From Infancy to Potency:

Lyapunov’s Second Method and the Past, Present, and Future of Control Theory

Dennis S. Bernstein
Aleksandr Mikhailovich Lyapunov

- Father astronomer, 7 children, 3 survived

- Professor of mechanics at Kharkov and St. Petersburg
  - Research on orbital mechanics and probability theory

- A. M. Lyapunov died tragically at age 61

- Completed his doctoral dissertation in 1892 under Chebyshev
  - Stability of rotating fluids applied to celestial bodies
  - Formulated his first and second methods (L1M and L2M)

- French translation appeared in 1907 = 1892 + 15
  - English translation didn’t appear until 1992 = 1892 + 100
How Successful Is L2M?

- Many areas have a strong interest in stability theory
  - Classical dynamics
  - Structural dynamics
  - Fluid mechanics
  - Astrodynamics
  - Chemical kinetics
  - Biology
  - Economics
  - Control

While there are some notable applications of L2M outside of control, there are surprisingly few overwhelming successes.
Let’s First Review the Basics of L2M
Basic L2M

- Consider $\dot{x} = f(x)$ with equilibrium $x_e$
- Assume $x_e$ is a strict local minimizer of $V$

\[ \dot{V} \leq 0 \text{ implies } x_e \text{ is Lyapunov stable (LS)} \]

\[ \dot{V} < 0 \text{ implies } x_e \text{ is asymptotically stable (AS)} \]

- Want $V(x(t))$ nonincreasing or decreasing
  - $V$ “keeps” $x(t)$ bounded or “makes” $x(t) \to 0$
  - In fact, $V$ merely predicts the behavior of $x(t)$
Suppose $V$ is Radially Unbounded

- If $\dot{V} \leq 0$ and $V$ is radially unbounded, then all trajectories are bounded
  - No finite escape, and thus global existence

- If $\dot{V} < 0$ and $V$ is radially unbounded, then $x_\circ$ is globally AS (GAS)
  - GAS $\iff$ LS + global convergence

How can we construct useful $V$’s? 2 ways.
1) Use $x(t)$ to Construct $V$

- Persidskii, 1938; Massera, 1949; Malkin, 1952; Ura, 1959 (converse theory)

- For AS or GAS, if $f$ is locally Lipschitz, then we can construct $C^\infty$ $V$ with $\dot{V} < 0$

- For LS, continuous $V$ may NOT exist even if $f$ is $C^\infty$
  - But, if $f$ is locally Lipschitz, then we can construct lower SEMIcontinuous $V$ with $\dot{V} \leq 0$

- How is trajectory-based construction useful if $x(t)$ is not available?
  - Consider an approximate system with KNOWN trajectories
  - For example, linearize the system and construct

\[
V(x_0) = \int_0^\infty x^T(t)x(t)dt = x_0^TPx_0
\]

▲ This is L1M
2) Use $f$ to Construct $V$

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- There aren’t really very many useful methods!!

This explains the lack of success of L2M outside of control.
How to Define $\dot{V}$? 2 Ways.

1) Use the trajectory $x(t)$: Need $V$ lower semicontinuous

$$\dot{V}(\xi) \triangleq \limsup_{h \to 0} \frac{1}{h} [V(x(h, \xi)) - V(\xi)]$$

2) Use the vector field $f$: Need $V$ locally Lipschitz

$$\dot{V}(x) = \limsup_{h \to 0} \frac{1}{h} [V(x + hf(x)) - V(x)]$$

If $V$ is $C^1$ then

$$\dot{V}(x) = V'(x)f(x)$$
Stability is qualitative.

How Can We Quantify It?
How Can We Quantify LS?

- Suppose we can invert $\delta(\varepsilon)$ to obtain $\varepsilon(\delta)$
- If $\|x(0)\| < \delta$, then $\|x(t)\| < \varepsilon(\delta)$
  - This quantifies LS by means of a trajectory bound
  - But doesn’t use L2M
How Can We Quantify AS?

1) Use a sublevel set \( \{ x : V(x) \leq c \} \) to estimate the domain of attraction

- Based on L2M

2) Estimate the speed of convergence

- Can we use L2M?
  - Yes, ...
How Can We Improve Our Stability Predictions?
Use Upgrades!
How Can We Upgrade from LS to AS?

