Proceedings of the 30th Conference
on Decision and Control
Brighton, England • December 1991

Explicit Construction of Quadratic Lyapunov Functions for the Small Gain, Positivity, Circle, and Popov Theorems and Their Application to Robust Stability

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ABSTRACT

Lyapunov Function proofs of sufficient conditions for asymptotic stability are given for feedback interconnections of bounded real and positive real transfer functions. Two cases are considered: 1) a proper bounded real (resp., positive real) transfer function with a bounded real (resp., positive real) time-varying memoryless nonlinearity; and 2) two strictly proper bounded real (resp., positive real) transfer functions. A similar treatment is given for the circle and Popov theorems. Application of these results to robust stability with time-varying bounded real, positive real, and sector-bounded uncertainty is discussed.

1. INTRODUCTION

One of the most basic issues in system theory is stability of feedback interconnections. Two of the most fundamental results concerning stability of feedback systems are the small-gain theorem and the positivity theorem [1–12]. Here we focus (in Sections 3 and 4) on the sufficiency aspect of these results. The small gain theorem implies that if G and G_c are asymptotically stable bounded-gain transfer functions such that $\|G\|_\infty \|G_c\|_\infty < 1$, then the feedback interconnection of G and G_c is asymptotically stable. Furthermore, the positivity theorem states that if G and G_c are (square) positive real transfer functions, one of which is strictly positive real, then the negative feedback interconnection of G and G_c is asymptotically stable.

For robust stability, if G_c represents an uncertain perturbation, then it follows from the small gain theorem that an H_∞-norm bound on G implies robust stability in the presence of an H_∞ norm bound on G_c. Similarly, if the system uncertainty G_c can be cast as a positive real transfer function and G is strictly positive real, then the positivity theorem implies robust stability. Although the small gain theorem and positivity theorem are equivalent via a suitable transformation [7], positive real modeling of system uncertainty may be significantly less conservative than small gain modeling of system uncertainty. This improvement is due to the fact that the small gain theorem is a normed-based result which captures gain uncertainty but ignores phase information. Since positive real transfer functions are phase bounded, the positivity theorem can exploit phase characteristics within a feedback interconnection.

Although the predominant approach to stability theory is Lyapunov's method, most of the available proofs of the small gain and positivity theorems are based upon input-output properties and function-analytic methods [1–3,6–8]. The purpose of this paper is thus to explicitly construct quadratic Lyapunov functions to prove sufficiency in special cases of the small gain and positivity theorems. Specifically, sufficient conditions for asymptotic stability are addressed for two cases of feedback interconnections. The first case involves a proper, but not necessarily strictly proper, bounded real (resp., strongly positive real) transfer function in a positive feedback (resp., negative feedback) configuration with a bounded real (resp., positive real) time-varying memoryless nonlinearity. The second case addresses the same problem with two strictly proper systems. Specialization of these results to robust stability with linear time-varying bounded real and positive real (but otherwise unknown) plant uncertainty is also discussed.

Having addressed the small gain and positivity theorems, we then turn our attention (in Section 5) to the well-known circle criterion or circle theorem [14–29]. In a multivariable setting this result applies to sector-bounded nonlinearities and thus, upon appropriate specialization, generalizes (and includes as special cases) both the small gain and positivity results. Thus, for practical purposes, the circle theorem provides the means for incorporating both gain and phase aspects. The proof of the circle theorem given here is completely consistent with the proofs of the small gain and positivity results, thus providing a unified treatment of these classical results.

Next we focus (in Section 6) on the Popov stability criterion [30–49]. Although often discussed in juxtaposition with the circle criterion, the Popov criterion is quite distinct with regard to its Lyapunov function foundation. Whereas the small gain, positivity, and circle results are based upon fixed quadratic Lyapunov functions, the Popov result is based upon a quadratic Lyapunov function that is a *function* of the sector-bounded nonlinearity. Thus, in effect, the Popov result guarantees stability by means of a *family* of Lyapunov functions. For robust stability, this situation corresponds to the construction of a parameter-dependent Lyapunov function as proposed in [50,51]. A key aspect of the Popov result is the fact that it does *not* apply to time-varying uncertainties, which renders it less conservative than fixed quadratic Lyapunov function results (such as the small gain, positivity, and circle results) in the presence of real, constant parameter uncertainty.

Our proof of the Popov criterion is given in a form that is similar to the proofs of the small gain, positivity, and circle theorems. This unified presentation is intended to clarify relationships among these results.

There are several reasons for seeking Lyapunov-function proofs of the small gain and positivity theorem. For example, these proofs help to build stronger ties between state space and frequency domain approaches to feedback system theory. Furthermore, these quadratic Lyapunov functions may be useful for extending previous results on the synthesis of robust feedback controllers [52–59].

2. PRELIMINARIES

In this section we establish definitions and notation. Let $\mathbb{R}$ and $\mathbb{C}$ denote the real and complex numbers, let $(\ )^T$ denote transpose, and let I_n or I denote the $n \times n$ identity matrix. Furthermore, we write $\|\cdot\|_2$ for Euclidean norm and $\sigma_{\max}(\cdot)$ for the maximum singular value and $M \geq 0$ ($M > 0$) to denote the fact that the Hermitian matrix M is nonnegative (positive) definite. In this paper a *real-rational matrix function* is a matrix whose elements are rational functions with real coefficients. Furthermore, a *transfer function* is a real-rational matrix function each of whose elements is *proper*, i.e., finite at $s = \infty$. A *strictly proper transfer function* is a transfer function that is zero at infinity. Finally, an *asymptotically stable transfer function* is a transfer function each of whose poles is in the open left half plane. The space of asymptotically stable transfer functions is denoted by RH_∞, i.e., the real-rational subset of H_∞ [10]. Let

$$G(s) \sim \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

denote a state space realization of a transfer function $G(s)$, that is, $G(s) = C(sI - A)^{-1}B + D$. The notation "$\overset{\min}{\sim}$" is used to denote a minimal realization.

