Semiparametric Identification of Wiener Systems

Using a Single Harmonic Input and Retrospective Cost Optimization

by

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Abstract

We present a two-step method for identifying SISO Wiener systems. First, using a single harmonic input, we estimate a nonparametric model of the static nonlinearity, which is assumed to be only piecewise continuous. Second, using the identified nonparametric map, we use retrospective cost optimization to identify a parametric model of the linear dynamic system. This method is demonstrated on several examples of increasing complexity.

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1 Introduction

Block-structured models are widely used for system identification [5, 10, 11]. These models provide useful information concerning the dynamic and static components of a system, and thus constitute grey-box models in which the block structure is ascribed physical meaning. The goal of system identification is to model the internal structure of each block from available data.

Among the most widely studied block-structured models are the Wiener [1, 3, 4, 6, 12, 16, 17, 21] and Hammerstein [1, 2, 6, 9, 18, 23] models. Each model structure involves a single linear dynamic block and a single nonlinear static block. For these two-block structures, the difficulty of the identification problem typically depends on a priori assumptions made about the components, for example, FIR-versus-IIR dynamics, and invertible-versus-noninvertible nonlinearities [16]. Furthermore, identification of Wiener systems is generally considered to be more challenging than identification of Hammerstein systems due to the fact that the input to the nonlinear block is available for Hammerstein systems but not for Wiener systems. In the present paper, we focus on Wiener systems.

The methods for identifying Wiener systems developed in [1, 4, 12, 21] assume that the nonlinear block is invertible. To overcome this requirement, nonparametric probabilistic methods are used in [10]. Alternatively, frequency-domain methods that apply multiple harmonic inputs are employed in [3, 6]. In [6], the multiple harmonic inputs are assigned random phase shifts, and a nonparametric model of the nonlinearity is obtained using the identified linear dynamic model, which is previously estimated in the frequency domain. In [3], the phase shift between the output of the linear dynamic block and the output is exploited in the frequency domain, for each harmonic input.

In the present paper we develop a novel technique for identifying single-input, single-output (SISO) Wiener systems. The proposed approach is semiparametric, which, as described in [10], refers to the fact that the nonlinear block is estimated nonparametrically, whereas the linear dy-
dynamics are identified parametrically. To do this, we consider a two-step procedure. In the first step, we apply a single harmonic input signal, and measure the output once the trajectory of the system reaches harmonic steady state. We then examine the output of the system (which is not harmonic due to the nonlinearity) relative to the input, and use the symmetry properties of these signals to estimate the nonharmonic phase shift. This estimate allows us to infer the phase shift of the unmeasured intermediate signal (that is, the output of the linear block) and thus reconstruct this signal up to an arbitrary amplitude. By plotting the output versus the reconstructed intermediate signal, we thus obtain a nonparametric approximation of the nonlinear block of the system.

The second step of the algorithm uses a sufficiently rich signal to estimate the linear dynamics of the system. Since we do not assume that the nonlinear block is invertible, we do not have an estimate of the output of the linear block. To overcome this difficulty, we apply retrospective cost optimization, which uses the available output signal (in this case, the output of the nonlinear block) to recursively update the linear dynamics. This technique is inspired by retrospective-cost-based adaptive control [14, 19, 22], which is used for model updating in [7, 18, 20].

As alluded to above, the two-step identification algorithm described herein does not require invertibility of the nonlinear block as assumed in [1, 4, 12, 21]. In fact, we do not require that the nonlinear block be either one-to-one, onto, or continuous, nor do we assume as in [4] that any specific value of the nonlinearity be known.

The contents of the paper are as follows. In Section 2 we define the Wiener identification problem. A method for nonparametric identification of the static nonlinearity using a single harmonic input is presented in Section 3, while a method for parametric identification of the linear time-invariant dynamics using retrospective cost optimization is reviewed in Section 4. These methods are demonstrated on several examples of increasing complexity in sections 5, 6, and 7. Concluding remarks are presented in Section 8. A preliminary version of the results of this paper appears as [8].
2 Problem Formulation

Consider the block-structured Wiener model shown in Figure 1a, where \( L \) is the SISO discrete-time linear time-invariant dynamic system

\[
\begin{align*}
x(k + 1) &= Ax(k) + Bu(k), \\
v(k) &= Cx(k),
\end{align*}
\]

(2.1)

(2.2)

with input \( u(k) \in \mathbb{R} \) and intermediate signal \( v(k) \in \mathbb{R} \), where \( k \) is the sample index, and \( y(k) \in \mathbb{R} \) is the output given by

\[
y(k) = W(v(k)),
\]

(2.3)

where \( W : \mathbb{R} \rightarrow \mathbb{R} \) is the static nonlinearity. We assume that \( L \) is asymptotically stable and \( W \) is piecewise continuous. Note that we do not assume that \( W \) is invertible, one-to-one, continuous, or (as in [4]) \( W(0) = 0 \). Also, we assume that \( v(k) \) is not accessible, and that \( x(0) \) is unknown and possibly nonzero.