- Barbashin/Krasovskii, 1952; LaSalle, 1967

- DAMPED nonlinear oscillator

\[ m\ddot{q} + c\dot{q} + kq^3 = 0 \]

- Energy

\[ V(q, \dot{q}) = \frac{m}{2}\dot{q}^2 + \frac{k}{4}q^4 \]  
and  
\[ \dot{V}(q, \dot{q}) = -c\dot{q}^2 \leq 0 \]

- So we have LS

- The INVARIANCENCE PRINCIPLE

  upgrades LS to AS

- But \( V \) is “defective”

  - That is, \( \dot{V} \leq 0 \) but we DON’T have \( \dot{V} < 0 \)

But converse theory guarantees that a NONdefective \( V \) exists!!! : – ( 
How Can We Upgrade from AS to GAS?

Teel, 1992

- GAS is equivalent to AS + global FINITE-TIME convergence to a domain of attraction
- Nested saturation controller for the double integrator

\[ u = \psi(q, \dot{q}) = -\text{sat}_\varepsilon(q + \text{sat}_{\varepsilon/2}(q + \dot{q})) \]

- Splice the trajectories

- Note that we did NOT construct a radially unbounded $V$
  - Hence local $V$ is defective for GAS! : – (}
How Can We Upgrade the Speed of Convergence?

1) Show $V$ satisfies a differential inequality
\[ \dot{V} \leq \rho(V) \]

2) Then construct the COMPARISON system
\[ \dot{\eta} = \rho(\eta) \]

- If $\rho'(0) < 0$ then $V(x(t)) \rightarrow 0$ exponentially
  - If $\alpha \|x\|^2 \leq V(x) \leq \beta \|x\|^2$ then $x(t)$ converges exponentially

- If $\eta \rho(\eta) < 0$ and $\int_0^\xi \frac{d\eta}{\rho(\eta)} < \infty$ then $V(x(t)) \rightarrow 0$ in finite time
  - And thus $x(t)$ converges in finite time

Next: L2M and Control
What Was the Impact of L2M on CLASSICAL Control?
Maxwell’s Work on Stability

- Stability of Saturn’s rings
  - Quartic linearization
    - Fortunately, biquadratic – trivial

- Stability of governors
  - 5th-order linearization – not trivial
  - Obtained only necessary conditions
  - Motivated the Adams prize competition at Cambridge in 1875

Who won?
Routh

- Routh, 1877 = 1892 - 15

- Routh considered the stability of a general polynomial
  
  \[ p(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_0 \]

  - Derived a necessary and sufficient condition for stability
  - Note: No mention of control

- His derivation was based on the Cauchy index theorem
  - Not based on L2M
  - Predates L2M

Is there an L2M proof of the Routh test?
Yes! Parks, 1962 = 1892 + 70

1) Compute Routh table parameters \( 1, b_1, b_2, b_1b_3, b_2b_4, b_1b_3b_5, \ldots \)

2) Construct the tridiagonal Schwarz matrix \( A = \begin{bmatrix} 0 & 1 & 0 \\ -b_3 & 0 & 1 \\ 0 & -b_2 & -b_1 \end{bmatrix} \)

3) Solve the Lyapunov equation \( A^T P + PA + R = 0 \) with

\[
R = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2b_1^2 \end{bmatrix}
\]

for

\[
P = \begin{bmatrix} b_1b_2b_3 & 0 & 0 \\ 0 & b_1b_2 & 0 \\ 0 & 0 & b_1 \end{bmatrix}
\] > 0

4) Define \( V(x) = x^T P x \), which implies \( \dot{V}(x) = -x^T R x \)

5) Since \( R \geq 0 \), get \( \dot{V} \leq 0 \), which proves LS

6) Use invariance principle to upgrade to AS

\[
\text{Since } R \neq 0, \text{ after all this, } V \text{ is defective!!} : -(\]
Nyquist Test

- Nyquist, 1932 = 1892 + 40

- In 1927, Harold Black invented the negative feedback amplifier
  - Unlike positive feedback amplifiers, his circuit was stable
  - Patent office was skeptical and treated negative feedback like perpetual motion
    - They demanded a prototype!