A transfer function $G(s)$ is *bounded real* [8] if 1) $G(s)$ is asymptotically stable and 2) $I - G^*(j\omega)G(j\omega)$ is nonnegative definite for all real ω. Equivalently, 2) can be replaced by [8, p. 307] 2′) $I - G^*(s)G(s)$ is nonnegative definite for $\mathrm{Re}[s] > 0$. Alternatively, a transfer function $G(s)$ is bounded real if and only if $G(s)$ is asymp-

* This research was supported in part by the Air Force Office of Scientific Research under contract F49620-89-C-0011 and the Florida Space Grant Consortium under grant NGT-40015.

totically stable and $\|G(s)\|_\infty \leq 1$. Furthermore, $G(s)$ is called *strictly bounded real* if 1) $G(s)$ is asymptotically stable and 2) $I - G^*(j\omega)G(j\omega)$ is positive definite for all real ω. Finally, $G(s)$ is *strongly bounded real* if it is strictly bounded real and $I - D^{\mathrm{T}}D > 0$, where $D \triangleq G(\infty)$.

A square transfer function $G(s)$ is called *positive real* [8, p. 216] if 1) all poles of $G(s)$ are in the closed left half plane and 2) $G(s)+G^*(s)$ is nonnegative definite for $\mathrm{Re}[s] > 0$. A square transfer function $G(s)$ is called *strictly positive real* [9,11,12] if 1) $G(s)$ is asymptotically stable and 2) $G(j\omega)+G^*(j\omega)$ is positive definite for all real ω. Finally, a square transfer function $G(s)$ is *strongly positive real* if it is strictly positive real and $D + D^{\mathrm{T}} > 0$, where $D \triangleq G(\infty)$. Recall that the minimal realization of a positive real transfer function is stable in the sense of Lyapunov [8]. Furthermore, strongly positive real implies strictly positive real, which further implies positive real.

For notational convenience in the paper, G will denote an $\ell \times m$ transfer function with input $u \in \mathbb{R}^m$, output $y \in \mathbb{R}^\ell$, and internal state $x \in \mathbb{R}^n$.

Next we give two key lemmas concerning bounded real and positive real matrices.

Lemma 2.1. *Let $M \in \mathbb{C}^{\ell\times m}$ and $N \in \mathbb{C}^{m\times \ell}$ be such that $\sigma_{\max}(M) \leq 1$ and $\sigma_{\max}(N) < 1$. Then $[I_\ell - MN] \neq 0$.*

Proof. Since $\sigma_{\max}(M) \leq 1$ and $\sigma_{\max}(N) < 1$ it follows that $\rho(MN) \leq \sigma_{\max}(MN)$
$\leq \sigma_{\max}(M)\sigma_{\max}(N) < 1$, where $\rho(\cdot)$ denotes spectral radius. Hence $\det[I_\ell - MN] \neq 0$. Q.E.D.

Lemma 2.2. *Let $M, N \in \mathbb{C}^{m\times m}$ be such that $M + M^* \geq 0$ and $N + N^* > 0$. Then $(I_m + MN) \neq 0$.*

Proof. First we show that N is invertible. Let $x \in \mathbb{C}^m$, $x \neq 0$, and $\lambda \in \mathbb{C}$ be such that $Nx = \lambda x$ and hence $x^*N^* = \bar{\lambda}x^*$. Then $x^*(N + N^*)x > 0$ implies that $\mathrm{Re}\,\lambda > 0$. Hence $\det N \neq 0$. Now define $S \triangleq N^{-1} + M$. Now, since $N^{-1} + N^{-*} = N^{-1}(N + N^*)N^{-*}$ it follows that $S + S^* > 0$. Thus $\det S \neq 0$. Consequently, $\det(I_m + MN) = \det NS = (\det N)(\det S) \neq 0$. Q.E.D.

3. THE SMALL GAIN THEOREM

In this section we use quadratic Lyapunov functions to prove sufficiency in the small gain theorem in two cases. First, recall the bounded real lemma [8].

Lemma 3.1 (Bounded Real Lemma). *$G(s) \overset{\min}{\sim} \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is bounded real if and only if there exist real matrices P, L, and W with P positive definite such that*

$$0 = A^{\mathrm{T}}P + PA + C^{\mathrm{T}}C + L^{\mathrm{T}}L, \tag{3.1}$$

$$0 = PB + C^{\mathrm{T}}D + L^{\mathrm{T}}W, \tag{3.2}$$

$$0 = I - D^{\mathrm{T}}D - W^{\mathrm{T}}W. \tag{3.3}$$

Proof. Sufficiency follows from algebraic manipulation of (3.1)–(3.3) while necessity follows from spectral factorization theory. For details see [8]. Q.E.D.

Remark 3.1. If (3.1) is replaced by

$$0 = A^{\mathrm{T}}P + PA + C^{\mathrm{T}}C + L^{\mathrm{T}}L + R, \tag{3.1$'$}$$

where $R \geq 0$, then (3.1)$'$–(3.3) imply that $G(s)$ is bounded real.

Suppose in Lemma 3.1 $\sigma_{\max}(D) < 1$. Then since $I - D^{\mathrm{T}}D > 0$ and

$$W^{\mathrm{T}}W = I - D^{\mathrm{T}}D, \tag{3.4}$$

$W^{\mathrm{T}}W$ is nonsingular. Furthermore, (3.2) is equivalent to

$$L^{\mathrm{T}}W = -(PB + C^{\mathrm{T}}D). \tag{3.5}$$

Using (3.5) and noting that $W(W^{\mathrm{T}}W)^{-1}W^{\mathrm{T}}$ is an orthogonal projection so that $L^{\mathrm{T}}L \geq L^{\mathrm{T}}W(W^{\mathrm{T}}W)^{-1}WL$, it follows from (3.1) that

$$0 \geq A^{\mathrm{T}}P+PA+(PB+C^{\mathrm{T}}D)(W^{\mathrm{T}}W)^{-1}(B^{\mathrm{T}}P+D^{\mathrm{T}}C)+C^{\mathrm{T}}C \tag{3.6}$$

or, since $(W^{\mathrm{T}}W)^{-1} = (I - D^{\mathrm{T}}D)^{-1}$,

$$0 \geq A^{\mathrm{T}}P + PA + (PB + C^{\mathrm{T}}D)(I - D^{\mathrm{T}}D)^{-1}(B^{\mathrm{T}}P + D^{\mathrm{T}}C) + C^{\mathrm{T}}C. \tag{3.7}$$

Thus, in this case conditions (3.1)–(3.3) are equivalent to the single Riccati inequality (3.7).

