From Infancy to Potency:

1857

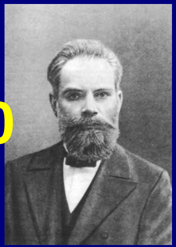
1918

Lyapunov’s Second Method and the Past, Present, and Future of Control Theory

Dennis S. Bernstein

Aleksandr Mikhailovich Lyapunov

- Father astronomer, 7 children, 3 survived
- Professor of mechanics at Kharkov and St. Petersburg
  - Research on orbital mechanics and probability theory
- A. M. Lyapunov died tragically at age 61
- Completed his doctoral dissertation in 1892 under Chebyshev
  - Stability of rotating fluids applied to celestial bodies
  - Formulated his first and second methods (L1M and L2M)
- French translation appeared in 1907 = 1892 + 15
  - English translation didn't appear until 1992 = 1892 + 100



How Successful Is L2M?

- Many areas have a strong interest in stability theory
  - Classical dynamics
  - Structural dynamics
  - Fluid mechanics
  - Astrodynamics
  - Chemical kinetics
  - Biology
  - Economics
  - Control

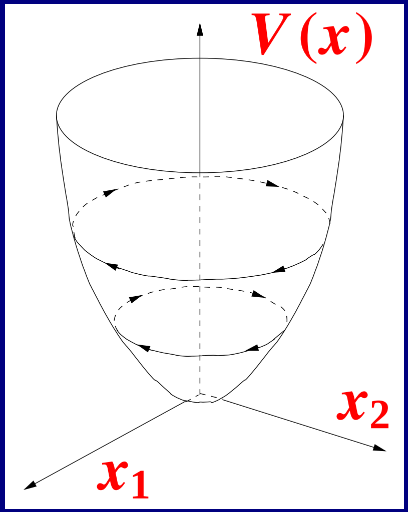
While there are some notable applications of L2M outside of control, there are surprisingly few overwhelming successes

How successful is L2M in control?

Let’s First Review the Basics of L2M

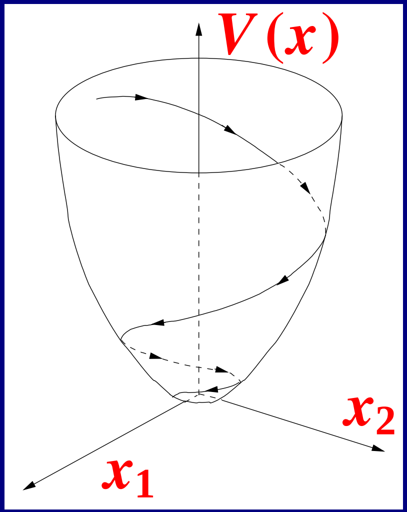
Basic L2M

- Consider $\dot{x} = f(x)$ with equilibrium x_e
- Assume x_e is a strict local minimizer of V

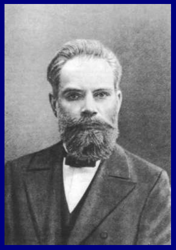


$\boxed{\dot{V} \le 0 \text{ implies } x_e \text{ is Lyapunov stable (LS)}}$

$\boxed{\dot{V} < 0 \text{ implies } x_e \text{ is asymptotically stable (AS)}}$



- Want $V(x(t))$ nonincreasing or decreasing
  - V "keeps" $x(t)$ bounded or "makes" $x(t) \to 0$
  - In fact, V merely predicts the behavior of $x(t)$



Suppose V is Radially Unbounded

- If $\dot{V} \le 0$ and V is radially unbounded, then all trajectories are bounded
  - No finite escape, and thus global existence
- If $\dot{V} < 0$ and V is radially unbounded, then x_e is globally AS (GAS)
  - GAS $\Longleftrightarrow$ LS + global convergence

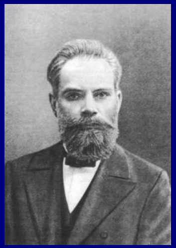
How can we construct useful V's? 2 ways.

1) Use $x(t)$ to Construct V

- Persidskii, 1938; Massera, 1949; Malkin, 1952; Ura, 1959 (converse theory)

- For AS or GAS, if f is locally Lipschitz, then we can construct C^∞ V with $\dot{V} < 0$
- For LS, continuous V may NOT exist even if f is C^∞
  - ♦ But, if f is locally Lipschitz, then we can construct lower SEMIcontinuous V with $\dot{V} \leq 0$
- How is trajectory-based construction useful if $x(t)$ is not available?
  - ♦ Consider an approximate system with KNOWN trajectories
  - ♦ For example, linearize the system and construct

$$\boxed{V(x_0) = \int_0^\infty x^{\mathrm{T}}(t)x(t)\mathrm{d}t = x_0^{\mathrm{T}} P x_0}$$



▲ This is L1M

2) Use f to Construct V

- Lagrange-Dirichlet Method $\boxed{V(q,\dot{q}) = T(q,\dot{q}) + U(q)}$
  - Predates L2M, 1788/1848
- Krasovskii's Method $\boxed{V(x) = f^{\mathrm{T}}(x)Pf(x)}$
- Variable Gradient Method $\boxed{\dot{V}(x) = g^{\mathrm{T}}(x)f(x)}$
- Constants of Motion $\boxed{V(x) = \sum[\lambda_i h_i(x) + \mu_i h_i^2(x)]}$
  - Energy-Casimir Method
- Zubov's Method $\boxed{V'(x)f(x) = -h(x)[1 - V(x)]}$
  - PDE

- There aren't really very many useful methods!!

This explains the lack of success of L2M outside of control

How to Define $\dot{V}$? 2 Ways.

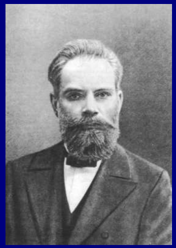
1) Use the trajectory $x(t)$: Need V lower semicontinuous

$$\boxed{\dot{V}(\xi) \triangleq \limsup_{h\to 0} \tfrac{1}{h}[V(x(h,\xi)) - V(\xi)]}$$

2) Use the vector field f: Need V locally Lipschitz

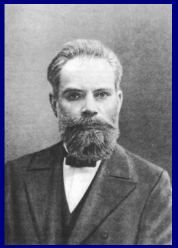
$$\boxed{\dot{V}(x) = \limsup_{h\to 0} \tfrac{1}{h}[V(x+hf(x)) - V(x)]}$$

If V is C^1 then $\boxed{\dot{V}(x) = V'(x)f(x)}$

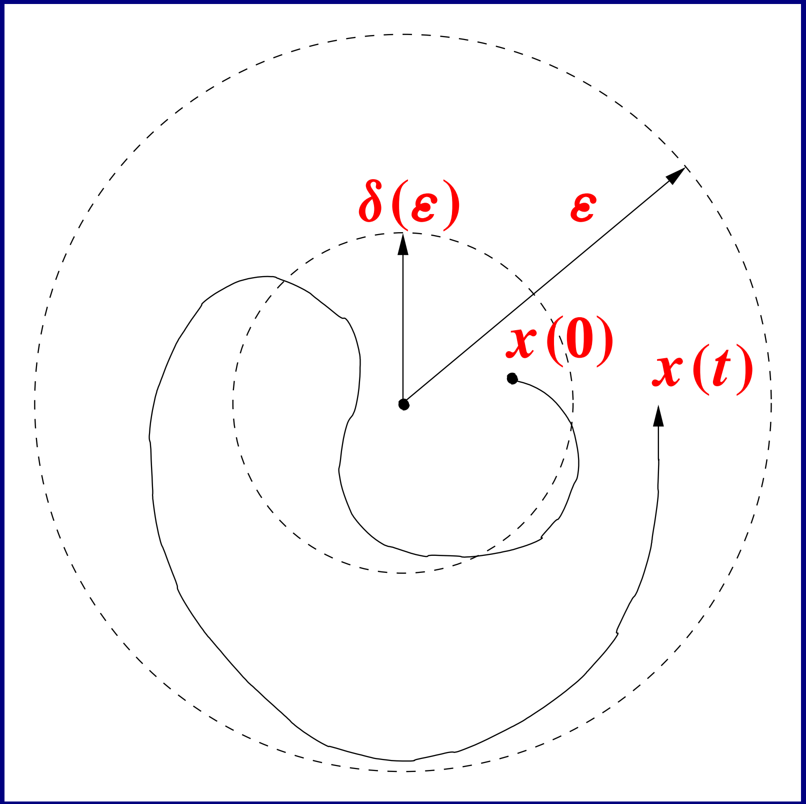


Stability is qualitative.

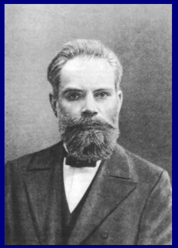
How Can We Quantify It?



How Can We Quantify LS?

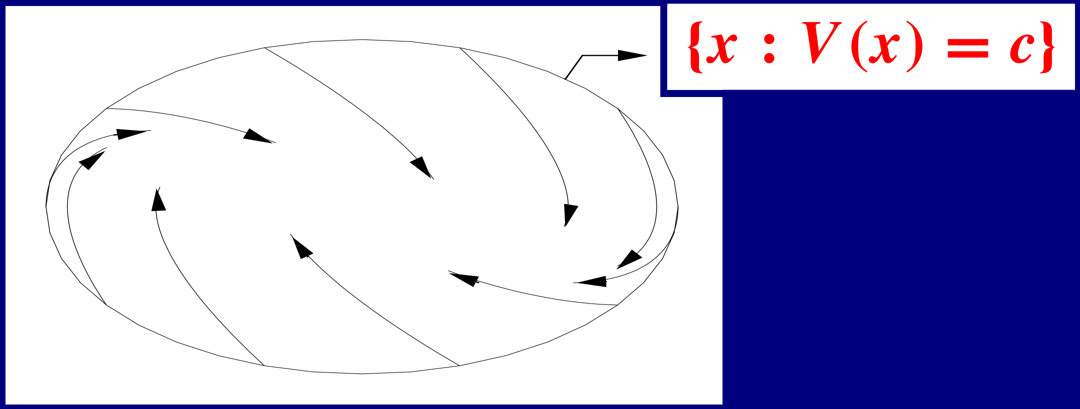


- Suppose we can invert $\delta(\varepsilon)$ to obtain $\varepsilon(\delta)$
- If $\|x(0)\| < \delta$, then $\|x(t)\| < \varepsilon(\delta)$
  - This quantifies LS by means of a trajectory bound
  - But doesn't use L2M



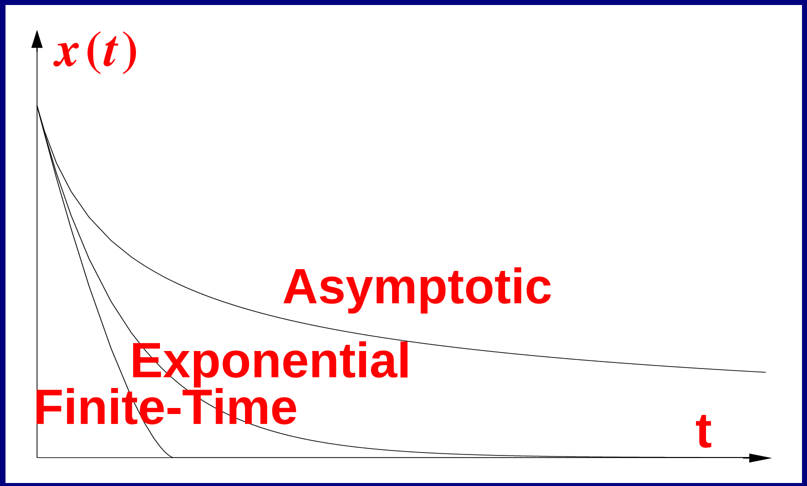
How Can We Quantify AS?