- Nyquist test provided the crucial frequency domain insight
  - Note: Loop closure stability test

- Like Routh, his proof was based on the Cauchy index theorem
  - **NOT on L2M**

Why not?
Let’s Review Absolute Stability

- Lur’e/Postnikov, 1944

- Feedback interconnection

- $\phi(y, t)$ is a memoryless time-varying nonlinearity in a sector $\Phi$
Absolute Stability Tests

- **Bounded real (small gain)**
  \[ |G| < \frac{1}{F_1} \implies \text{GAS for all NLTV } \phi \in \Phi_{br} \]

- **Positivity**
  \[ \text{Re } G > -\frac{1}{F_1} \implies \text{GAS for all NLTV } \phi \in \Phi_{pr} \]

- **Circle**
  \[ \text{Re } \frac{G}{1+F_1G} > \frac{1}{F_1-F_0} \implies \text{GAS for all NLTV } \phi \in \Phi_c \]

- For each test \[ V(x) = x^T Px \]
  - Get \emph{P} from KYP conditions or a Riccati equation
  - \ldots and it’s the SAME \emph{V} for ALL \phi in the sector \phi

How do we REDUCE conservatism for time-INVARIANT \phi(\gamma)?
Introduce a Multiplier!

- Popov, 1961

- Insert $Z(s) = 1 + \alpha s$ to restrict the time variation of $\phi$
  - $\Re Z(s)G(s) \geq -1/F_1 \implies$ GAS for all NLTI $\phi \in \Phi$

- $V_\phi(x) = x^T P x + \alpha \int_0^y \phi(\sigma) d\sigma$
  - $V_\phi$ depends on $\phi$ so we actually have a FAMILY of $V$’s

- Furthermore, we can construct $Z$ to further restrict $\phi$
  - Slope bounded
  - Monotonic
  - Odd
How Can We Allow Only LINEAR $\phi(y) = Fy$

- Narendra, 1966; Brockett/Willems, 1967; Thathachar/Srinath, 1967

- Construct a SPECIAL $Z_G$ that DEPENDS on $G$

  - $\text{Re } Z_GG \geq -1/F_1 \iff \text{GAS for all } \phi(y) = Fy, F \in [0, F_1]$

- $V_F(x) = x^TPx + Fy^Ty$

  - A FAMILY of $V$'s

- This $V$ proves the Nyquist test,
  - and completes a long and fruitful application of L2M : – )

- But a L2M proof of MULTIVARIABLE Nyquist is open!

Next: Let’s include performance
Use Absolute Stability to Bound Performance

- Bernstein/Haddad, 1989

\[ z = G_F w \]

- Small gain uncertainty \( \sigma_{\max}(F') \leq \gamma \)
- \( G_F \sim (A + BFC, D, E) \)

- Construct \( V(x) = x^T P x \)
  - \( 0 = A^T P + PA + \gamma^2 PB B^T P + C^T C + E^T E \)
  - Then \( \|G_F\|_2 \leq \text{tr} \ D^T P D \) for all uncertain \( F \)
  - Guarantees robust stability with a bound on worst-case \( H_2 \) performance

Speaking of \( H_2 \), let’s turn to MODERN control
How HAS L2M Contributed to Modern Control?
Stabilization Based on Linearization Only

- \( \dot{x} = f(x, u), \quad C^1 f, \quad u = \psi(x), \quad \dot{x} = f(x, \psi(x)) \)
  - \( f(x_e, u_e) = 0 \) with linearization \( \dot{x} = Ax + Bu \)

- **Sufficient condition**
  - If \((A, B)\) is stabilizable, then \(x_e\) is AS’ble with \(C^\infty \psi\)
  - Not necessary: \( \dot{x} = -x^3 + xu \) is AS with \( u = \psi(x) = 0 \)

- **Necessary condition**
  - If \(x_e\) is AS’ble with \(C^1 \psi\), then \((A, B)\) is CLHP stabilizable
  - Not sufficient: \( \dot{x} = x^2 + x^3u \) has \( A = 0 \) and \( B = 0 \)

To do better, let’s use \( f \) directly
Stabilization Based on the Vector Field

- Brockett, 1983

- Necessary condition
  - If $x_e$ is AS’ble with $C^0 \psi$, then $0 \in \text{int } f(\mathcal{N}(x_e, u_e))$
  - Not sufficient: $\dot{x} = x + x^3 u$ (need discontinuous $\psi$)

- How can we use this result?
  - If $0 \notin \text{int } f(\mathcal{N}(x_e, u_e))$, then stabilization is impossible with continuous feedback control
  - And the same result applies to DISCONTINUOUS controllers with Fillipov solutions