Lemma 3.2. *Let $G(s) \overset{\min}{\sim} \begin{bmatrix} A & B \\ C & D \end{bmatrix}$. Then the following statements are equivalent:*

i) A is asymptotically stable and $G(s)$ is strongly bounded real;
ii) $I - D^{\mathrm{T}}D > 0$ and there exist positive-definite matrices P and R such that

$$0 = A^{\mathrm{T}}P+PA+(PB+C^{\mathrm{T}}D)(I-D^{\mathrm{T}}D)^{-1}(B^{\mathrm{T}}P+D^{\mathrm{T}}C)+C^{\mathrm{T}}C+R.$$

Now we prove sufficiency of the small gain theorem for the feedback interconnection of a bounded real transfer function and a norm-bounded memoryless time-varying nonlinearity. Thus define the set

$$\Phi \triangleq \{\phi : \mathbb{R}^\ell\times\mathbb{R}^+ \to \mathbb{R}^m : \|\phi(y,t)\|_2 \leq \|y\|_2, \quad y \in \mathbb{R}^\ell,$$
$$\text{a.a.} \quad t \geq 0, \text{and } \phi(y,\cdot) \text{ is Lebesgue measurable for all } y \in \mathbb{R}^\ell\}.$$

Theorem 3.1. *If $G(s) \overset{\min}{\sim} \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is strongly bounded real, then the feedback interconnection of $G(s)$ and ϕ is asymptotically stable for all $\phi \in \Phi$.*

Proof. First note that the feedback interconnection of $G(s)$ and ϕ corresponds to the state space representation

$$\dot{x}(t) = Ax(t) + B\phi(y,t), \tag{3.21}$$

$$y(t) = Cx(t) + D\phi(y,t). \tag{3.22}$$

Since $G(s)$ is strongly bounded real it follows from Lemma 3.2 that there exist positive-definite matrices P and R such that

$$0 = A^{\mathrm{T}}P+PA+(PB+C^{\mathrm{T}}D)(I-D^{\mathrm{T}}D)^{-1}(B^{\mathrm{T}}P+D^{\mathrm{T}}C)+C^{\mathrm{T}}C+R. \tag{3.23}$$

Next, we use the Lyapunov function $V(x) = x^{\mathrm{T}}Px$ to show that the feedback interconnection (3.21), (3.22) is asymptotically stable. The corresponding Lyapunov derivative is given by

$$\dot{V}(x) = x^{\mathrm{T}}(A^{\mathrm{T}}P + PA)x + \phi^{\mathrm{T}}B^{\mathrm{T}}Px + x^{\mathrm{T}}PB\phi \tag{3.24}$$

or, equivalently, using (3.23)

$$\begin{aligned}\dot{V}(x) = &-x^{\mathrm{T}}Rx - x^{\mathrm{T}}(PB + C^{\mathrm{T}}D)(I - D^{\mathrm{T}}D)^{-1}(B^{\mathrm{T}}P + D^{\mathrm{T}}C)x - \\ &x^{\mathrm{T}}C^{\mathrm{T}}Cx + \phi^{\mathrm{T}}B^{\mathrm{T}}Px + x^{\mathrm{T}}PB\phi.\end{aligned} \tag{3.25}$$

Next, add and subtract $\phi^{\mathrm{T}}\phi$, $2x^{\mathrm{T}}C^{\mathrm{T}}D\phi$, and $\phi^{\mathrm{T}}D^{\mathrm{T}}D\phi$ to (3.25) so that

$$\begin{aligned}\dot{V}(x) = &-x^{\mathrm{T}}Rx - x^{\mathrm{T}}(PB + C^{\mathrm{T}}D)(I - D^{\mathrm{T}}D)^{-1}(B^{\mathrm{T}}P + D^{\mathrm{T}}C)x - x^{\mathrm{T}}C^{\mathrm{T}}Cx \\ &+ \phi^{\mathrm{T}}B^{\mathrm{T}}Px + x^{\mathrm{T}}PB\phi + \phi^{\mathrm{T}}\phi - \phi^{\mathrm{T}}\phi + x^{\mathrm{T}}C^{\mathrm{T}}D\phi + \phi^{\mathrm{T}}D^{\mathrm{T}}Cx \\ &- x^{\mathrm{T}}C^{\mathrm{T}}D\phi - \phi^{\mathrm{T}}D^{\mathrm{T}}Cx + \phi^{\mathrm{T}}D^{\mathrm{T}}D\phi - \phi^{\mathrm{T}}D^{\mathrm{T}}D\phi\end{aligned} \tag{3.26}$$

or, equivalently,

$$\begin{aligned}\dot{V}(x) = &-x^{\mathrm{T}}Rx - x^{\mathrm{T}}(PB + C^{\mathrm{T}}D)(I - D^{\mathrm{T}}D)^{-1}(B^{\mathrm{T}}P + D^{\mathrm{T}}C)x \\ &+ x^{\mathrm{T}}(PB + C^{\mathrm{T}}D)\phi + \phi^{\mathrm{T}}(B^{\mathrm{T}}P + D^{\mathrm{T}}C)x - \phi^{\mathrm{T}}(I - D^{\mathrm{T}}D)\phi \\ &+ \phi^{\mathrm{T}}\phi - x^{\mathrm{T}}C^{\mathrm{T}}Cx - \phi^{\mathrm{T}}D^{\mathrm{T}}D\phi - x^{\mathrm{T}}C^{\mathrm{T}}D\phi - \phi^{\mathrm{T}}D^{\mathrm{T}}D\phi.\end{aligned} \tag{3.27}$$

Grouping the appropriate terms in (3.27) yields

$$\begin{aligned}\dot{V}(x) = &-x^{\mathrm{T}}Rx-[(I - D^{\mathrm{T}}D)^{-1/2}(B^{\mathrm{T}}P + D^{\mathrm{T}}D)x - (I - D^{\mathrm{T}}D)^{1/2}\phi]^{\mathrm{T}} \\ &\cdot[(I - D^{\mathrm{T}}D)^{-1/2}(B^{\mathrm{T}}P + D^{\mathrm{T}}D)x - (I - D^{\mathrm{T}}D)^{1/2}\phi] \\ &+ \phi^{\mathrm{T}}\phi - y^{\mathrm{T}}y\end{aligned} \tag{3.28}$$

which is negative definite since $\phi \in \Phi$ implies $\phi^{\mathrm{T}}\phi - y^{\mathrm{T}}y \leq 0$. Q.E.D.

4. THE POSITIVITY THEOREM

In this section we use quadratic Lyapunov functions to prove the positivity theorem in two cases as in Section 3.