1) Use a sublevel set $\{x : V(x) \leq c\}$ to estimate the domain of attraction

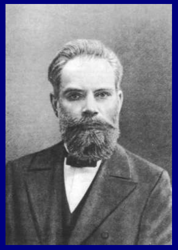


- Based on L2M

2) Estimate the speed of convergence



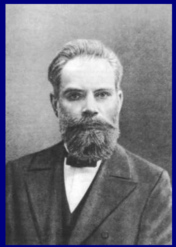
- Can we use L2M?
  - ♦ Yes, . . .



How Can We Improve Our

Stability Predictions?

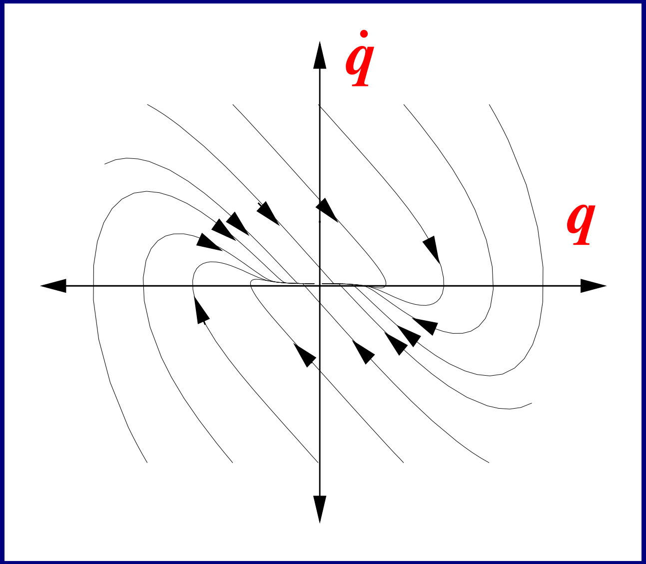
Use Upgrades!



How Can We Upgrade from LS to AS?

- Barbashin/Krasovskii, 1952; LaSalle, 1967

- DAMPED nonlinear oscillator $\boxed{m\ddot{q} + c\dot{q} + kq^3 = 0}$
  - Energy $\boxed{V(q,\dot{q}) = \frac{m}{2}\dot{q}^2 + \frac{k}{4}q^4 \text{ and } \dot{V}(q,\dot{q}) = -c\dot{q}^2 \leq 0}$
  - So we have LS



- The INVARIANCE PRINCIPLE upgrades LS to AS

- But V is “defective"
  - That is, $\dot{V} \leq 0$ but we DON'T have $\dot{V} < 0$

But converse theory guarantees that a NONdefective V exists!!! : – (

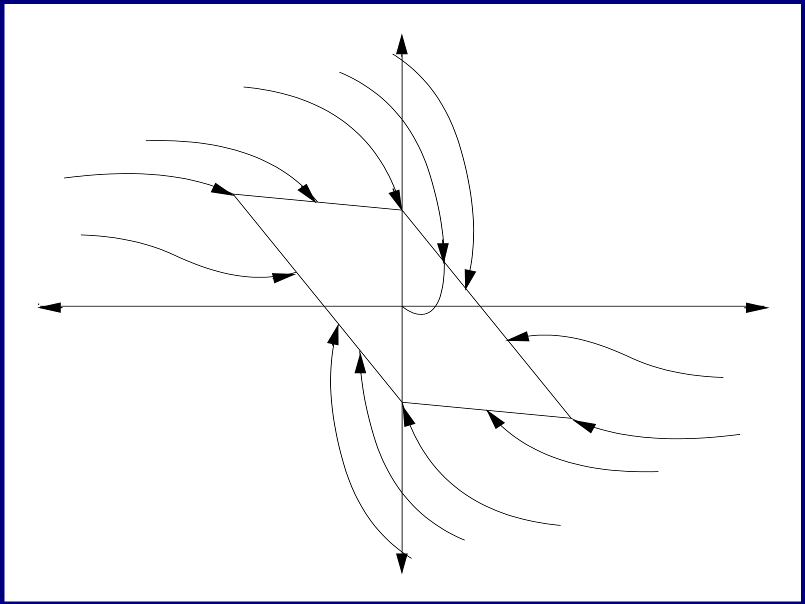
How Can We Upgrade from AS to GAS?

- Teel, 1992

- GAS is equivalent to AS + global FINITE-TIME convergence to a domain of attraction
- Nested saturation controller for the double integrator

$$\boxed{u = \psi(q, \dot{q}) = -\mathrm{sat}_{\varepsilon}(\dot{q} + \mathrm{sat}_{\varepsilon/2}(q + \dot{q}))}$$

- Splice the trajectories



- Note that we did NOT construct a radially unbounded V
  - Hence local V is defective for GAS!   : – (

How Can We Upgrade the Speed of Convergence?

1) Show V satisfies a differential inequality $\boxed{\dot{V} \leq \rho(V)}$

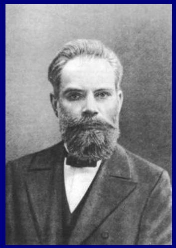
2) Then construct the COMPARISON system $\boxed{\dot{\eta} = \rho(\eta)}$

- If $\rho'(0) < 0$ then $V(x(t)) \longrightarrow 0$ exponentially
  - If $\alpha\|x\|^2 \leq V(x) \leq \beta\|x\|^2$ then $x(t)$ converges exponentially

- If $\eta\rho(\eta) < 0$ and $\int_0^{\xi} \frac{d\eta}{\rho(\eta)} < \infty$ then $V(x(t)) \longrightarrow 0$ in finite time
  - And thus $x(t)$ converges in finite time

Next: L2M and Control

What Was the Impact of L2M on CLASSICAL Control?

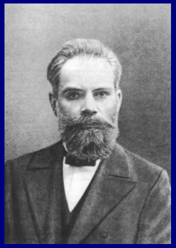


Maxwell's Work on Stability

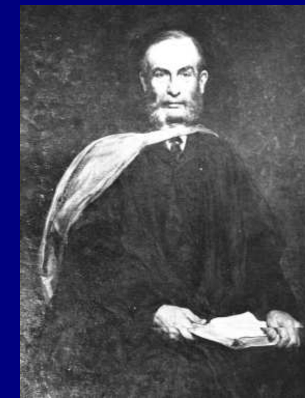
- Stability of Saturn's rings
  - Quartic linearization
    - Fortunately, biquadratic – trivial

- Stability of governors
  - 5th-order linearization – not trivial
  - Obtained only necessary conditions
  - Motivated the Adams prize competition at Cambridge in 1875

Who won?



Routh



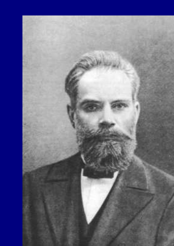
- Routh, 1877 = 1892 - 15

- Routh considered the stability of a general polynomial
  - $p(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_0$
  - Derived a necessary and sufficient condition for stability
  - Note: No mention of control

$$\begin{array}{c|ccccc} s^n & a_0 & a_2 & a_4 & a_6 & \cdots \\ s^{n-1} & a_1 & a_3 & a_5 & a_7 & \cdots \\ \hline s^{n-2} & b_0 & b_2 & b_4 & b_6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & & \end{array}$$

- His derivation was based on the Cauchy index theorem
  - Not based on L2M
    - Predates L2M

Is there an L2M proof of the Routh test?



Yes! Parks, 1962 = 1892 + 70

1) Compute Routh table parameters $1,\ b_1, b_2, b_1 b_3, b_2 b_4, b_1 b_3 b_5, \ldots$

2) Construct the tridiagonal Schwarz matrix $A = \begin{bmatrix} 0 & 1 & 0 \\ -b_3 & 0 & 1 \\ 0 & -b_2 & -b_1 \end{bmatrix}$

3) Solve the Lyapunov equation $A^{\mathrm{T}} P + P A + R = 0$ with

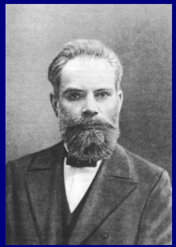
$$R = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2b_1^2 \end{bmatrix} \quad \text{for} \quad P = \begin{bmatrix} b_1 b_2 b_3 & 0 & 0 \\ 0 & b_1 b_2 & 0 \\ 0 & 0 & b_1 \end{bmatrix} > 0$$

4) Define $V(x) = x^{\mathrm{T}} P x$, which implies $\dot{V}(x) = -x^{\mathrm{T}} R x$

5) Since $R \geq 0$, get $\dot{V} \leq 0$, which proves LS

6) Use invariance principle to upgrade to AS

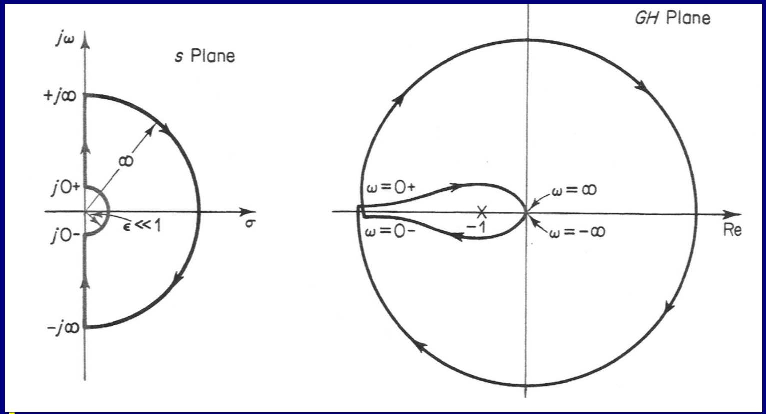
$\boxed{\text{Since } R \not> 0\text{, after all this, } V \text{ is defective!!} \quad : - (}$



Nyquist Test

- Nyquist, 1932 = 1892 + 40

- In 1927, Harold Black invented the negative feedback amplifier
  - ♦ Unlike positive feedback amplifiers, his circuit was stable
  - ♦ Patent office was skeptical and treated negative feedback like perpetual motion
    - ▲ They demanded a prototype!