- Hence we get convergence but not LS with continuous control or Fillipov solutions

So what do we do?
Use Time-Varying Feedback!


\[
\begin{align*}
\dot{x}_1 &= u_1, \\
\dot{x}_2 &= u_2, \\
\dot{x}_3 &= x_2u_1 - x_1u_2
\end{align*}
\]

- Since \((0, 0, x_3) \notin f(\mathbb{R}^3)\), origin not AS’ble by CONTINUOUS \(\psi\)

- Origin is locally ATTRACTIVE but not LS using DISCONT \(\psi\)
  \[\psi_1 = -\alpha x_1 + \beta x_2 \text{sign}(x_3)\]
  \[\psi_2 = -\alpha x_2 - \beta x_1 \text{sign}(x_3)\]

- Also, origin is AS’ble by continuous TIME-VARYING \(\psi\)
  \[\psi_1 = -x_1 + (x_3 - x_1x_2)(\sin t - \cos t)\]
  \[\psi_2 = -2x_2 + x_1(x_3 - x_1x_2) + (\cos t)x_1(x_1 + (\cos t)(x_3 - x_1x_2))\]
  \[V(x, t) = [x_1 + (\cos t)(x_3 - x_1x_2)]^2 + 4x_2^2 + (x_3 - x_1x_2)^2\]

What about discontinuous and TV controllers in applications?
A Multibody Attitude Control Problem

\[ \dot{z}_1 = u_1, \quad \dot{z}_2 = u_2, \quad \dot{\theta} = \frac{m_1 l_1 u_1}{J + m_1 z_1^2 + m_2 z_2^2} + \frac{m_2 l_2 u_2}{J + m_1 z_1^2 + m_2 z_2^2} \]

- The system is controllable but \((z_1, z_2, 0) \notin \text{int} f(\mathcal{N}(x_e, u_e))\)
  - Requires either DISCONTINUOUS feedback or TIME-VARYING feedback

Now, let’s use L2M for stabilization
Idea: Use $u$ to make $\dot{V} < 0$

- Jurdjevic/Quinn, 1978; Artstein, 1983; Sontag 1989; Tsinias, 1989

Consider $\dot{x} = f(x) + g(x)u, \quad u = \psi(x)$

If $\exists u : V'(x)[f(x) + g(x)u] < 0$, then $x_e$ is AS’ble with $C^\infty \backslash \{0\} \psi$

$\psi = -\frac{V'f + \sqrt{(V'f)^2 + (V'g)^4}}{V'g}$ is a universal controller

$\dot{V} = -\sqrt{(V'f)^2 + (V'g)^4} < 0$

$\dot{x} = x + x^3u$
$u = \psi(x) = -x^{-2}(1 + \sqrt{1 + x^{12}})$

$V$ is a CONTROL LYAPUNOV FUNCTION (CLF)

Are CLF controllers optimal?
Minimize $J(x_0, u) = \int_0^T L(x, u) dt$ with $\dot{x} = f(x, u)$

- Kalman, 1964; Moylan/Anderson, 1973; Freeman/Kokotovic, 1996

- Hamilton-Jacobi-Bellman yields the feedback control
  \[ \psi(x) = \arg\min_u [L(x, u) + V'(x)f(x, u)] \]

- The cost-to-go $V(x_0) = \int_0^T L(x, \psi(x)) dt$ is a VALUE FUNCTION

- Value functions $\leftrightarrow$ control Lyapunov functions
  - Quadratic $L$ yields LQR or $H_2$ synthesis (linear controllers)
  - Exponential-of-quadratic $L$ yields worst-case or $H_\infty$ synthesis (linear controllers)
  - $L \equiv 1$ yields minimum time synthesis (nonlinear controllers)

What’s special about minimum-time control?
Two Things!

