1_y 1_z = 1_z 1_y$, $1_z 1_x = 1_x 1_z$ (commuting property)
$1_x^2 = 1_y^2 = 1_z^2 = 1_0$ (involutory property)
$1_x 1_y = 1_z$, $1_z 1_x = 1_y$, $1_z 1_x = 1_y$ (cyclic property)

Which Group Is This?

$\{1_0, 1_x, 1_y, 1_z\}$ is the *Klein 4-group*——the mattress flipping group

Trigonometry and Orientation

Orientation Notation

$\langle x \rangle$ denotes the orientation of x

What about Angles?

$\langle \theta \rangle = \langle m_x \rangle / \langle m_y \rangle = 1_x / 1_y = 1_z$ Reminiscent of angle vector

$\langle \sin\theta \rangle = \langle \theta - \frac{1}{3!}\theta^3 \rangle = 1_z = 1_z^3 = 1_z$

$\langle \cos\theta \rangle = \langle 1 - \frac{1}{2}\theta^2 \rangle = 1_0 = 1_z^2 = 1_0$

$\Longrightarrow \sin\theta$ has orientation; $\cos\theta$ has no orientation

$\Longrightarrow \langle \sin^2\theta + \cos^2\theta \rangle$ exists

$\langle \sin\theta + \cos\theta \rangle$ does NOT exist

Complex Exponential

$e^{\jmath\theta} = \cos\theta + \jmath\cos\theta$

$\langle e^{\jmath\theta} \rangle = \langle \cos\theta \rangle + \langle \jmath \rangle \langle \cos\theta \rangle = 1_z + \langle \jmath \rangle$

$\Longrightarrow \jmath$ has orientation $1_z \Longrightarrow$ consistent with rotation about an angle vector

e^{θ} does not exist—–must form $e^{\jmath\theta}$ since $\langle \jmath\theta \rangle = 1_0$

Orientations

Area and Volume

Area has orientation: $1_x 1_y = 1_z$ normal to surface
Volume does NOT have orientation: $1_x 1_y 1_z = 1_z^2 = 1_0$

Velocity

Velocity has vector orientation: $\langle \text{velocity} \rangle = \left\langle \begin{bmatrix} Z1_x \\ Z1_y \\ Z1_z \end{bmatrix} \right\rangle = \begin{bmatrix} 1_x \\ 1_y \\ 1_z \end{bmatrix}$

Force, Moment, and Pressure

Force has vector orientation: $\langle \text{force} \rangle = \left\langle \begin{bmatrix} N1_x \\ N1_y \\ N1_z \end{bmatrix} \right\rangle = \begin{bmatrix} 1_x \\ 1_y \\ 1_z \end{bmatrix}$

Moment has vector orientation: $\langle \vec{M} \rangle = \left\langle \begin{bmatrix} \text{N-m}1_x \\ \text{N-m}1_y \\ \text{N-m}1_z \end{bmatrix} \right\rangle = \begin{bmatrix} 1_x \\ 1_y \\ 1_z \end{bmatrix}$

Pressure $= \frac{\text{normal force}}{\text{area}}$ does not have orientation: $1_z / 1_z = 1_0$

Energy and Moment

Orientation of Energy

$$\langle \text{energy} \rangle = \langle \vec{f} \cdot \vec{x} \rangle = \left\langle \vec{f}\Big|_{\text{A}} \cdot \vec{x}\Big|_{\text{A}} \right\rangle = \left\langle \begin{bmatrix} \text{N}1_x \\ \text{N}1_y \\ \text{N}1_z \end{bmatrix} \right\rangle \cdot \left\langle \begin{bmatrix} \text{m}1_x \\ \text{m}1_y \\ \text{m}1_z \end{bmatrix} \right\rangle = 1_x^2 + 1_y^2 + 1_z^2 = 1_0$$

Moment Versus Energy Revisited

$$\langle \text{energy} \rangle = \langle \vec{M} \cdot \vec{\theta} \rangle = \left\langle \begin{bmatrix} \text{N-m}1_x \\ \text{N-m}1_y \\ \text{N-m}1_z \end{bmatrix} \right\rangle \cdot \left\langle \begin{bmatrix} 1_x \\ 1_y \\ 1_z \end{bmatrix} \right\rangle = 1_x^2 + 1_y^2 + 1_z^2 = 1_0$$

$[\text{energy}] = [\text{moment}]$ but $\langle \text{energy} \rangle \neq \langle \text{moment} \rangle$

Buckingham Pi Theorem with Orientations

Principle

Dimensionless parameters should also be orientationless
Yields parity conditions on the exponents in the dimensionless/orientationless parameters

Projectile Example Third Try

$\Pi = h^a v_0^b R^c g^d$

$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & -2 \end{bmatrix} \Longrightarrow a+b+c+d=0$ and $b+2d=0$

Orientations: $1_0 = 1_z^a 1_x^b 1_x^c 1_z^d = 1_x^{b+c} 1_z^{a+d} \Longrightarrow a+d$ and $b+c$ are even

This eliminates one of the two restrictive solutions, namely, $(1, 0, -1, 0)$

The other restrictive solution $(a, b, c, d) = (-1, 2, 0, -1)$ is NOT ruled out!

The least restrictive solution is $(1, 2, -2, -1)$——How to determine this one uniquely?

Siano uses additional physical reasoning to determine the “correct” exponents

Remarks and Questions

Phase Angle

Apparently $\omega_{\mathrm{n}} = \sqrt{\frac{k}{m}}$ is the rate of change of a phase angle, which is not a geometric angle

ω_{n} has no orientation but has units rad/sec (!)

Electromagnetism

Electromagnetic parameters that involve length and area have orientation (e.g., current density)

Is “Orientation” the Final Word on “Vector Dimension”?

Orientation is one approach—perhaps not the whole story

Multivectors in geometric algebra may be another

- Vectors, bivectors, trivectors, exterior algebra due to Grassmann

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Final Remarks I

Dimensions, Units, Orientation

Dimensions connect mathematics to reality and vice versa
Discovering the “rules” and “laws” of dimensional analysis is an ongoing process
Some of these rules are quite subtle

What Did We Do?

Nothing we have said suggests that any grievous errors have been committed!
The goal is to *understand* how dimensions work as an aid to modeling the real world

Final Remarks II

Main Insight

Physics seems to greatly constrain the mathematical form of models
Apparently no physical law can have e^{θ} or $\cos(\theta) + \sin(\theta)$—and we have an idea why

Mathematics versus the Real World

- Mathematics can help us formulate rules for working with dimensions, but only the real world can reveal what those rules are
- We need to be wary about claiming what these “rules” are since we may never know what is or is not physically possible
- This applies to all “laws” of nature–Newton, thermodynamic, etc.