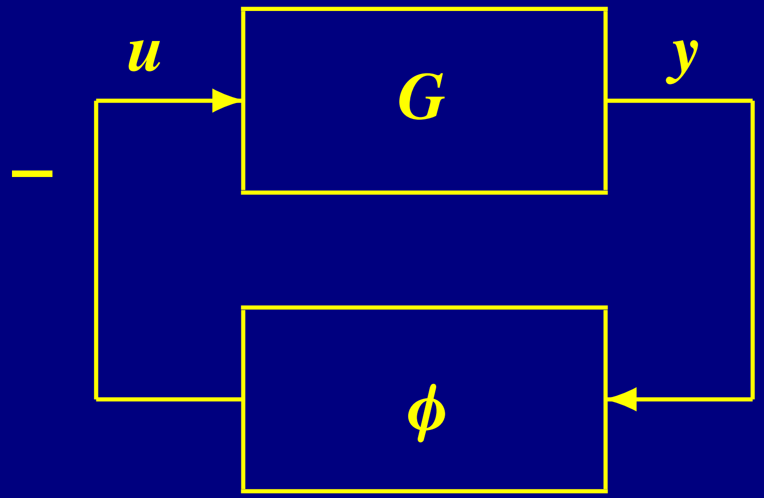
- Nyquist test provided the crucial frequency domain insight
  - ♦ Note: Loop closure stability test
- Like Routh, his proof was based on the Cauchy index theorem
  - ♦ $\boxed{\text{NOT on L2M}}$

Why not?

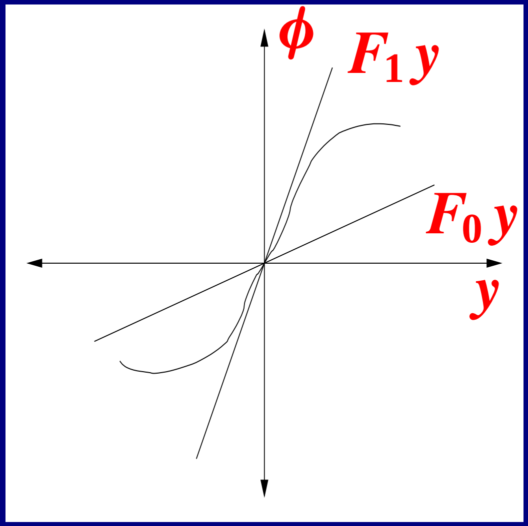
Let's Review Absolute Stability

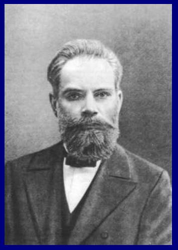
- Lur'e/Postnikov, 1944

- Feedback interconnection



- $\phi(y,t)$ is a memoryless time-varying nonlinearity in a sector Φ

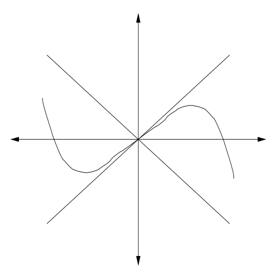




Absolute Stability Tests

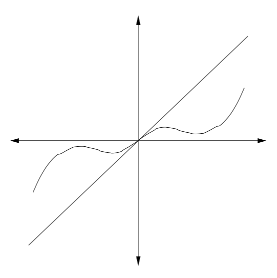
- Bounded real (small gain)

  $\boxed{|G| < 1/F_1}$ $\Longrightarrow$ $\boxed{\text{GAS for all NLTV } \phi \in \Phi_{\mathrm{br}}}$



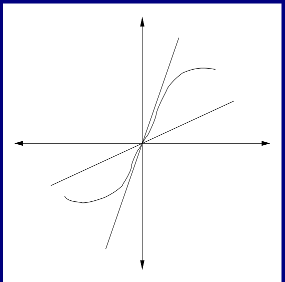
- Positivity

  $\boxed{\mathrm{Re}\, G > -1/F_1}$ $\Longrightarrow$ $\boxed{\text{GAS for all NLTV } \phi \in \Phi_{\mathrm{pr}}}$



- Circle

  $\boxed{\mathrm{Re}\, \frac{G}{1+F_1 G} > \frac{1}{F_1 - F_0}}$ $\Longrightarrow$ $\boxed{\text{GAS for all NLTV } \phi \in \Phi_{\mathrm{c}}}$



- For each test $\boxed{V(x) = x^{\mathrm{T}} P x}$
  - Get P from KYP conditions or a Riccati equation
  - $\boxed{\text{. . . and it's the SAME } V \text{ for ALL } \phi \text{ in the sector } \Phi}$

How do we REDUCE conservatism for time-INVARIANT $\phi(y)$?

Introduce a Multiplier!

- Popov, 1961

- Insert $Z(s) = 1 + \alpha s$ to restrict the time variation of ϕ
  - $\boxed{\mathrm{Re}\, Z(s)G(s) \geq -1/F_1} \implies \boxed{\text{GAS for all NLTI } \phi \in \Phi}$

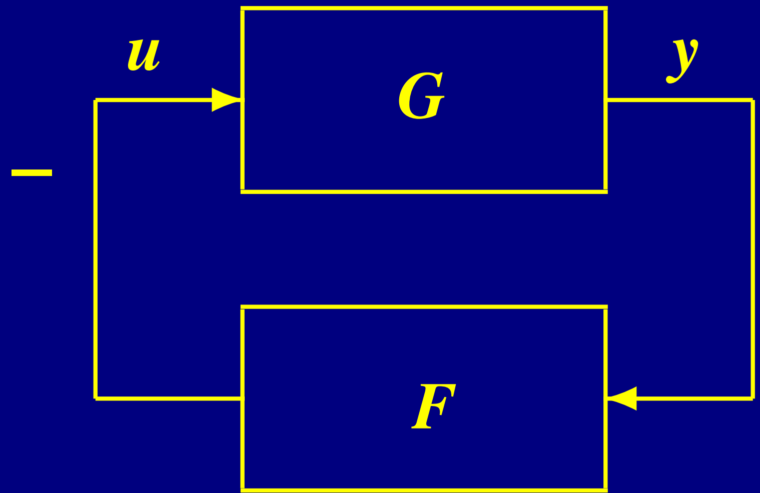
- $\boxed{V_\phi(x) = x^{\mathrm{T}} P x + \alpha \int_0^y \phi(\sigma)\mathrm{d}\sigma}$
  - V_ϕ depends on ϕ so we actually have a FAMILY of V's

- Furthermore, we can construct Z to further restrict ϕ
  - Slope bounded
  - Monotonic
  - Odd

How Can We Allow Only LINEAR $\phi(y) = Fy$

- Narendra, 1966; Brockett/Willems, 1967; Thathachar/Srinath, 1967

- Construct a SPECIAL Z_G that DEPENDS on G
  - ♦ $\boxed{\mathrm{Re}\, Z_G G \geq -1/F_1}$ $\Longleftrightarrow$ $\boxed{\text{GAS for all } \phi(y) = Fy,\ \ F \in [0, F_1]}$
- $\boxed{V_F(x) = x^{\mathrm{T}} P x + F y^{\mathrm{T}} y}$
  - ♦ A FAMILY of V's



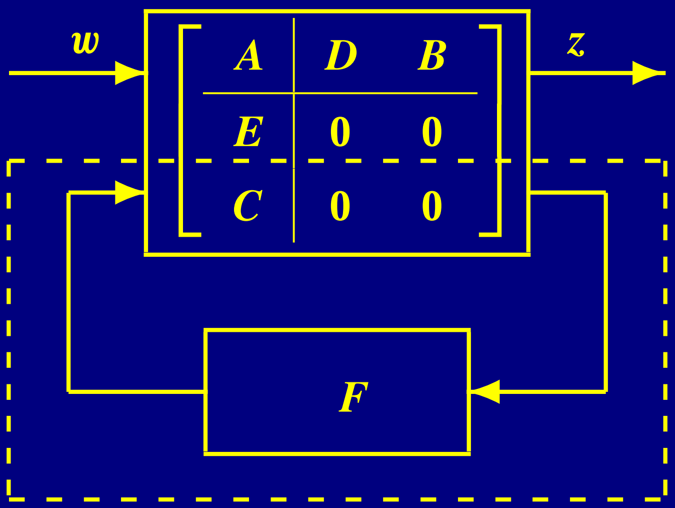
- This V proves the Nyquist test,
  - ♦ and completes a long and fruitful application of L2M :–)
- But a L2M proof of MULTIVARIABLE Nyquist is open!

Next: Let's include performance

Use Absolute Stability to Bound Performance

- Bernstein/Haddad, 1989

- $z = G_F w$
  - ♦ Small gain uncertainty $\sigma_{\max}(F) \leq \gamma$
  - ♦ $G_F \sim (A + BFC,\, D,\, E)$



- Construct $V(x) = x^{\mathrm{T}} P x$
  - ♦ $0 = A^{\mathrm{T}} P + PA + \boxed{\gamma^2 PBB^{\mathrm{T}} P + C^{\mathrm{T}} C} + E^{\mathrm{T}} E$
  - ♦ Then $\|G_F\|_2 \leq \mathrm{tr}\; D^{\mathrm{T}} P D$ for all uncertain F
  - ♦ Guarantees robust stability with a bound on worst-case H_2 performance

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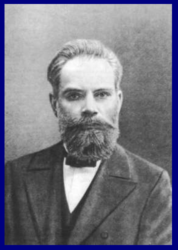
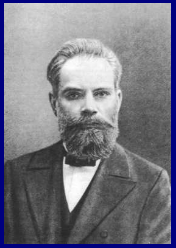
Speaking of H_2, let’s turn to MODERN control



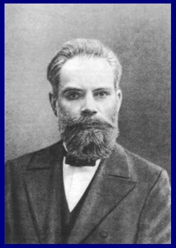
How HAS L2M Contributed to Modern Control?