- Consider $m\ddot{q} = u$ with bounded $u$

1) Minimum-time control $u = \psi(q, \dot{q})$ is **DISCONTINUOUS** (in fact, it’s bang bang)

2) And the states converge in **FINITE TIME**

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What else is finite-time convergent?
The Oscillator with Coulomb Friction

\[ m\ddot{q} + c\text{sign}(\dot{q}) + kq = 0 \]

- These dynamics are DISCONTINUOUS
  - And the states converge in FINITE TIME

Are CONTINUOUS dynamics ever finite-time convergent?
Consider a Leaky Bucket
\[
\dot{h} = -\beta \sqrt{h}
\]

where

\[
\beta \triangleq \sqrt{\frac{2g}{(D/d)^4-1}}
\]

- Vector field is continuous, but NOT Lipschitzian at \( h = 0 \)
  - Nevertheless the solution is unique
    \[
    h(t) = \left(\sqrt{h_0} - \frac{1}{2} \beta t\right)^2
    \]
  - And all trajectories converge to zero in FINITE TIME
    ▲ As expected!!

- Note that finite-time convergence REQUIRES non-Lipschitzian dynamics . . .
  - . . . since trajectories are NOT UNIQUE in reverse time

Let’s use L2M to PROVE finite-time convergence
To Do This, We Need a “Natural” $V$

- $T(x_0) = \text{time to converge from } x_0 = \text{time to go}$
  
  - $T(x(t)) = T(x_0) - t$ and therefore $\dot{T}(x(t)) = -1$
  
  ▲ But $\dot{T}(0) = 0$ and thus $\dot{T}$ is not continuous

- However, let $V(x) = T^2(x)$ so that $\dot{V} = -2T$, which IS continuous and negative definite

- Now use the comparison lemma
  
  - $V$ satisfies $\dot{V} = -2\sqrt{V} < 0$
  
  - $\int_0^\xi \frac{d\eta}{\sqrt{\eta}} < \infty$ upgrades to finite-time convergence via $V = T^2$

Next, let’s finite-time STABILIZE with a continuous $\psi$
Consider the Double Integrator Again

- CONTINUOUS controller \( u = \psi(q, \dot{q}) = -q^{1/5} - \dot{q}^{1/3} \)
  - Closed-loop system \( m\ddot{q} + \dot{q}^{1/3} + q^{1/5} = 0 \)

- \( V(q, \dot{q}) = \frac{5}{6} q^{6/5} + \frac{1}{2} m\dot{q}^2 \)
  - \( \dot{V} = -\dot{q}^{4/3} \leq 0 \) gives LS
  - Invariance principle upgrades to AS
    - But \( V \) is defective : (−)

Let's prove finite-time convergence
The system \( \dot{x} = f(x) \) is HOMOGENEOUS of DEGREE \( r \) with respect to the DILATION \( \Delta = \text{diag}(\alpha^{r_1}, \ldots, \alpha^{r_n}) \) if

\[
f(\Delta x) = \alpha^r \Delta f(x)
\]

Familiar case \( \Delta = \alpha I \) yields \( f(\alpha x) = \alpha^{r+1} f(x) \)

---

**Theorem** \( \text{FTC} \iff \text{AS and } r < 0 \)

- \( m\ddot{q} + \dot{q}^{1/3} + q^{1/5} = 0 \) is AS
- With \( \Delta = \begin{bmatrix} \alpha^5 & 0 \\ 0 & \alpha^3 \end{bmatrix} \), \( r = -2 < 0 \)
- Hence finite-time convergent

**Negative-degree homogeneity upgrades from AS to FTC**

- Bhat/Bernstein, 1997
What Do We Do If Some States Don’t Converge?

Idea #1: Ignore Them
Orbital Motion

\[ \ddot{r} - r \dot{\theta}^2 + \frac{\mu}{r^2} = 0 \]

\[ \ddot{\theta} = \frac{-2 \dot{\theta} \dot{r}}{r} \]

- \( \theta(t) \to \infty \) but luckily \( \theta(t) \) doesn’t appear
  - So ignore it!

- Constants of motion
  - Energy \( E = \frac{\dot{r}^2}{2} + \frac{r^2 \dot{\theta}^2}{2} - \frac{\mu}{r} \)
  - Angular momentum \( h = r^2 \dot{\theta} \)

- For CIRCULAR orbit define
  - \( V(r, \dot{r}, \dot{\theta}) = (E - E_c)^2 + (h - h_c)^2 \)
  - \( \dot{V} = 0 \) and thus the circular orbit is LS

Can we always ignore states that don’t converge?
No: Consider Time-Varying Systems

- \( \dot{x} = f(x, t) \)

- Introducing \( x_{n+1} \equiv 1 \) yields \( \tilde{x} = \tilde{f}(\tilde{x}) \triangleq \left[ \begin{array}{c} f(x, x_{n+1}) \\ 1 \end{array} \right] \)
  - which “looks” time-invariant

- But \( x_{n+1}(t) = t \longrightarrow \infty \) as \( t \longrightarrow \infty \) (fairly obvious)
  - AND, you can NOT ignore \( t \)!