Lemma 4.1 (Positive Real Lemma). *$G(s) \overset{\min}{\sim} \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is positive real if and only if there exist matrices P, L, and W with P positive definite such that*

$$0 = A^{\mathrm{T}}P + PA + L^{\mathrm{T}}L, \tag{4.1}$$
$$0 = PB - C^{\mathrm{T}} + L^{\mathrm{T}}W, \tag{4.2}$$
$$0 = D + D^{\mathrm{T}} - W^{\mathrm{T}}W. \tag{4.3}$$

Lemma 4.2. *Let $G(s) \overset{\min}{\sim} \begin{bmatrix} A & B \\ C & D \end{bmatrix}$. Then the following statements are equivalent:*

i) A is asymptotically stable and $G(s)$ is strongly positive real;

ii) $D+D^{\mathrm{T}} > 0$ and there exist positive-definite matrices P and R such that

$$0 = A^{\mathrm{T}}P + PA + (C - B^{\mathrm{T}}P)^{\mathrm{T}}(D + D^{\mathrm{T}})^{-1}(C - B^{\mathrm{T}}P) + R. \tag{4.8}$$

We now prove the positivity theorem for the negative feedback interconnection of a strongly positive real transfer function and an odd memoryless time-varying nonlinearity. For the statement of the next result we define the set

$$\hat{\Phi} \triangleq \{\phi: \ \mathbb{R}^{\ell} \times \mathbb{R}^{+} \to \mathbb{R}^{\ell}: \quad \phi^{\mathrm{T}}(y,t)y \geq 0, \quad y \in \mathbb{R}^{m}, \text{ a.a. } t \geq 0, \text{ and } \phi(y,\cdot) \text{ is Lebesgue measurable for all } y \in \mathbb{R}^{m}\}.$$

Theorem 4.1. *If $G(s) \overset{\min}{\sim} \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is strongly positive real, then the negative feedback interconnection of $G(s)$ and ϕ is asymptotically stable for all $\phi \in \hat{\Phi}$.*

Proof. First, note that the negative feedback interconnection of $G(s)$ and $\phi(\cdot,\cdot)$ corresponds to the state space representation

$$\dot{x}(t) = Ax(t) - B\phi(y,t), \tag{4.20}$$
$$y(t) = Cx(t) - D\phi(y,t). \tag{4.21}$$

Now it follows from Lemma 4.2 that if $G(s)$ is strongly positive real then there exist positive-definite matrices P and R such that

$$0 = A^{\mathrm{T}}P + PA + (C - B^{\mathrm{T}}P)^{\mathrm{T}}(D + D^{\mathrm{T}})^{-1}(C - B^{\mathrm{T}}P) + R. \tag{4.22}$$

Next, we use the Lyapunov function $V(x) = x^{\mathrm{T}}Px$ to show that the feedback interconnection (4.20), (4.21) is asymptotically stable. The corresponding Lyapunov derivative is given by

$$\dot{V}(x) = x^{\mathrm{T}}(A^{\mathrm{T}}P + PA^{\mathrm{T}})x - \phi^{\mathrm{T}}B^{\mathrm{T}}Px - x^{\mathrm{T}}PB\phi. \tag{4.24}$$

Add and subtract $2\phi^{\mathrm{T}}Cx$ and $2\phi^{\mathrm{T}}D\phi$ to (4.24) so that

$$\begin{aligned}\dot{V}(x) = &- x^{\mathrm{T}}Rx - x^{\mathrm{T}}(C - B^{\mathrm{T}}P)^{\mathrm{T}}(D + D^{\mathrm{T}})^{-1}(C - B^{\mathrm{T}}P)x \\ &- \phi^{\mathrm{T}}B^{\mathrm{T}}Px - x^{\mathrm{T}}PB\phi + 2\phi^{\mathrm{T}}D\phi - \phi^{\mathrm{T}}D\phi - \phi^{\mathrm{T}}D^{\mathrm{T}}\phi \\ &- 2\phi^{\mathrm{T}}Cx + \phi^{\mathrm{T}}Cx + x^{\mathrm{T}}C^{\mathrm{T}}\phi\end{aligned} \tag{4.25}$$

or, equivalently,

$$\begin{aligned}\dot{V}(x) = &- x^{\mathrm{T}}Rx - x^{\mathrm{T}}(C - B^{\mathrm{T}}P)^{\mathrm{T}}(D + D^{\mathrm{T}})^{-1}(C - B^{\mathrm{T}}P)x \\ &+ x^{\mathrm{T}}(C^{\mathrm{T}} - PB)\phi + \phi^{\mathrm{T}}(C^{\mathrm{T}} - PB)^{\mathrm{T}}x - \phi^{\mathrm{T}}(D + D^{\mathrm{T}})\phi \\ &- 2\phi^{\mathrm{T}}(D\phi - Cx).\end{aligned} \tag{4.26}$$

Grouping the appropriate terms in (4.26) yields

$$\begin{aligned}\dot{V}(x) = &-x^{\mathrm{T}}Rx - [(D + D^{\mathrm{T}})^{-1/2}(C - B^{\mathrm{T}}P)x - (D + D^{\mathrm{T}})^{1/2}\phi]^{\mathrm{T}} \\ &\cdot[(D + D^{\mathrm{T}})^{-1/2}(C - B^{\mathrm{T}}P)x - (D + D^{\mathrm{T}})^{1/2}\phi] - 2\phi^{\mathrm{T}}y,\end{aligned} \tag{4.27}$$

which is negative definite since $\phi^{\mathrm{T}}(y,t)y \geq 0$ for all $\phi(\cdot,\cdot) \in \hat{\Phi}$. Q.E.D.