Stabilization Based on Linearization Only

- $\dot{x} = f(x, u), \quad C^1\ f, \quad u = \psi(x), \quad \dot{x} = f(x, \psi(x))$
  - ♦ $f(x_e, u_e) = 0$ with linearization $\dot{x} = Ax + Bu$
- Sufficient condition
  - ♦ $\boxed{\text{If } (A, B) \text{ is stabilizable, then } x_e \text{ is AS'ble with } C^\infty\ \psi}$
  - ♦ Not necessary: $\dot{x} = -x^3 + xu$ is AS with $u = \psi(x) = 0$
- Necessary condition
  - ♦ $\boxed{\text{If } x_e \text{ is AS'ble with } C^1\ \psi\text{, then } (A, B) \text{ is CLHP stabilizable}}$
  - ♦ Not sufficient: $\dot{x} = x^2 + x^3 u$ has $A = 0$ and $B = 0$

To do better, let's use f directly



Stabilization Based on the Vector Field

- Brockett, 1983

- Necessary condition
  - ♦ $\boxed{\text{If } x_e \text{ is AS'ble with } C^0 \ \psi \text{, then } 0 \in \operatorname{int} f(\mathcal{N}(x_e, u_e))}$
  - ♦ Not sufficient: $\dot{x} = x + x^3 u$ (need discontinuous ψ)

- How can we use this result?
  - ♦ If $0 \notin \operatorname{int} f(\mathcal{N}(x_e, u_e))$, then stabilization is impossible with continuous feedback control
  - ♦ And the same result applies to DISCONTINUOUS controllers with Fillipov solutions

- Hence we get convergence but not LS with continuous control or Fillipov solutions

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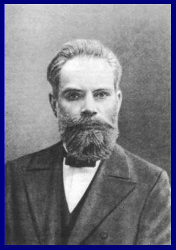


So what do we do?

Use Time-Varying Feedback!

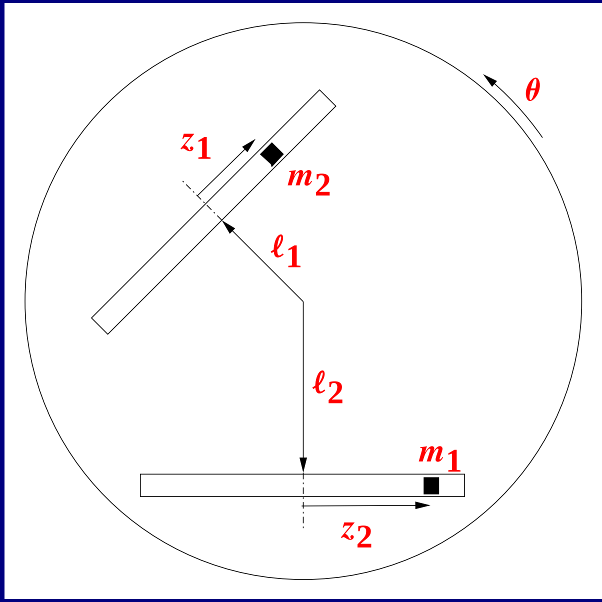
- Nonholonomic integrator: Brockett 1983, Pomet, 1992, Bloch/Drakunov, 1996

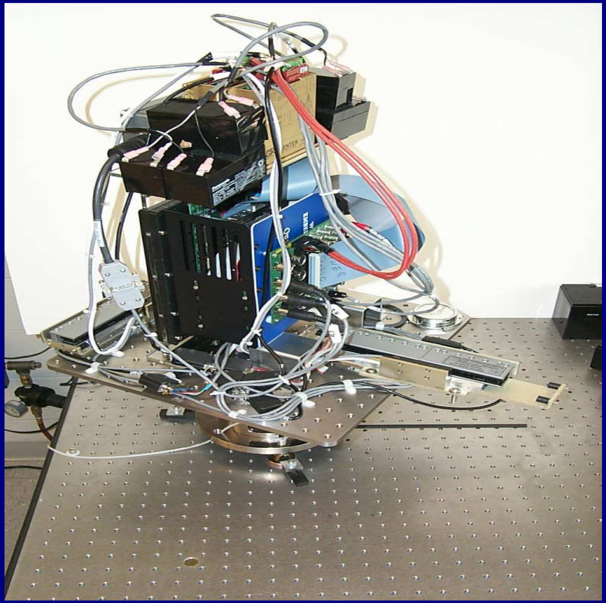
$$\boxed{\dot{x}_1 = u_1, \quad \dot{x}_2 = u_2, \quad \dot{x}_3 = x_2 u_1 - x_1 u_2}$$

- Since $(0, 0, x_3) \notin f(\mathbb{R}^3)$, origin not AS'ble by CONTINUOUS ψ
- Origin is locally ATTRACTIVE but not LS using DISCONT ψ
  - $\boxed{\psi_1 = -\alpha x_1 + \beta x_2 \mathrm{sign}(x_3)}$ $\quad$ $\boxed{\psi_2 = -\alpha x_2 - \beta x_1 \mathrm{sign}(x_3)}$
- Also, origin is AS'ble by continuous TIME-VARYING ψ
  - $\boxed{\psi_1 = -x_1 + (x_3 - x_1 x_2)(\sin t - \cos t)}$
  - $\boxed{\psi_2 = -2x_2 + x_1(x_3 - x_1 x_2) + (\cos t) x_1 (x_1 + (\cos t)(x_3 - x_1 x_2))}$
  - $\boxed{V(x, t) = [x_1 + (\cos t)(x_3 - x_1 x_2)]^2 + 4x_2^2 + (x_3 - x_1 x_2)^2}$

What about discontinuous and TV controllers in applications?

A Multibody Attitude Control Problem





$$\boxed{\dot{z}_1 = u_1, \quad \dot{z}_2 = u_2, \quad \dot{\theta} = \frac{m_1 l_1 u_1}{J + m_1 z_1^2 + m_2 z_2^2} + \frac{m_2 l_2 u_2}{J + m_1 z_1^2 + m_2 z_2^2}}$$

- The system is controllable but $(z_1, z_2, 0) \notin \operatorname{int} f(\mathcal{N}(x_e, u_e))$
  - ♦ Requires either DISCONTINUOUS feedback or TIME-VARYING feedback

Now, let's use L2M for stabilization

Idea: Use u to make $\dot{V} < 0$

- Jurdjevic/Quinn, 1978; Artstein, 1983; Sontag 1989; Tsinias, 1989

- Consider $\dot{x} = f(x) + g(x)u, \quad u = \psi(x)$

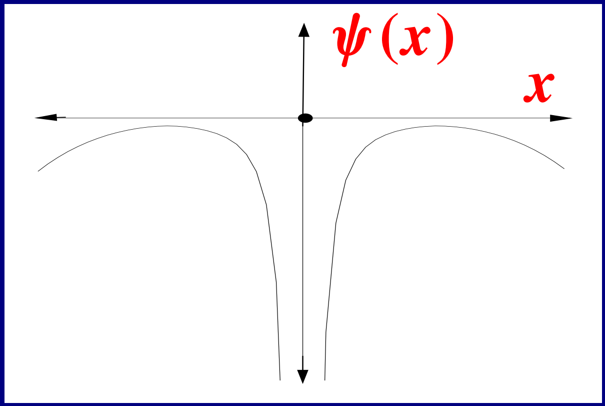
$$\boxed{\text{If } \exists\, u : V'(x)[f(x) + g(x)u] < 0, \text{ then } x_e \text{ is AS'ble with } C^{\infty}\backslash\{0\}\ \psi}$$

- $\psi = -\dfrac{V'f + \sqrt{(V'f)^2 + (V'g)^4}}{V'g}$ is a universal controller

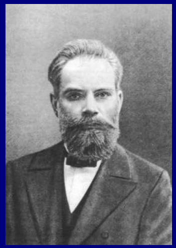
  ♦ $\dot{V} = -\sqrt{(V'f)^2 + (V'g)^4} < 0$

- $\dot{x} = x + x^3 u$

  $u = \psi(x) = -x^{-2}(1 + \sqrt{1 + x^{12}})$



- V is a CONTROL LYAPUNOV FUNCTION (CLF)



Are CLF controllers optimal?

Minimize $J(x_0, u) = \int_0^T L(x,u)\mathrm{d}t$ with $\dot{x} = f(x,u)$

- Kalman, 1964; Moylan/Anderson, 1973; Freeman/Kokotovic, 1996

- **Hamilton-Jacobi-Bellman yields the feedback control**

$$\boxed{\psi(x) = \mathrm{argmin}_u \, [L(x,u) + V'(x) f(x,u)]}$$

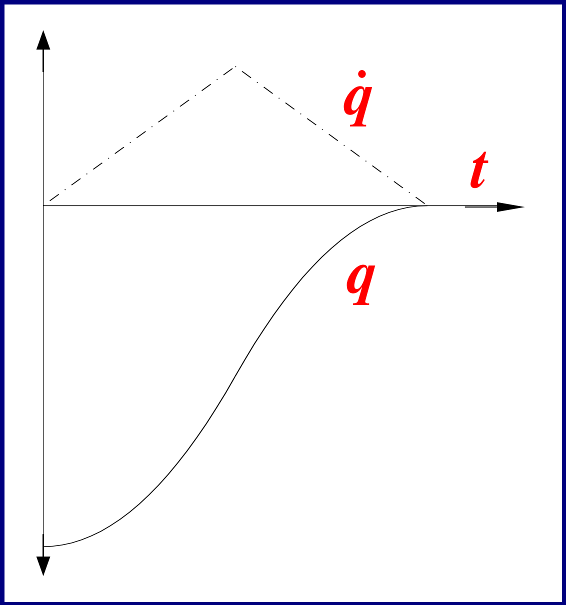
- **The cost-to-go** $\boxed{V(x_0) = \int_0^T L(x, \psi(x))\mathrm{d}t}$ **is a VALUE FUNCTION**

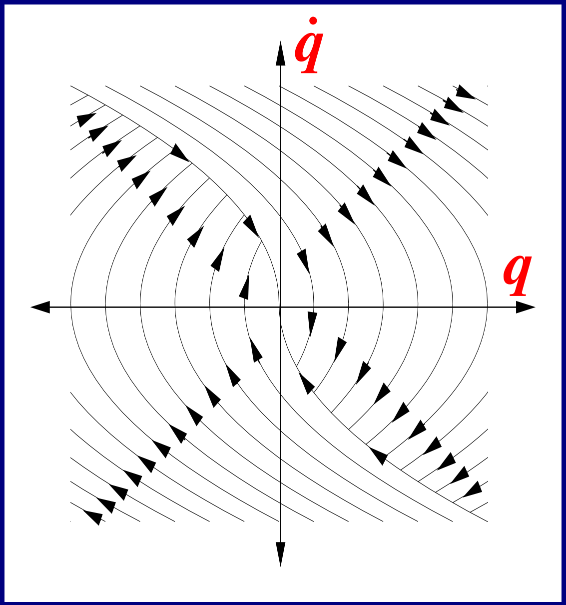
- **Value functions** $\longleftrightarrow$ **control Lyapunov functions**
  - ♦ **Quadratic L yields LQR or H_2 synthesis (linear controllers)**
  - ♦ **Exponential-of-quadratic L yields worst-case or H_∞ synthesis (linear controllers)**
  - ♦ **$L \equiv 1$ yields minimum time synthesis (nonlinear controllers)**

What's special about minimum-time control?