Idea #2: Require stability wrto only PART of \( x \)
Partition \( x = (x_1, x_2) \)

- Rumyantsev, 1970; Vorotnikov, 1998

- Get partitioned dynamics
  \[
  \dot{x}_1 = f_1(x_1, x_2) \\
  \dot{x}_2 = f_2(x_1, x_2)
  \]

- Define a PARTIAL equilibrium \( x_{1e} \) satisfying
  \[ f_1(x_{1e}, x_2) \equiv 0 \text{ for all } x_2 \]
  
  ♦ Consider ONLY \( x_1(t) - x_{1e} \)

- Theorem: Assume \( x_{1e} \) is a strict minimizer of \( V(x_1) \)
  
  ♦ \[ V'(x_1) f_1(x_1, x_2) \leq 0 \implies \text{PARTIAL Lyapunov stability} \]
  
  ♦ \[ V'(x_1) f_1(x_1, x_2) < 0 \implies \text{PARTIAL asymptotic stability} \]

Are there any useful applications of partial stability?
The Controlled Slider-Crank!

Choose \( u = \psi(\theta, \dot{\theta}) \) so that \( \dot{\theta}(t) \rightarrow \dot{\theta}_{\text{des}} \) (constant angular velocity)

- But this implies \( \theta(t) \approx \dot{\theta}_{\text{des}} t \rightarrow \infty \)

However, since \( m(\theta) \) and \( c(\theta) \) depend on \( \theta \), we can\textbf{NOT} ignore \( \theta \)

- AND, since \( \theta(t) \rightarrow \infty \),

- we can\textbf{NOT} get AS but we \textbf{CAN} get PARTIAL AS

Next, let’s require that all states converge to \textbf{SOMETHING}
Consider the Damped Rigid Body

- NONE of equilibria are AS!!
- Since $\dot{q} \to 0$, we have partial AS wrto $\dot{q}$
- Also $q \to q_\infty$, where $q_\infty$ is determined by initial conditions
- So all states converge to SOMETHING

What kind of stability is THIS?
Suppose we have a continuum of equilibria.

A LS equilibrium is SEMISTABLE if every nearby trajectory converges to a (possibly different) LS equilibrium.

Sandwich property: AS $\implies$ semistability $\implies$ LS.
How Can We Analyze Semistability?

- Bhat/Bernstein, 2001

Assume \( x_e \) is a local minimizer of \( V \) and \( \dot{V} \leq 0 \)

- If \( f \) is NONTANGENT to the 0 level set of \( \dot{V} \) near \( x_e \), then \( x_e \) is semistable

- Note that \( V \) need only be POSITIVE SEMIDEFINITE at \( x_e \)

Are there any interesting applications of semistability?
Michaelis-Menten reaction

\[ S + E \xrightleftharpoons[k_1]{k_2} C \xrightarrow{k_3} P + E \]

\[
\begin{align*}
x_1 & \triangleq [S] & \dot{x}_1 &= k_2 x_2 - k_1 x_1 x_3 \\
x_2 & \triangleq [E] & \dot{x}_2 &= -(k_2 + k_3) x_2 + k_1 x_1 x_3 \\
x_3 & \triangleq [C] & \dot{x}_3 &= (k_2 + k_3) x_2 - k_1 x_1 x_3 \\
x_4 & \triangleq [P] & \dot{x}_4 &= k_3 x_2 
\end{align*}
\]

- All states are nonnegative and all \((0, 0, x_3, x_4)\) are equilibria
- Let's choose \( V = \alpha x_1 + x_2 \geq 0 \) (semidefinite) and thus \( \dot{V} \leq 0 \)
  - Note that LINEAR \( V \) is allowed since all states are NONNEGATIVE
- Nontangency implies that all equilibria are semistable
  - and \([S] \to 0, \ [E] \to 0, \ [C] \to [C]_\infty, \ [P] \to [P]_\infty\)
  - where \([C]_\infty\) and \([P]_\infty\) depend on initial conditions
Is Anything Else Semistable?