Next, we specialize Theorem 4.1 to the feedback interconnection of a strongly positive real transfer function and a linear gain $F(t)$ satisfying $F(t) + F^{\mathrm{T}}(t) \geq 0$. Hence define

$$\hat{\mathcal{F}} \triangleq \{F: \ \mathbb{R}^{+} \to \mathbb{R}^{m \times m}: \quad F(\cdot) \text{ is Lebesgue measurable and } F(t) + F^{\mathrm{T}}(t) \geq 0, \quad \text{a.a. } t \geq 0\}.$$

Corollary 4.1. *If $G(s) \overset{\min}{\sim} \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is strongly positive real, then the negative feedback interconnection of $G(s)$ and ϕ is asymptotically stable for all $F \in \hat{\mathcal{F}}$.*

As in the bounded real case, Corollary 4.1 guarantees robust stability for the system

$$\dot{x}(t) = (A + \Delta A(t))x(t), \tag{4.28}$$

where $\Delta A(\cdot) \in \hat{\mathcal{U}}$ and $\hat{\mathcal{U}}$ is the uncertainty set

$$\hat{\mathcal{U}} \triangleq \{\Delta A(\cdot): \ \Delta A(t) = -BF(t)(I + DF(t))^{-1}C, \ F(t) + F(t) \geq 0, \quad F(\cdot) \text{ is Lebesgue measurable}\}.$$

The key feature of the uncertainty set $\hat{\mathcal{U}}$ is the fact that $BF(I + DF)^{-1}C$ also involves a positive real condition. To see this note that if $D + D^{\mathrm{T}} > 0$ and $F + F^{\mathrm{T}} \geq 0$, then

$$\begin{aligned}&F(I + DF)^{-1} + [F(I + DF)^{-1}]^{\mathrm{T}} \\ &\quad = (I + DF)^{-\mathrm{T}}[F + F^{\mathrm{T}} + F^{\mathrm{T}}(D + D^{\mathrm{T}})F](I + DF)^{-1} \geq 0.\end{aligned}$$

As shown in [13,58], a natural characterization of uncertainty that can be captured by $\hat{\mathcal{U}}$ arises in lightly damped structures with uncertain modal data.

5. THE CIRCLE CRITERION

In this section we use quadratic Lyapunov functions to prove the circle criterion. Application of this result to robust stability with respect to sector-bounded time-varying uncertainty is also discussed. Although proofs of the circle criterion appear in the literature [27,28] using quadratic Lyapunov functions, these proofs are confined to strictly proper systems with a single loop non-linearity. We remove these constraints and address the MIMO case for proper SISO systems. To begin, we define the set Φ_c of sector-bounded time-varying memoryless nonlinearities. Let $K_1, K_2 \in \mathbb{R}^{m \times \ell}$ and define

$$\Phi_c \triangleq \{\phi: \ \mathbb{R}^{\ell} \times \mathbb{R}^{+} \to \mathbb{R}^{m}: \ [\phi(y,t) - K_1y]^{\mathrm{T}}[\phi(y,t) - K_2y] \leq 0, \ y \in \mathbb{R}^{m}, \text{a.a., } t \geq 0, \text{ and } \phi(y,\cdot) \text{ is Lebesgue measurable for all } y \in \mathbb{R}^{m}\}.$$

Note that for the scalar case, the sector condition characterizing Φ_c is equivalent to $k_1y \leq \phi(y,t) \leq k_2y, \quad y \in \mathbb{R}, \quad t \geq 0$.

Theorem 5.1. *Let $K_1, K_2 \in {}^{m \times \ell}$. If $\frac{1}{2}[I + K_2G(s)][I + K_1G(s)]^{-1}$ is strongly positive real, where $G(s) \overset{\min}{\sim} \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, then the negative feedback interconnection of $G(s)$ and ϕ is asymptotically stable for all $\phi \in \Phi_c$.*

Proof. First note that the negative feedback interconnection of $G(s)$ and $\phi(\cdot,\cdot)$ corresponds to the nonlinear state-space equations (4.44) and (4.45). Next, note that

$$[I + K_2G(s)][I + K_1G(s)]^{-1} = I + (K_2 - K_1)[I + K_1G(s)]^{-1}G(s). \tag{5.1}$$

Now, noting that $[I + K_1G(s)]^{-1}G(s)$ corresponds to a plant $G(s)$ with a feedback gain K_1, it follows from feedback interconnection manipulations that a minimal realization for $\frac{1}{2}I + \frac{1}{2}(K_2 - K_1)[I + K_1G(s)]^{-1}G(s)$ is given by

$$\begin{bmatrix} A - B(I + K_1D)^{-1}K_1C & B(I + K_1D)^{-1} \\ \frac{1}{2}(K_2 - K_1)(I + K_1D)^{-1}C & \frac{1}{2}I + \frac{1}{2}(K_2 - K_1)(I + K_1D)^{-1}D \end{bmatrix}.$$

Note that $(I+K_1D)^{-1}$ exists since $[I+K_2G(\infty)][I+K_1G(\infty)]^{-1} > 0$. Now it follows from Lemma 4.2 that if $\frac{1}{2}[I + K_2G(s)][I + K_1G(s)]^{-1}$ is strongly positive real then there exist positive-definite matrices P and R such that

$$\begin{aligned}0 = &[A - B(I + K_1D)^{-1}K_1C]^{\mathrm{T}}P + P[A - B(I + K_1D)^{-1}K_1C] \\ &+ [\tfrac{1}{2}(K_2 - K_1)(I + K_1D)^{-1}C - (I + K_1D)^{-\mathrm{T}}B^{\mathrm{T}}P]^{\mathrm{T}} \\ &\cdot [I + \tfrac{1}{2}(K_2 - K_1)[(I + K_1D)^{-1}D + D^{\mathrm{T}}(I + K_1D)^{-\mathrm{T}}]]^{-1} \\ &\cdot [\tfrac{1}{2}(K_2 - K_1)(I + K_1D)^{-1}C - (I + K_1D)^{-\mathrm{T}}B^{\mathrm{T}}P] + R.\end{aligned} \tag{5.2}$$

Next, define the Lyapunov function $V(x) = x^{\mathrm{T}}Px$ and let $\phi(\cdot,\cdot) \in \Phi_c$. Then we obtain

$$\dot{V}(x) = x^{\mathrm{T}}(A^{\mathrm{T}}P + PA)x - \phi^{\mathrm{T}}B^{\mathrm{T}}Px - x^{\mathrm{T}}PB\phi \tag{5.3}$$

or, equivalently, using (5.2)

$$\begin{aligned}\dot{V}(x) = &- x^{\mathrm{T}}Rx - x^{\mathrm{T}}Qx + x^{\mathrm{T}}C^{\mathrm{T}}K_1(I + K_1D)^{-\mathrm{T}}B^{\mathrm{T}}Px \\ &+ x^{\mathrm{T}}PB(I + K_1D)^{-1}K_1Cx - \phi^{\mathrm{T}}B^{\mathrm{T}}Px - x^{\mathrm{T}}PB\phi,\end{aligned} \tag{5.4}$$

where

$$\begin{aligned}Q \triangleq &[\tfrac{1}{2}(K_2 - K_1)(I + K_1D)^{-1}C - (I + K_1D)^{-\mathrm{T}}B^{\mathrm{T}}P]^{\mathrm{T}} \\ &\cdot[I + \tfrac{1}{2}(K_2 - K_1)[(I + K_1D)^{-1}D + D^{\mathrm{T}}(I + K_1D)^{-\mathrm{T}}]]^{-1} \\ &\cdot[\tfrac{1}{2}(K_2 - K_1)(I + K_1D)^{-1}C - (I + K_1D)^{-\mathrm{T}}B^{\mathrm{T}}P].\end{aligned} \tag{5.5}$$