Two Things!

- Consider $\boxed{m\ddot{q} = u}$ with bounded u





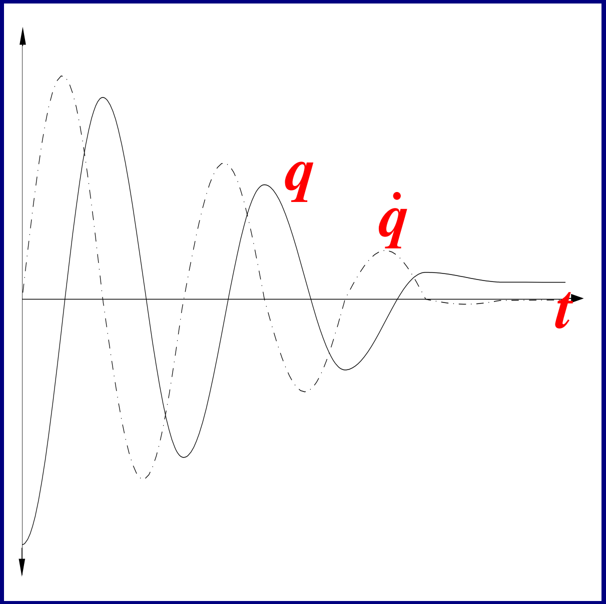
1) Minimum-time control $u = \psi(q, \dot{q})$ is DISCONTINUOUS
   (in fact, it's bang bang)

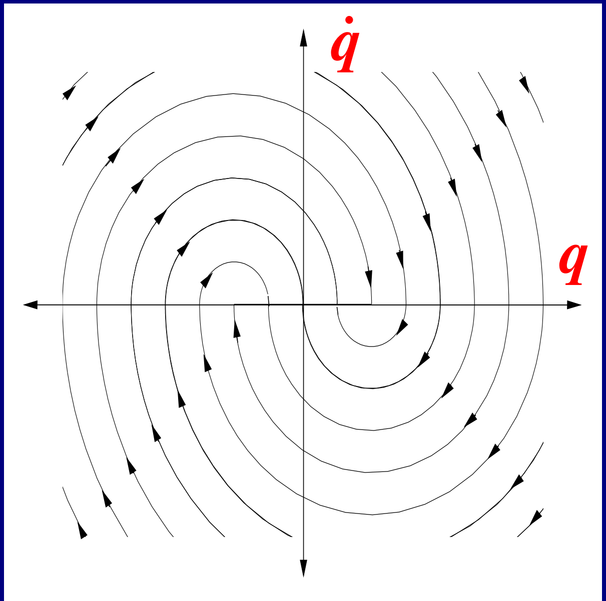
2) And the states converge in FINITE TIME

What else is finite-time convergent?

The Oscillator with Coulomb Friction

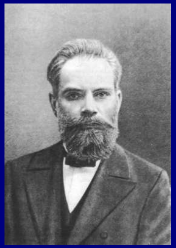
$$\boxed{m\ddot{q} + c\,\mathrm{sign}(\dot{q}) + kq = 0}$$



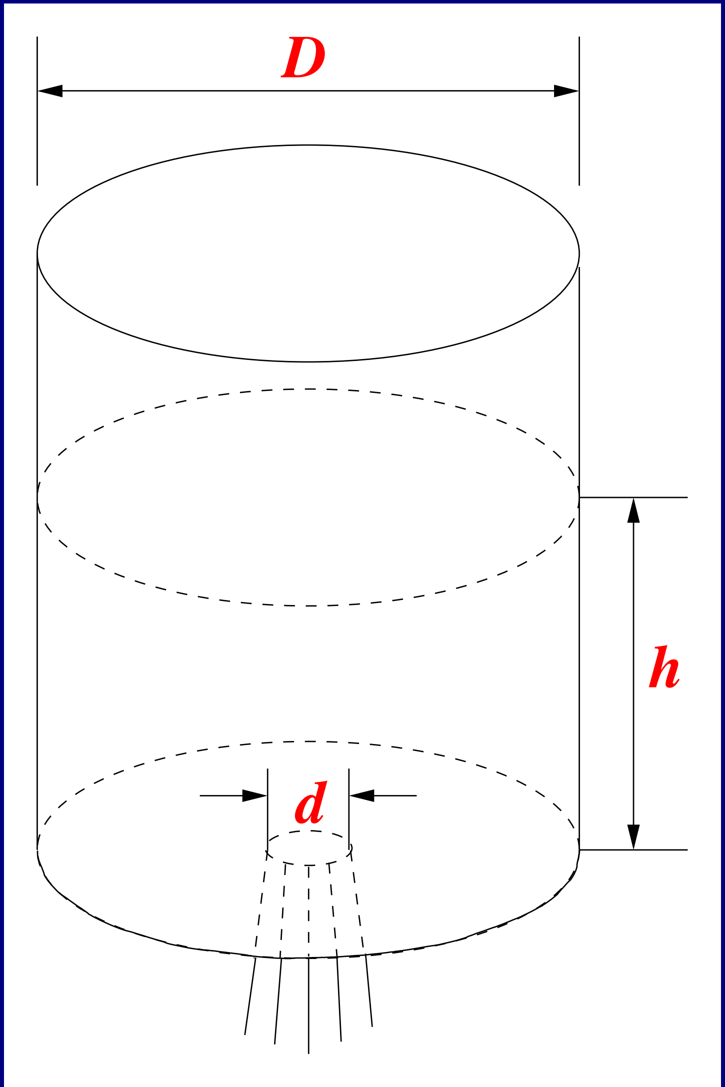


- **These dynamics are DISCONTINUOUS**
  - **And the states converge in FINITE TIME**

Are CONTINUOUS dynamics ever finite-time convergent?

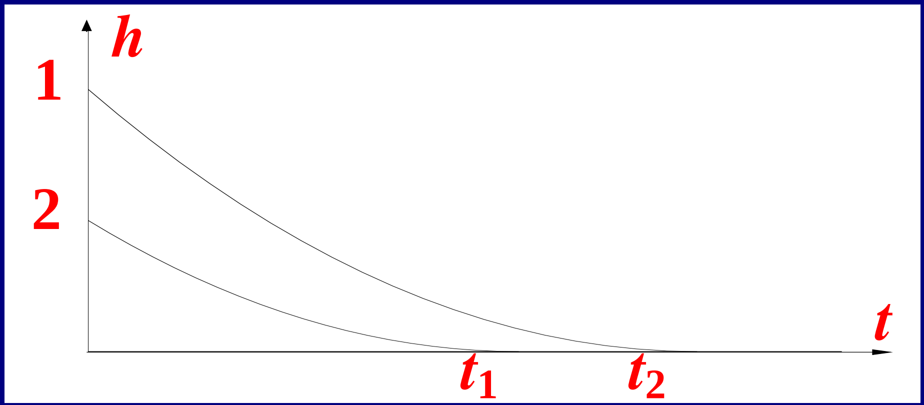


Consider a Leaky Bucket



$\boxed{\dot{h} = -\beta\sqrt{h}}$ where $\boxed{\beta \triangleq \sqrt{\frac{2g}{(D/d)^4-1}}}$

- Vector field is continuous, but NOT Lipschitzian at $h = 0$
  - Nevertheless the solution is unique $\boxed{h(t) = \left(\sqrt{h_0} - \frac{1}{2}\beta t\right)^2}$
  - And all trajectories converge to zero in FINITE TIME
    - As expected!!



- Note that finite-time convergence REQUIRES non-Lipschitzian dynamics . . .
  - . . . since trajectories are NOT UNIQUE in reverse time

Let's use L2M to PROVE finite-time convergence

To Do This, We Need a “Natural" V

- $T(x_0) =$ time to converge from $x_0 =$ time to go
  - $\boxed{T(x(t)) = T(x_0) - t \text{ and therefore } \dot{T}(x(t)) = -1}$
    - But $\dot{T}(0) = 0$ and thus $\dot{T}$ is not continuous

- However, let $\boxed{V(x) = T^2(x)}$ so that $\dot{V} = -2T$, which IS continuous and negative definite

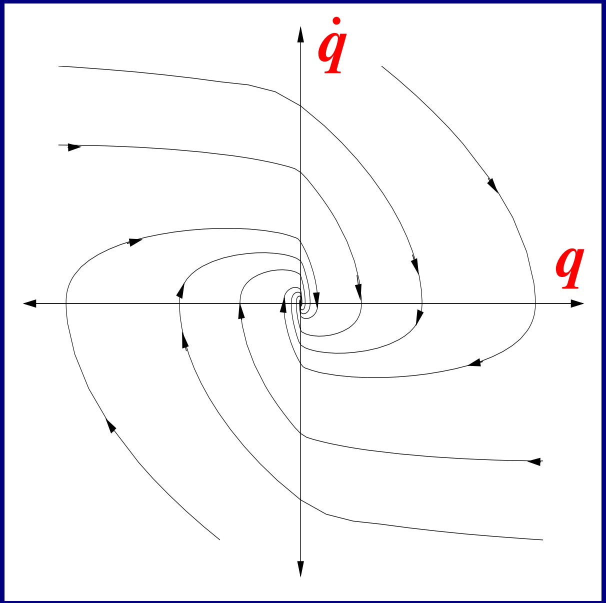
- Now use the comparison lemma
  - $\boxed{V \text{ satisfies } \dot{V} = -2\sqrt{V} < 0}$
  - $\boxed{\int_0^{\xi} \frac{\mathrm{d}\eta}{\sqrt{\eta}} < \infty \text{ upgrades to finite-time convergence via } V = T^2}$

Next, let’s finite-time STABILIZE with a continuous ψ

Consider the Double Integrator Again

- CONTINUOUS controller $u = \psi(q, \dot{q}) = -q^{1/5} - \dot{q}^{1/3}$
  - Closed-loop system $m\ddot{q} + \dot{q}^{1/3} + q^{1/5} = 0$

- $\boxed{V(q, \dot{q}) = \frac{5}{6}q^{6/5} + \frac{1}{2}m\dot{q}^2}$
  - $\dot{V} = -\dot{q}^{4/3} \leq 0$ gives LS
  - Invariance principle upgrades to AS
    - But V is defective : – (