• Compartamental models
  ♦ Mass transport
    ▲ Biological systems
  ♦ Energy transport
    ▲ Thermodynamics

Could semistability possibly be useful for ADAPTIVE control?
Simplest Adaptive Stabilization Problem

- Consider the uncertain system
  \[ \dot{x} = Ax + Bu \]
  \[ u = -K x \]
  - Assume there exists unknown \( K_s \) such that \( A + BK_s \) is AS

- Consider the control update
  \[ \dot{K} = B^T Pxx^T \]

- Let \( V(x, K) = x^T P x + \text{tr} \ (K - K_s)^T (K - K_s) \) so \( \dot{V}(x, K) = -x^T x \)
  - Hence we have LS
  - Invariance principle implies \( x \to 0 \)

- Nontangency implies semistability and \( K \to K_\infty \)
  - \( K_\infty \) depends on initial conditions

Note that \( B \) must be known
What Do We Do When $B$ is Unknown?

- Nussbaum, 1983; Morse, 1985

- Consider $\dot{x} = ax + bu$ where $\text{sign}(b)$ is unknown

- We can NOT use increasing gain $k$ with $\dot{k} = x^2$ and $u = kx$

- Instead we use increasing gain $k$ with $\dot{k} = x^2$ and OSCILLATING control amplitude $u = k^2(\cos k)x$

- Define INDEFINITE $V$

\[ V(x, k) = e^{-k} + \frac{1}{2}x^2 + (a + 2b \cos k)k + bk^2 \sin k - 2b \sin k \]

- Every sublevel set of $V$ is a union of disconnected compact sets

\[ \dot{V} = -e^{-k}x^2 \leq 0, \text{ which implies } x \text{ is bounded} \]

- Nontangency implies semistability and $k \rightarrow k_\infty$
Output-Feedback Adaptive Stabilization

- Consider minimum phase $G$ with relative degree 1

\[
y = Gu \quad u = -ky \quad \dot{k} = y^2
\]

- Scalar case: \( V(x, k) = e^{-k} + \frac{1}{2} x^2 + \frac{1}{2b} (a + bk)^2 \)
  
  \( \dot{V} = -e^{-k} y^2 \leq 0 \) implies $x$ is bounded and $y \to 0$
  
  $f$ is nontangent to the zero level set of $\dot{V}$
  
  Hence semistability holds and $k \to k_\infty$

What happens if $y$ is noisy?
Noise is the Scourge of Adaptive Control!!

- Consider minimum phase $G$ with relative degree 1

$$ y = Gu + w \quad u = -ky \quad \dot{k} = y^2 $$

- Noise $w$ causes $k \to \infty$!!

- Damped modification $\dot{k} = -\gamma k + y^2$ causes bursting

What can we do about this?
Use Chattering Control for Noise Rejection

\[ \ddot{q} + a_1 \dot{q} + a_2 q = \dot{u} - zu + w_1 \]

\[ y = q + w_2 \]

- \( a_1 \) and \( a_2 \) are unknown, \( z < 0 \) but otherwise unknown
- \( w_1 \) and \( w_2 \) are bounded with UNKNOWN bound

\[
\begin{align*}
\dot{q}_f &= -\lambda q_f + y & \text{filters } y \\
\dot{u}_f &= \ddot{z} q_f + u & \text{filters } u \\
\dot{a}_1 &= -\ddot{x}_f q_f & \text{estimates } a_1 \\
\dot{a}_2 &= -\ddot{x}_f q_f & \text{estimates } a_2 \\
\dot{\alpha} &= k \alpha^{3/2} |\ddot{x}_f| & \text{estimates bound on } w_1 \text{ and } w_2 \\
\dot{\zeta} &= -\ddot{x}_f u_f (-\dot{\zeta})^{3/2} & \text{estimates } z \\
u &= (\lambda + \ddot{z}) u_f + (\dot{a}_1 - f_1) \dot{q}_f + (\dot{a}_2 - f_2) q_f - \alpha \text{sign}(\ddot{x}_f) & \text{chattering}
\end{align*}
\]

So what’s \( V \)?
Here's $V$!