Next, add and subtract

$$[(I + K_1D)\phi - K_1Cx]^{\mathrm{T}}[(I + K_1D)\phi - K_1Cx],$$
$$(K_2 - K_1)[(I + K_1D)\phi - K_1Cx]^{\mathrm{T}}(I + K_1D)^{-1}Cx,$$
$$(K_2 - K_1)[(I + K_1D)\phi - K_1Cx]^{\mathrm{T}}(I + K_1D)^{-1}D[(I + K_1D)\phi - K_1Cx]$$

to (5.5) so that (after some algebraic manipulation)

$$\begin{aligned}\dot{V}(x) = &-x^{\mathrm{T}}Rx - x^{\mathrm{T}}Qx \\ &+ [\tfrac{1}{2}(K_2 - K_1)x^{\mathrm{T}}C^{\mathrm{T}}(I + K_1D)^{-\mathrm{T}} - x^{\mathrm{T}}PB(I + K_2D)^{-1}][(I + K_1D)\phi - K_1Cx] \\ &+ [\phi^{\mathrm{T}}(I + K_1D)^{\mathrm{T}} - K_1x^{\mathrm{T}}C^{\mathrm{T}}][\tfrac{1}{2}(K_2 - K_1)(I + K_1D)^{-1}Cx - (I + K_1D)^{-\mathrm{T}}B^{\mathrm{T}}Px] \\ &- [\phi^{\mathrm{T}}(I + K_1D)^{\mathrm{T}} - K_1x^{\mathrm{T}}C^{\mathrm{T}}][I + \tfrac{1}{2}(K_2 - K_1)[(I + K_1D)^{-1}D + D^{\mathrm{T}}(I + K_1D)^{-\mathrm{T}}]] \\ &\quad\cdot[(I + K_1D)\phi K_1Cx] \\ &+ [\phi^{\mathrm{T}}(I + K_1D)^{\mathrm{T}} - K_1x^{\mathrm{T}}C^{\mathrm{T}}]\{(I + K_1D)\phi - K_1Cx - (K_2 - K_1)(\\ &I + K_1D)^{-1}Cx + (K_2 - K_1)(I + K_1D)^{-1}D[(I + K_1D)\phi - K_1Cx]\}.\end{aligned} \tag{5.6}$$

Grouping the appropriate terms in (5.6) yields

$$\dot{V}(x) = -x^{\mathrm{T}}Rx - z^{\mathrm{T}}z + (\phi - K_1y)^{\mathrm{T}}(\phi - K_2y), \tag{5.7}$$

where

$$\begin{aligned}z \triangleq &[I + \tfrac{1}{2}(K_2 - K_1)[(I + K_1D)^{-1}D + D^{\mathrm{T}}(I + K_1D)^{-\mathrm{T}}]]^{-1/2} \\ &\cdot[\tfrac{1}{2}(K_2 - K_1)[(I + K_1D)^{-1}C \\ &-(I + K_1D)^{-\mathrm{T}}B^{\mathrm{T}}P]x[I + \tfrac{1}{2}(K_2 - K_1)[(I + K_1D)^{-1}D + D^{\mathrm{T}}(I + K_1D)^{-\mathrm{T}}]^{1/2} \\ &\cdot[(I + K_1D)\phi - K_1Cx].\end{aligned}$$

Since $\phi(\cdot,\cdot) \in \Phi_c$, it follows from (5.7) that $\dot{V}(x) < 0$ and thus the corresponding feedback interconnection is asymptotically stable for all $\phi(\cdot,\cdot) \in \Phi_c$. Q.E.D.

Remark 5.1. Note that the condition $\frac{1}{2}[I + K_2G(s)][I + K_1G(s)]^{-1}$ strongly positive real in the statement of Theorem 5.1 is equivalent to $\mathrm{Re}[I + K_2G(j\omega)][I + K_1G(j\omega)]^{-1} > 0$ for all $\omega \in$ which is the classical representation of the circle criterion [44]. Furthermore, if K_1 and K_2 are diagonal, then the conditions of Theorem 5.1 can be verified by using the multivariable Nyquist criterion. Specifically, by examining the number of counter-clockwise encirclements of the zero point of the image of the clockwise Nyquist contour under the mapping $\det[I + K_1G(s)]$, the stability of the closed-loop system can be related to the number of unstable poles of $G(s)$. For further details (in the SISO case) see [44].

Next, as in Sections 3 and 4, we specialize the results of Theorem 5.1 to robust stability of a linear time-invariant plant with a linear time-varying uncertainty. To this end we have the following immediate result. Define

$$\mathcal{F}_c \triangleq \{F:\ {}^+ \to {}^{m\times \ell}:\ F(\cdot) \text{ is Lebesgue measurable and } [F(t) - K_1]^{\mathrm{T}}[F(t) - K_2] \leq 0, \ \text{a.a. } t \geq 0\}$$

and consider the system

$$\dot{x}(t) = (A + \Delta A(t))x(t), \tag{5.8}$$

where $\Delta A(\cdot) \in \mathcal{U}_c$ and the uncertainty set $\mathcal{U}_c$ is defined by

$$\mathcal{U}_c \triangleq \{\Delta A(\cdot):\ \Delta A(t) = -BF(t)(I + DF(t))^{-1}C, \quad F(\cdot) \in \mathcal{F}_c\}.$$

Then it follows from Theorem 5.1, with $\phi(y,t) = F(t)y = F(t)(I + DF(t))^{-1}Cx$, that the zero solution to (5.8) is asymptotically stable for all $\Delta A(\cdot) \in \mathcal{U}_c$. Note that a simpler uncertainty structure can be achieved by setting $D = 0$ in $\mathcal{U}_c$. This results in considerable simplification of (5.2). Finally, it is interesting to note that if $K_1 = -I$ and $K_2 = I$, then $\mathcal{U}_c = \mathcal{U}$, while if $K_1 = 0$ and $K_2 = \infty$, then $\hat{\mathcal{U}} = \mathcal{U}$.