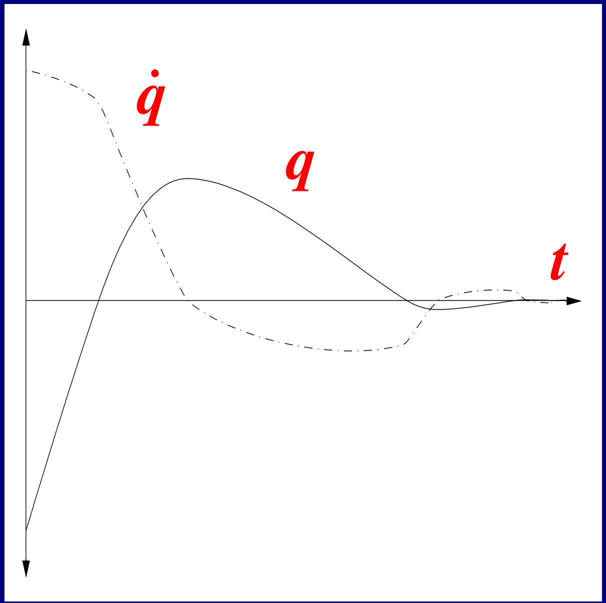
Let's prove finite-time convergence

- The system $\dot{x} = f(x)$ is HOMOGENEOUS of DEGREE r with respect to the DILATION $\Delta = \mathrm{diag}(\alpha^{r_1}, \ldots, \alpha^{r_n})$ if

$$\boxed{f(\Delta x) = \alpha^r \Delta f(x)}$$

  - Familiar case $\Delta = \alpha I$ yields $f(\alpha x) = \alpha^{r+1} f(x)$

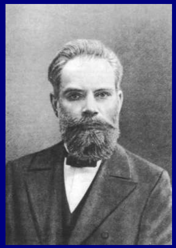
- Theorem $\boxed{\text{FTC} \iff \text{AS and } r < 0}$
  - $m\ddot{q} + \dot{q}^{1/3} + q^{1/5} = 0$ is AS
  - With $\Delta = \begin{bmatrix} \alpha^5 & 0 \\ 0 & \alpha^3 \end{bmatrix}$, $r = -2 < 0$
  - Hence finite-time convergent



- Negative-degree homogeneity upgrades from AS to FTC
  - Bhat/Bernstein, 1997

What Do We Do If Some States Don’t Converge?

Idea #1: Ignore Them

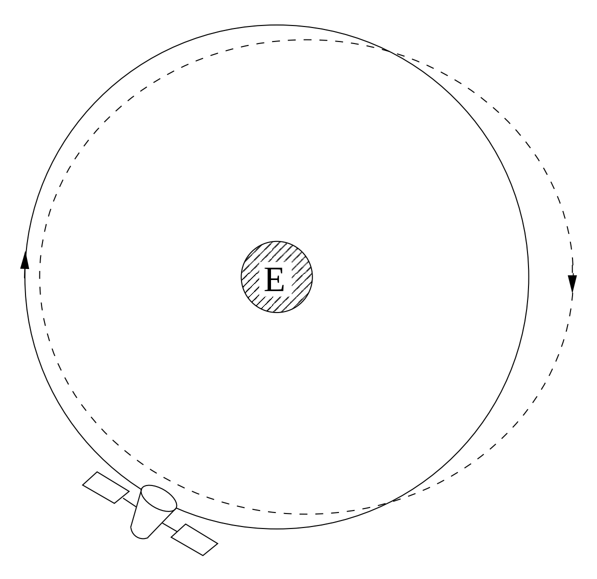


$$\boxed{\ddot{r} - r\dot{\theta}^2 + \frac{\mu}{r^2} = 0}$$

Orbital Motion

$$\boxed{\ddot{\theta} = \frac{-2\dot{\theta}\dot{r}}{r}}$$

- $\theta(t) \to \infty$ but luckily $\theta(t)$ doesn't appear
  - So ignore it!

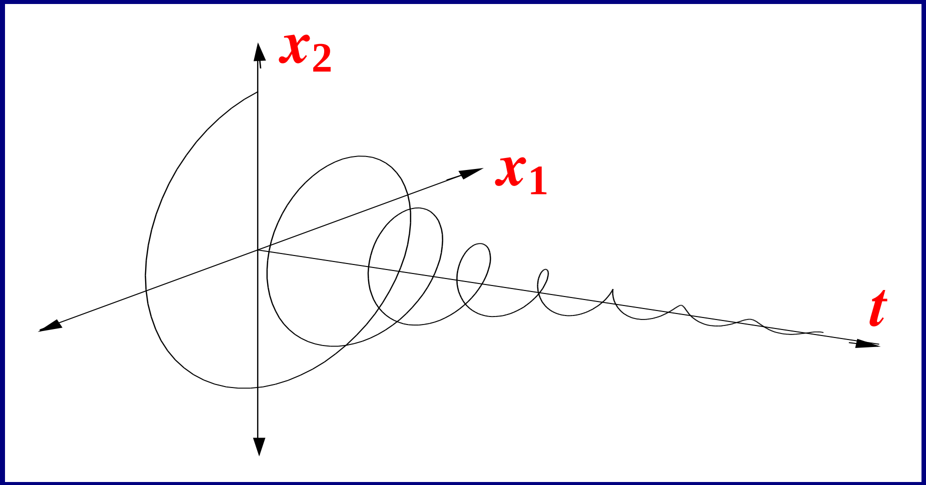


- Constants of motion
  - Energy $E = \frac{\dot{r}^2}{2} + \frac{r^2\dot{\theta}^2}{2} - \frac{\mu}{r}$
  - Angular momentum $h = r^2\dot{\theta}$
- For CIRCULAR orbit define $\boxed{V(r, \dot{r}, \dot{\theta}) = (E - E_c)^2 + (h - h_c)^2}$
  - $\dot{V} \equiv 0$ and thus the circular orbit is LS

Can we always ignore states that don't converge?

No: Consider Time-Varying Systems

- $\dot{x} = f(x, t)$
- Introducing $\dot{x}_{n+1} \equiv 1$ yields $\dot{\tilde{x}} = \tilde{f}(\tilde{x}) \triangleq \begin{bmatrix} f(x, x_{n+1}) \\ 1 \end{bmatrix}$
  - ♦ which “looks" time-invariant



- But $x_{n+1}(t) = t \longrightarrow \infty$ as $t \longrightarrow \infty$ (fairly obvious)
  - ♦ AND, you canNOT ignore t!

Idea #2: Require stability wrto only PART of x

Partition $x = (x_1, x_2)$

- Rumyantsev, 1970; Vorotnikov, 1998

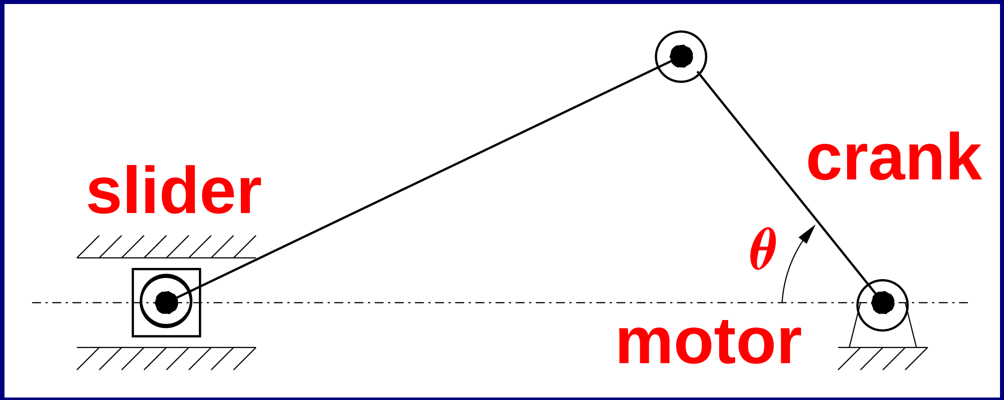
- Get partitioned dynamics $\boxed{\dot{x}_1 = f_1(x_1, x_2)}$ $\boxed{\dot{x}_2 = f_2(x_1, x_2)}$

- Define a PARTIAL equilibrium x_{1e} satisfying
  $f_1(x_{1e}, x_2) \equiv 0$ for all x_2
  - ♦ Consider ONLY $x_1(t) - x_{1e}$

- Theorem: Assume x_{1e} is a strict minimizer of $V(x_1)$
  - ♦ $\boxed{V'(x_1)\, f_1(x_1, x_2) \le 0 \implies \text{PARTIAL Lyapunov stability}}$
  - ♦ $\boxed{V'(x_1)\, f_1(x_1, x_2) < 0 \implies \text{PARTIAL asymptotic stability}}$

Are there any useful applications of partial stability?

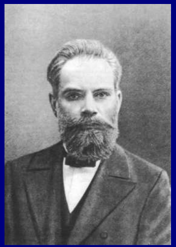
The Controlled Slider-Crank!