- Sane, Bernstein, Sussmann, 2001

$$V(x) = \begin{bmatrix} q_t \\ \dot{q}_t \\ \dot{a}_1 \\ \dot{a}_2 \\ \dot{\alpha} \end{bmatrix}^T P \begin{bmatrix} q_t \\ \dot{q}_t \\ \dot{a}_1 \\ \dot{a}_2 \\ \dot{\alpha} \end{bmatrix} + \sqrt{-\hat{z}} - \frac{z}{\sqrt{-\hat{z}}}$$

- $W(\hat{z})$ confines $\hat{z} < 0$ since $z < 0$
- $W$ is a LYAPUNOV WELL

- $\dot{V}$ is INDEFINITE
  - But $\dot{V}(x(t)) \leq \gamma e^{-2\lambda t}$ along trajectories
  - Hence $\dot{V}(x(t))$ is ASYMPTOTICALLY NONPOSITIVE

- Use BARBALAT'S LEMMA to prove $y \longrightarrow 0$
  - ... and all states are bounded : – )
Let’s Recapitulate
We Acknowledged the Weaknesses of L2M

- While Lyapunov-like ideas are the basis of classical stability analysis (e.g., the Lagrange-Dirichlet stability condition), L2M per se has had relatively few successes outside of control.

- In general, it’s simply too difficult to construct Lyapunov functions using only the vector field.
And We Celebrated Its Successes

- L2M is immensely successful in control theory
- While it had no impact on CLASSICAL control . . .
- . . . it’s the heart and soul of MODERN control, where we synthesize controllers to suit Lyapunov functions of CHOSEN form
  - The ability to construct the control and the Lyapunov function TOGETHER is what makes L2M so successful in control
  - L2M is the backbone of optimal, robust, and adaptive control
We Traveled from Infancy . . .

\[ \dot{V} \leq 0 \implies x_e \text{ is Lyapunov stable} \]

\[ \dot{V} < 0 \implies x_e \text{ is asymptotically stable} \]
Invariance principle
Comparison lemma
Control Lyapunov functions
Homogeneity
Partial stability
Semistability
Nontangency
Semidefinite and indefinite $V$’s
Asymptotic nonpositivity
Barbalat’s lemma
Lyapunov wells
What Lies in the Future for L2M?
L2M Beyond ODE’s

- Discontinuous dynamics and differential inclusions
  - Nonholonomic dynamics
  - Relay and sliding mode control
  - Essential in control

- PDE’s
  - Stability of solitons
  - Hysteresis in smart materials
  - Flow stabilization
  - Many other applications
Input-Output Analysis Based on L2M

● **Dissipativity** *(Willems)*
  ◇ Storage function $V_s$, supply rate $r(u, y)$
    ▲ $\dot{V}_s(x) \leq r(u, y)$
  ◇ Nonlinear positive real theory (passivity)
  ◇ Nonlinear bounded real/H$_\infty$ theory (nonexpansivity)

● **Input-to-state stability** *(Sontag)*
  ◇ GAS: $\|x(t)\| < b(\|x(0)\|, t) \iff \dot{V} < -a(x)$
  ◇ ISS: $\|x(t)\| < b(\|x(0)\|, t) + \sup |u| \iff \dot{V} < -a(x) + b(u)$
Trends in Nonlinear Control Based on L2M

- Receding horizon control
  - CLF’s to obtain suboptimal HJB solutions

- Problems with control and state constraints
  - Anti-windup and control saturation
  - Invariant set methods for state constraints (Gilbert/Kolmanovsky)

- Gain scheduling methods
  - LPV methods
  - Equilibrium switching methods (multiple V’s)

- Impulsive dynamics
  - Hybrid systems (Lakshmikanthanam, Haddad/Chellaboina/Bhat)
  - Resetting controllers (Hollot/Chait)
Specialized Applications of L2M

- Nonnegative systems
  - Chemical kinetics
    - Zero deficiency theorem for rate-independent semistability (Feinberg)

- Emergent behavior of large scale, interconnected systems
  - Thermodynamics
    - Analyze energy flow and entropy as emergent properties
    - Linear storage functions and supply rates (Haddad/Chellaboina)
  - Swarm dynamics
Some Research Questions

- Can we do more with L2M in discrete time?
  - Discrete-time adaptive control, especially for disturbance rejection (many patents due to lack of theory!)

- Can we use set stability (Zubov, Bhatia/Szego) to prove LS of an elliptical orbit?
  - Poisson and orbital stability

- Is there an L2M foundation for averaging?

- Is there an L2M proof of the Poincare stability theorem?
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What Hath Lyapunov Wrought!