6. THE POPOV CRITERION

In this section we use Lyapunov functions to prove the Popov criterion for a multivariable plant containing an arbitrary number of memoryless *time-invariant* nonlinearities. Specialization of this result to robust stability with respect to *time-invariant* plant uncertainty is also considered. To begin we define the set Φ_P characterizing a class of sector-bounded *time-invariant* nonlinearities. Let

$$\begin{aligned}\Phi_P \triangleq \{\phi:\ {}^m \to {}^m:\ &\phi^{\mathrm{T}}(y)[\phi(y) - Ky] \leq 0, \quad y \in {}^m, \\ &K = \mathrm{diag}[k_1, k_2, \ldots, k_m], \quad k_i > 0, \quad i = 1, \ldots, m, \text{ and} \\ &\phi(y) = [\phi_1(y_1), \phi_2(y_2), \ldots, \phi_m(y_m)]^{\mathrm{T}}\}.\end{aligned}$$

Note that $\phi \in \Phi_P$ implies that each component $\phi_i(y_i)$ of ϕ satisfies

$$0 \leq \phi_i(y_i) \leq k_iy_i, \text{ for } k_i > 0, \quad i = 1, 2, \ldots, m.$$

Note that the respective nonlinearities are assumed to be decoupled.

Theorem 6.1. *Let $K \in {}^{m\times m}$ be a positive-definite diagonal matrix. If there exists a nonnegative-definite diagonal matrix N such that $K^{-1} + (I + Ns)G(s)$ is strongly positive real, where $G(s) \overset{\min}{\sim} \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$, then the negative feedback connection of $G(s)$ and $\phi(\cdot)$ is asymptotically stable for all $\phi(\cdot) \in \Phi_P$.*

Proof. First note that the negative feedback interconnection of $G(s)$ and $\phi(\cdot)$ corresponds to the state-space equations

$$\dot{x}(t) = Ax(t) - B\phi(y), \tag{6.1}$$
$$y = Cx(t). \tag{6.2}$$

Next, note that $sG(s)$ has a realization

$$\begin{bmatrix} A & B \\ CA & CB \end{bmatrix}$$

so that $K^{-1} + (I + Ns)G(s)$ has minimal realization (using cascade state space manipulations)

$$\begin{bmatrix} A & B \\ C + NCA & NCB + K^{-1} \end{bmatrix}.$$

Now it follows from Lemma 4.2 that if $K^{-1} + (I + Ns)G(s)$ is strongly positive real then there exist positive-definite matrices P and R such that

$$\begin{aligned}0 = A^{\mathrm{T}}P + PA + &(C + NCA - B^{\mathrm{T}}P)^{\mathrm{T}}[(K^{-1} + NCB) + (K^{-1} + NCB)^{\mathrm{T}}]^{-} \\ &\cdot(C + NCA - B^{\mathrm{T}}P) + R.\end{aligned} \tag{6.3}$$

Next, for $\phi \in \Phi_P$ define the Lyapunov function

$$V(x) = x^{\mathrm{T}}Px + 2\sum_{i=1}^{m}\int_0^{y_i}\phi_i(\sigma)\mathrm{d}\sigma. \tag{6.4}$$

Note that since P is positive definite and $\phi(\cdot) \in \Phi_P$, $V(x)$ is positive definite for all nonzero x. Thus, the corresponding Lyapunov derivative is given by

$$\dot{V}(x) = x^{\mathrm{T}}(A^{\mathrm{T}}P + PA)x - \phi^{\mathrm{T}}B^{\mathrm{T}}Px - x^{\mathrm{T}}PB\phi + 2\sum_{i=1}^{m}\phi_i(y_i)\dot{y}_i$$

or, equivalently, using (6.3)

$$\dot{V}(x) = -x^{\mathrm{T}}Rx - x^{\mathrm{T}}(C + NCA - B^{\mathrm{T}}P)^{\mathrm{T}}[(K^{-1} + NCB) + (K^{-1} + NCB)^{\mathrm{T}}]^{-1}(C + NCA - B^{\mathrm{T}}P)x + 2\phi^{\mathrm{T}}(y)\dot{y}. \tag{6.5}$$

Next, since $\dot{y} = C\dot{x} = CAx - CB\phi$, (6.4) becomes $\dot{V}(x) =$

$$- x^{\mathrm{T}}Rx - x^{\mathrm{T}}(C + NCA - B^{\mathrm{T}}P)^{\mathrm{T}}[(K^{-1} + NCB) + (K^{-1} + NCB)^{\mathrm{T}}]^{-1} \cdot (C + NCA - B^{\mathrm{T}}P)x - x^{\mathrm{T}}(PB - A^{\mathrm{T}}C^{\mathrm{T}}N)\phi - \phi^{\mathrm{T}}(B^{\mathrm{T}}P - NCA)x - \phi^{\mathrm{T}}(NCB + B^{\mathrm{T}}C^{\mathrm{T}}N)\phi. \tag{6.6}$$

Adding and subtracting $2\phi^{\mathrm{T}}Cx$ and $\phi^{\mathrm{T}}K^{-1}\phi$ to (6.5) yields $\dot{V}(x) =$

$$- x^{\mathrm{T}}Rx - x^{\mathrm{T}}(C + NCA - B^{\mathrm{T}}P)^{\mathrm{T}}[(K^{-1} + NCB) + (K^{-1} + NCB)^{\mathrm{T}}]^{-1} \cdot (C+NCA - B^{\mathrm{T}}P)x - \phi^{\mathrm{T}}[B^{\mathrm{T}}P - (C + NCA)]x - x^{\mathrm{T}}[PB - (C^{\mathrm{T}} + A^{\mathrm{T}}C^{\mathrm{T}}N)]\phi - \phi^{\mathrm{T}}[(K^{-1} + NCB) + (K + NCB)^{\mathrm{T}}]\phi^{\mathrm{T}} - \phi^{\mathrm{T}}(K^{-1}\phi - Cx) \tag{6.7}$$

or, equivalently,

$$\begin{aligned}\dot{V}(x) = {} & -x^{\mathrm{T}}Rx \\ & - \{[(K^{-1} + NCB) + (K^{-1} + NCB)^{\mathrm{T}}]^{-1/2}(C + NCA - B^{\mathrm{T}}P)x \\ & - [(K^{-1} + NCB) + (K^{-1} + NCB)^{\mathrm{T}}]^{1/2}\phi\}^{\mathrm{T}} \\ & \cdot \{[(K^{-1} + NCB) + (K^{-1} + NCB)^{\mathrm{T}}]^{-1/2}(C + NCA - B^{\mathrm{T}}P)x \\ & - [(K^{-1} + NCB) + (K^{-1} + NCB)]^{1/2}\phi\} \\ & + \phi^{\mathrm{T}}[K^{-1}\phi - y].\end{aligned} \tag{6.8}$$