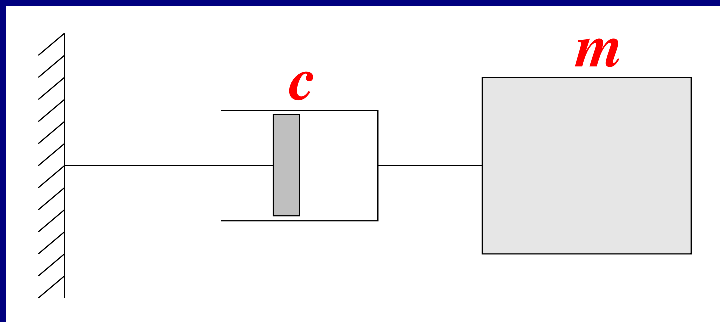
$$m(\theta)\ddot{\theta} + c(\theta)\dot{\theta}^2 = u$$

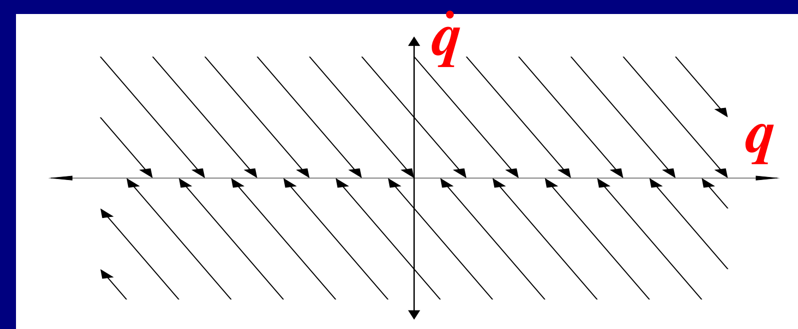
- Choose $u = \psi(\theta, \dot{\theta})$ so that $\dot{\theta}(t) \longrightarrow \dot{\theta}_{\text{des}}$ (constant angular velocity)
  - But this implies $\theta(t) \approx \dot{\theta}_{\text{des}} t \longrightarrow \infty$
- However, since $m(\theta)$ and $c(\theta)$ depend on θ, we canNOT ignore θ
  - AND, since $\theta(t) \to \infty$,
  - we canNOT get AS but we CAN get PARTIAL AS

Next, let's require that all states converge to SOMETHING



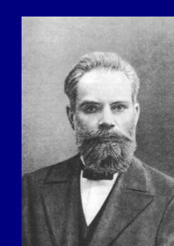
Consider the Damped Rigid Body





- $\boxed{\text{NONE of equilibria are AS!!}}$
- Since $\dot{q} \longrightarrow 0$, we have partial AS wrto $\dot{q}$
- Also $q \longrightarrow q_\infty$, where q_∞ is determined by initial conditions
- So all states converge to SOMETHING

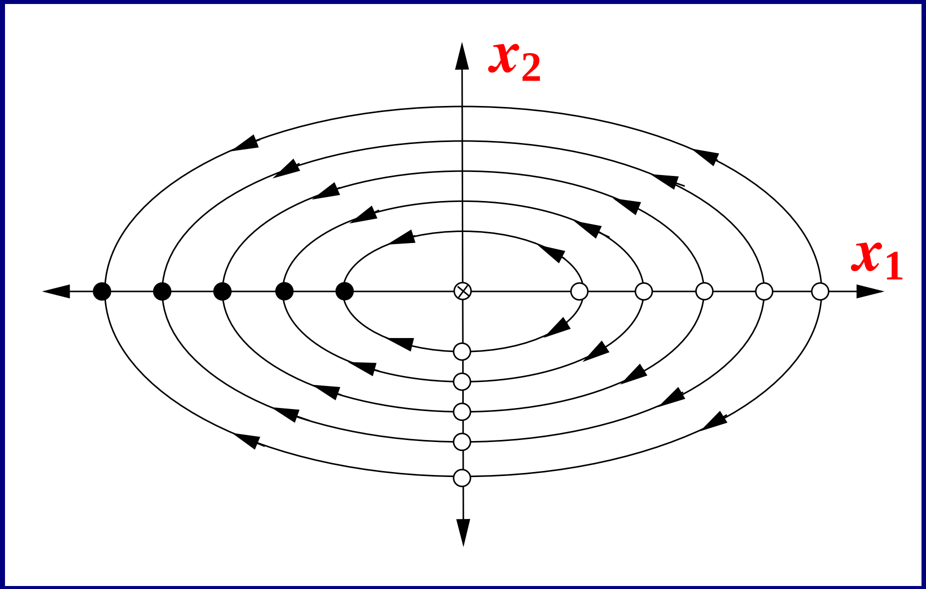
What kind of stability is THIS?



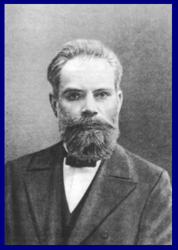
This is SEMISTABILITY

- Campbell, 1980

- Suppose we have a CONTINUUM of equilibria



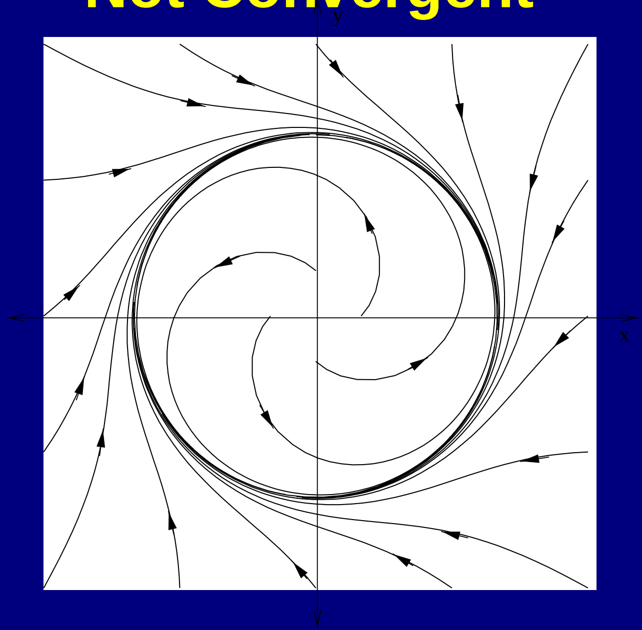
- A LS equilibrium is SEMISTABLE if every nearby trajectory converges to a (possibly different) LS equilibrium
- Sandwich property: AS $\Longrightarrow$ semistability $\Longrightarrow$ LS



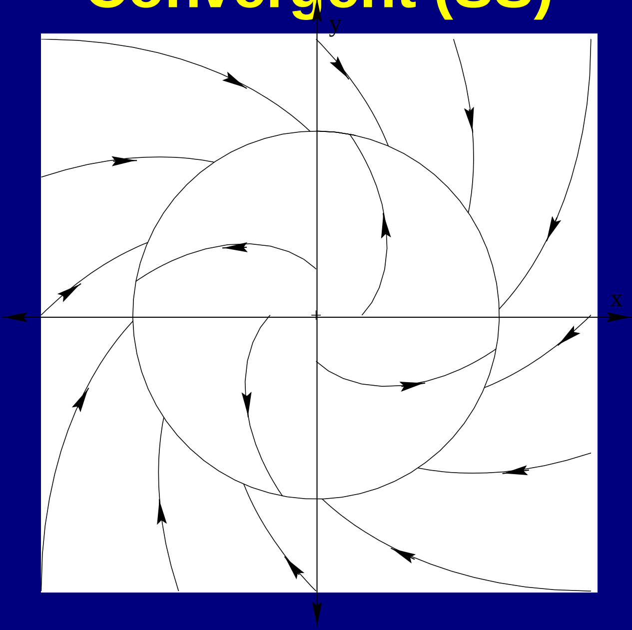
How Can We Analyze Semistability?

- Bhat/Bernstein, 2001

Not Convergent

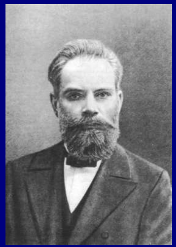


Convergent (SS)



- Assume x_e is a local minimizer of V and $\dot{V} \leq 0$
  - If f is NONTANGENT to the 0 level set of $\dot{V}$ near x_e, then x_e is semistable
- Note that V need only be POSITIVE SEMIDEFINITE at x_e

Are there any interesting applications of semistability?



Michaelis-Menten reaction

$$\boxed{\mathrm{S} + \mathrm{E} \underset{k_2}{\overset{k_1}{\rightleftharpoons}} \mathrm{C} \xrightarrow{k_3} \mathrm{P} + \mathrm{E}}$$

$$\boxed{\begin{array}{lll lll} x_1 & \triangleq & [\mathrm{S}] & \dot{x}_1 & = & k_2 x_2 - k_1 x_1 x_3 \\ x_2 & \triangleq & [\mathrm{E}] & \dot{x}_2 & = & -(k_2 + k_3) x_2 + k_1 x_1 x_3 \\ x_3 & \triangleq & [\mathrm{C}] & \dot{x}_3 & = & (k_2 + k_3) x_2 - k_1 x_1 x_3 \\ x_4 & \triangleq & [\mathrm{P}] & \dot{x}_4 & = & k_3 x_2 \end{array}}$$

- **All states are nonnegative and all $(0, 0, x_3, x_4)$ are equilibria**
- **Let's choose $V = \alpha x_1 + x_2 \geq 0$ (semidefinite) and thus $\dot{V} \leq 0$**
  - **Note that LINEAR V is allowed since all states are NONNEGATIVE**
- **Nontangency implies that all equilibria are semistable**
  - **and $[\mathrm{S}] \to 0, \quad [\mathrm{E}] \to 0, \quad [\mathrm{C}] \to [\mathrm{C}]_\infty, \quad [\mathrm{P}] \to [\mathrm{P}]_\infty$**
  - **where $[\mathrm{C}]_\infty$ and $[\mathrm{P}]_\infty$ depend on initial conditions**

Is Anything Else Semistable?

- Compartmental models
  - Mass transport
    - Biological systems
  - Energy transport
    - Thermodynamics

Could semistability possibly be useful for ADAPTIVE control?

Simplest Adaptive Stabilization Problem

- Consider the uncertain system $\boxed{\dot{x} = Ax + Bu}$ $\boxed{u = -Kx}$
  - Assume there exists unknown K_s such that $A + BK_s$ is AS
- Consider the control update $\boxed{\dot{K} = B^{T}Pxx^{T}}$
- Let $V(x, K) = x^{T}Px + \mathrm{tr}\,(K - K_s)^{T}(K - K_s)$ so $\dot{V}(x, K) = -x^{T}x$
  - Hence we have LS
  - Invariance principle implies $x \to 0$
- $\boxed{\text{Nontangency implies semistability and } K \longrightarrow K_\infty}$
  - K_∞ depends on initial conditions

Note that B must be known

What Do We Do When B is Unknown?