Thus, since $\phi(\cdot) \in \Phi_P$, and $K > 0$, it follows from (6.8) that $\dot{V}(x) < 0$ and thus the corresponding feedback interconnection is asymptotically stable for all $\phi(\cdot) \in \Phi_P$. Q.E.D.

Remark 6.1. A similar proof of the generalized Popov criterion is given in [44] using the three equation form of the positive real lemma.

Finally, as in the previous sections, we specialize the results of Theorem 6.1 to robust stability with linear parameter uncertainty. Now, however, such uncertainty is assumed to be constant. Specifically, consider the system

$$\dot{x}(t) = (A + \Delta A)x(t), \tag{6.9}$$

where $\Delta A \in \mathcal{U}_P$ and $\mathcal{U}_P$ is defined by

$$\mathcal{U}_P \triangleq \{\Delta A\colon \Delta A = -BFC,\ F = \mathrm{diag}[F_1, F_2, \ldots, F_m],\ 0 \le F_i \le k_i,\ i = 1, \ldots, m\},$$

B, and C are given matrices denoting the structure of the uncertainty, F is a constant diagonal uncertain matrix, and K is a diagonal matrix denoting the sector boundaries of the uncertain parameters. It now follows from Theorem 6.1 by setting $\phi(y) = Fy = FCx$ that $A+\Delta A$ is asymptotically stable for all $\Delta A \in \mathcal{U}_P$. Note that if $k \to \infty$ then the set becomes

$$\mathcal{U}_P = \{\Delta A\colon\ \Delta A = -BFC, \quad F_i \ge 0,\ i = 1, \ldots, m\}.$$

The main difference between the result of this section and the previous sections is that the elements of the set $\mathcal{U}_p$ are constant rather than time-varying. This is due to the Lyapunov function that establishes robust stability, i.e.,

$$V(x) = x^{\mathrm{T}}Px + 2\sum_{i=1}^{m}\int_0^{y_i} F_i\sigma \,\mathrm{d}\sigma, \quad y_i = (Cx)_i,$$

or, equivalently,

$$V(x) = x^{\mathrm{T}}Px + x^{\mathrm{T}}C^{\mathrm{T}}FCx = x^{\mathrm{T}}Px + \sum_{i=1}^{m} F_i x^{\mathrm{T}}C_i^{\mathrm{T}}C_i x.$$

Note that this Lyapunov function is parameter dependent, i.e., it is a function of the uncertain parameters. Consequently the uncertain parameters are not allowed to be time-varying. Such Lyapunov functions are generally less conservative than constant Lyapunov functions [50,51] when the uncertain parameters are known to be constant. In contrast, the results of the previous sections are established by parameter independent quadratic Lyapunov functions that guarantee robust stability with respect to *time-varying* parameter variations.

7. Conclusions

Special cases of the small gain and positivity theorems were proved by explicitly constructing quadratic Lyapunov functions. Application of these results to robust stability with bounded real and positive real uncertainty was discussed. Similar techniques were used to prove multivariable versions of the circle and Popov theorems. It was stressed that the Popov theorem is based upon a parameter-dependent quadratic Lyapunov function in the case of linear uncertainty.

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APPENDIX:

Next, we specialize Theorem 3.1 to the feedback interconnection of a strongly bounded real transfer function and a linear bounded real gain. Hence consider the $\mathcal{F}$ defined by

$$\mathcal{F} \triangleq \{F: \mathbb{R}^+ \to \mathbb{R}^{m\times \ell}: \quad F(\cdot) \text{ is Lebesgue measurable and } \sigma_{\max}(F(t)) < 1, \quad \text{a.a. } t \geq 0\}.$$

That is, $\mathcal{F}$ includes those ϕ in Φ of the form $\phi(y,t) = F(t)y$. The following corollary is thus immediate.

Corollary 3.1. *If $G(s) \overset{\min}{\sim} \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is strongly bounded real, then the feedback interconnection of $G(s)$ and $F(\cdot)$ is asymptotically stable for all $F(\cdot) \in \mathcal{F}$.*

Corollary 3.1 implies that $A + BF(\cdot)(I - DF(\cdot))^{-1}C$ is asymptotically stable in the sense that the zero solution of the linear time-varying system

$$\dot{x}(t) = (A + BF(t)(I - DF(t))^{-1}C)x(t) \tag{3.29}$$

is asymptotically stable. This result thus implies robust stability with time-varying bounded real (but otherwise unknown) uncertainty. Specifically, consider the system

$$\dot{x}(t) = (A + \Delta A(t))x(t), \tag{3.29$'$}$$

where $\Delta A(\cdot) \in \mathcal{U}$ and $\mathcal{U}$ is the uncertainty set

$$\mathcal{U} \triangleq \{\Delta A(\cdot): \ \Delta A(t) = BF(t)(I - DF(t))^{-1}C, \ F^{\mathrm{T}}(t)F(t) \leq I, \ F(\cdot) \text{ is Lebesgue measurable}\}.$$

Then it follows from Corollary 3.1 and (3.29) that the zero solution to (3.29)$'$ is asymptotically stable for all $\Delta A(\cdot) \in \mathcal{U}$. The set $\mathcal{U}$ is a generalization of the uncertainty sets appearing in [52,53,56,57] for robust controller analysis and synthesis. These uncertainty structures can be recovered by setting $D = 0$ in $\mathcal{U}$. The case $D \neq 0$ has not been treated previously. Finally, if we restrict our attention to constant matrices, then Corollary 3.1 implies that if $G(s)$ is strongly bounded real, then $A + BF(I - DF)^{-1}C$ is asymptotically stable for all F satisfying $\sigma_{\max}(F) \leq 1$.