- Nussbaum, 1983; Morse, 1985

- Consider $\boxed{\dot{x} = ax + bu}$ where $\mathrm{sign}(b)$ is unknown

- We canNOT use increasing gain k with $\dot{k} = x^2$ and $u = kx$

- Instead we use increasing gain k with $\dot{k} = x^2$ and OSCILLATING control amplitude $\boxed{u = k^2(\cos k)x}$

- Define INDEFINITE V

  $$V(x, k) = e^{-k} + \tfrac{1}{2}x^2 + (a + 2b\cos k)k + bk^2 \sin k - 2b \sin k$$

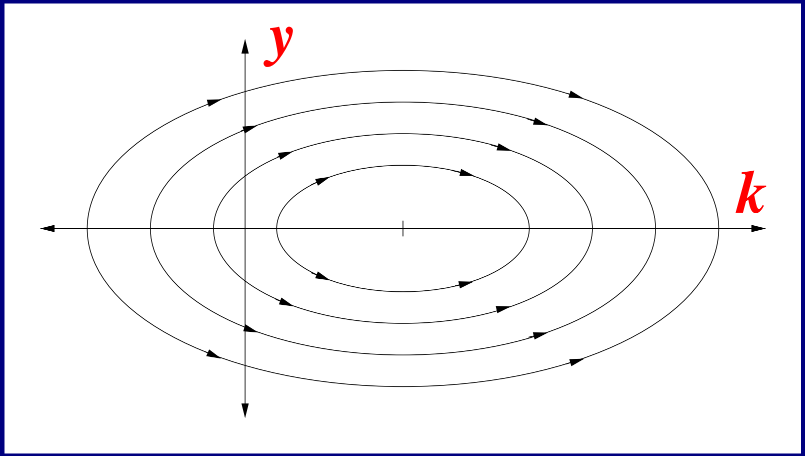
  - ♦ Every sublevel set of V is a union of disconnected compact sets
  - ♦ $\dot{V} = -e^{-k}x^2 \leq 0$, which implies x is bounded
  - ♦ $\boxed{\text{Nontangency implies semistability and } k \longrightarrow k_\infty}$

Output-Feedback Adaptive Stabilization

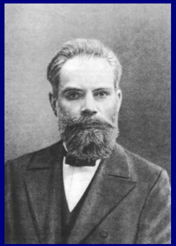
- Consider minimum phase G with relative degree 1

$$\boxed{y = Gu} \qquad \boxed{u = -ky} \qquad \boxed{\dot{k} = y^2}$$

- Scalar case: $V(x, k) = e^{-k} + \frac{1}{2}x^2 + \frac{1}{2b}(a + bk)^2$
  - ♦ $\dot{V} = -e^{-k}y^2 \le 0$ implies x is bounded and $y \longrightarrow 0$
  - ♦ f is nontangent to the zero level set of $\dot{V}$
    - ▲ Hence semistability holds and $k \longrightarrow k_\infty$



What happens if y is noisy?



Noise is the Scourge of Adaptive Control!!

- Consider minimum phase G with relative degree 1

$$\boxed{y = Gu + w} \qquad \boxed{u = -ky} \qquad \boxed{\dot{k} = y^2}$$

- Noise w causes $k \longrightarrow \infty$!!

- Damped modification $\dot{k} = -\gamma k + y^2$ causes bursting

What can we do about this?

Use Chattering Control for Noise Rejection

$$\boxed{\ddot{q} + a_1\dot{q} + a_2 q = \dot{u} - zu + w_1} \qquad \boxed{y = q + w_2}$$

- a_1 and a_2 are unknown, $z < 0$ but otherwise unknown
- w_1 and w_2 are bounded with UNKNOWN bound

$$\dot{q}_f = -\lambda q_f + y \qquad \text{filters } y$$

$$\dot{u}_f = \hat{z} q_f + u \qquad \text{filters } u$$

$$\dot{\hat{a}}_1 = -\tilde{x}_f \dot{q}_f \qquad \text{estimates } a_1$$

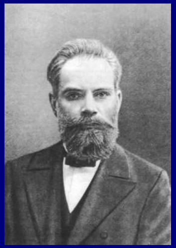
$$\dot{\hat{a}}_2 = -\tilde{x}_f q_f \qquad \text{estimates } a_2$$

$$\dot{\hat{\alpha}} = k\hat{\alpha}^{3/2}|\tilde{x}_f| \qquad \text{estimates bound on } w_1 \text{ and } w_2$$

$$\dot{\hat{z}} = -\tilde{x}_f u_f (-\hat{z})^{3/2} \qquad \text{estimates } z$$

$$u = (\lambda + \hat{z})u_f + (\hat{a}_1 - f_1)\dot{q}_f + (\hat{a}_2 - f_2)q_f - \underbrace{\alpha \mathrm{sign}(\tilde{x}_f)}_{\text{chattering}}$$

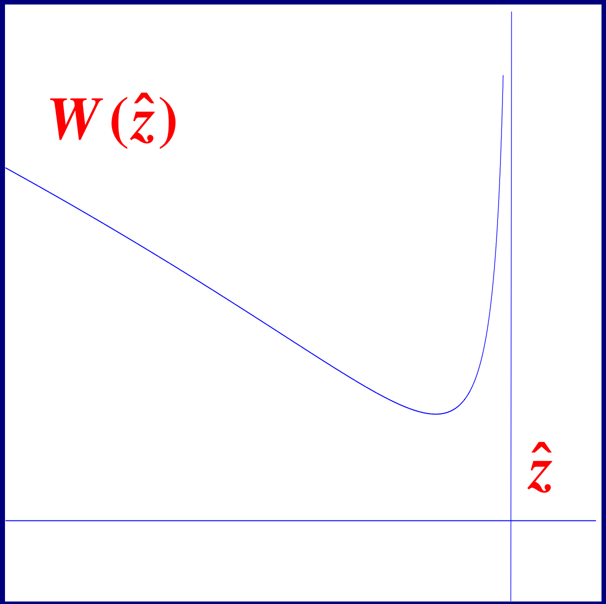
So what's V?



Here's V!

- Sane, Bernstein, Sussmann, 2001

- $$V(x) = \begin{bmatrix} q_\mathrm{f} \\ \dot{q}_\mathrm{f} \\ \hat{a}_1 \\ \hat{a}_2 \\ \hat{\alpha} \end{bmatrix}^\mathrm{T} P \begin{bmatrix} q_\mathrm{f} \\ \dot{q}_\mathrm{f} \\ \hat{a}_1 \\ \hat{a}_2 \\ \hat{\alpha} \end{bmatrix} + \underbrace{\sqrt{-\hat{z}} - \frac{z}{\sqrt{-\hat{z}}}}_{W(\hat{z})}$$



  - ♦ $W(\hat{z})$ confines $\hat{z} < 0$ since $z < 0$
  - ♦ W is a LYAPUNOV WELL

- $\dot{V}$ is INDEFINITE
  - ♦ But $\dot{V}(x(t)) \leq \gamma e^{-2\lambda t}$ along trajectories
  - ♦ Hence $\dot{V}(x(t))$ is ASYMPTOTICALLY NONPOSITIVE

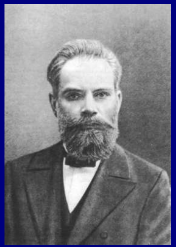
- Use BARBALAT'S LEMMA to prove $y \longrightarrow 0$
  - ♦ . . . and all states are bounded : –)

Let’s Recapitulate

We Acknowledged the Weaknesses of L2M

- While Lyapunov-like ideas are the basis of classical stability analysis (e.g., the Lagrange-Dirichlet stability condition), L2M per se has had relatively few successes outside of control

- In general, it's simply too difficult to construct Lyapunov functions using only the vector field



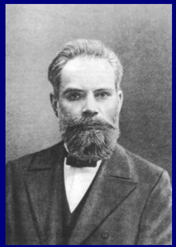
And We Celebrated Its Successes

- L2M is immensely successful in control theory
- While it had no impact on CLASSICAL control . . .
- . . . it's the heart and soul of MODERN control, where we synthesize controllers to suit Lyapunov functions of CHOSEN form
  - ♦ The ability to construct the control and the Lyapunov function TOGETHER is what makes L2M so successful in control
  - ♦ L2M is the backbone of optimal, robust, and adaptive control

We Traveled from Infancy . . .

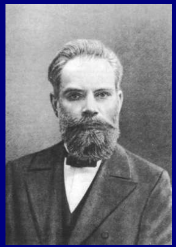
$$\boxed{\dot{V} \le 0 \implies x_e \text{ is Lyapunov stable}}$$

$$\boxed{\dot{V} < 0 \implies x_e \text{ is asymptotically stable}}$$



. . . to Potency

- Invariance principle
- Comparison lemma
- Control Lyapunov functions
- Homogeneity
- Partial stability
- Semistability
- Nontangency
- Semidefinite and indefinite V's
- Asymptotic nonpositivity
- Barbalat's lemma
- Lyapunov wells



What Lies in the Future for L2M?

L2M Beyond ODE’s

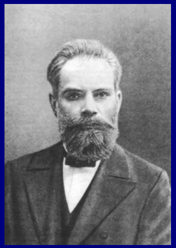
- **Discontinuous dynamics and differential inclusions**
  - ♦ **Nonholonomic dynamics**
  - ♦ **Relay and sliding mode control**
  - ♦ **Essential in control**

- **PDE’s**
  - ♦ **Stability of solitons**
  - ♦ **Hysteresis in smart materials**
  - ♦ **Flow stabilization**
  - ♦ **Many other applications**

Input-Output Analysis Based on L2M

- **Dissipativity** (Willems)
  - Storage function V_s, supply rate $r(u, y)$
    - $\dot{V}_s(x) \le r(u, y)$
  - Nonlinear positive real theory (passivity)
  - Nonlinear bounded real/H_∞ theory (nonexpansivity)

- **Input-to-state stability** (Sontag)
  - GAS: $\|x(t)\| < b(\|x(0)\|, t) \iff \dot{V} < -a(x)$
  - ISS: $\|x(t)\| < b(\|x(0)\|, t) + \sup|u| \iff \dot{V} < -a(x) + b(u)$



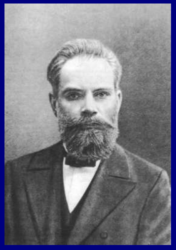
Trends in Nonlinear Control Based on L2M

- **Receding horizon control**
  - ♦ **CLF's to obtain suboptimal HJB solutions**

- **Problems with control and state constraints**
  - ♦ **Anti-windup and control saturation**
  - ♦ **Invariant set methods for state constraints** (Gilbert/Kolmanovsky)

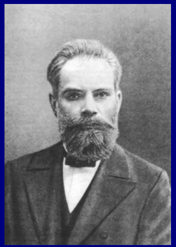
- **Gain scheduling methods**
  - ♦ **LPV methods**
  - ♦ **Equilibrium switching methods (multiple V's)**

- **Impulsive dynamics**
  - ♦ **Hybrid systems** (Lakshmikantham, Haddad/Chellaboina/Bhat)
  - ♦ **Resetting controllers** (Hollot/Chait)



Specialized Applications of L2M

- Nonnegative systems
  - Chemical kinetics
    - Zero deficiency theorem for rate-independent semistability (Feinberg)
- Emergent behavior of large scale, interconnected systems
  - Thermodynamics
    - Analyze energy flow and entropy as emergent properties
    - Linear storage functions and supply rates (Haddad/Chellaboina)
  - Swarm dynamics



Some Research Questions

- Can we do more with L2M in discrete time?
  - Discrete-time adaptive control, especially for disturbance rejection (many patents due to lack of theory!)
- Can we use set stability (Zubov, Bhatia/Szego) to prove LS of an elliptical orbit?
  - Poisson and orbital stability
- Is there an L2M foundation for averaging?
- Is there an L2M proof of the Poincare stability theorem?

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What Hath Lyapunov Wrought!