Moreover, Figure 1b shows the scaled-domain modification \( W_\lambda(v) \triangleq W\left( \frac{v}{\lambda} \right) \) of \( W \), where \( \lambda \) is a nonzero real number. Therefore, \( W_\lambda(\lambda v) = W(v) \). Each value of \( \lambda \) scales both the gain of \( L \) and the domain of \( W \). However, \( \lambda \) is not identifiable.

3 Nonparametric Identification of the Static Nonlinearity

Consider the harmonic input signal

\[
u(k) = A_0 \sin(\omega_0 k T_s) = A_0 \sin(\Omega_0 k),
\]

(3.1)

where \( A_0 \) is the amplitude, \( \omega_0 \) is the angular frequency in rad/sec, \( T_s \) is the sample period in sec/sample, and \( \Omega_0 \triangleq \omega_0 T_s \) is the angular sample frequency in rad/sample. Since \( L \) is asymptotically stable, it follows that, for large values of \( k \), the intermediate signal \( v \) is given approximately by the
harmonic steady-state signal

\[ v(k) = |G(e^{j\omega_0})|A_0 \sin(\Omega_0 k + \angle G(e^{j\omega_0})) , \]  

(3.2)

where \( |G(e^{j\omega_0})| \) and \( \angle G(e^{j\omega_0}) \) are, respectively, the magnitude and phase shift of the frequency response of \( G(z) = C(zI - A)^{-1}B \) at the angular sample frequency \( \Omega_0 \). Therefore,

\[ y(k) = \mathcal{W}(|G(e^{j\omega_0})|A_0 \sin(\Omega_0 k + \angle G(e^{j\omega_0}))). \]  

(3.3)

Next, note that the continuous-time harmonic signal \( \sin(\omega_0 t) \) is symmetric in the intervals \([0, \frac{1}{2}T_0] \) and \([\frac{1}{2}T_0, T_0] \) about the points \( \frac{1}{4}T_0 \) and \( \frac{3}{4}T_0 \), respectively, where \( T_0 = \frac{2\pi}{\omega_0} \) is the period of the harmonic input. To preserve symmetry for the sampled signal (3.1) about the points \( \frac{1}{4}T_0 \) and \( \frac{3}{4}T_0 \), we assume that

\[ \Omega_0 = \frac{\pi}{2m} , \]  

(3.4)

where \( m \) is a positive integer. Thus \( N_0 = 4m = \frac{T_0}{T_s} \) is the period of the discrete-time input (3.1).

With this choice of \( \Omega_0 \), the sampled signal \( u(k) \) is symmetric in the intervals \([0, \frac{1}{2}N_0] \) and \([\frac{1}{2}N_0, N_0] \) about the points \( \frac{1}{4}N_0 \) and \( \frac{3}{4}N_0 \), respectively. Furthermore, assuming that \( q = \frac{\angle G(e^{j\Omega_0})}{\pi} \) is an integer, that is, \( \frac{\angle G(e^{j\Omega_0})}{\pi} \) is an integer, the intermediate signal \( v(k) \), which is shifted relative to \( u(k) \) due to \( \angle G(e^{j\Omega_0}) \), is symmetric about \( \frac{1}{4}N_0 + q \) in the interval \([q, \frac{1}{2}N_0 + q] \) and about \( \frac{3}{4}N_0 + q \) in the interval \([\frac{1}{2}N_0 + q, N_0 + q] \). If \( q \) is not an integer, then \( v(k) \) is only approximately symmetric.

Next, we note that the output signal \( y \), which is not generally harmonic, possesses the same symmetry as \( v \) on the same intervals. By exploiting knowledge of this symmetry, we can identify the nonharmonic phase shift of \( y \) relative to \( u \), and thus the phase shift of \( v \) relative to \( u \). Since \( y \) is not sinusoidal, the nonharmonic phase shift of \( y \) relative to \( u \) refers to the shifting of the symmetric portions of \( y \) relative to the symmetric portions of \( u \). Knowledge of this nonharmonic phase shift allows us to determine \( v \) up to a constant multiple, specifically, \( v \) is a sinusoid that is shifted relative to \( u \) by a known number of samples.

To clarify the above discussion, we present two examples using \( A_0 = 1 \), \( m = 18 \) (so that
\[ \Omega_0 = \pi/36, \] and \[ G(z) = \frac{0.0685}{z - 0.9164}. \] First, consider the polynomial nonlinearity \[ y = W(v) = 0.6(v + 1)^3 - 1, \] which is neither even nor odd. Figure 2a illustrates the resulting signals \( u(k), v(k), \) and \( y(k) \) in harmonic steady state. Note that \( u \) is symmetric about the discrete-time index \( \delta \) in the interval \([ \delta - \frac{1}{4}N_0, \delta + \frac{1}{4}N_0 ]\) and about \( \delta + \frac{1}{2}N_0 \) in the interval \([ \delta + \frac{3}{4}N_0, \delta + \frac{5}{4}N_0 ]\). Likewise, \( v \) is symmetric about the discrete-time index \( \epsilon \) in the interval \([ \epsilon - \frac{1}{4}N_0, \epsilon + \frac{1}{4}N_0 ]\) and about \( \epsilon + \frac{1}{2}N_0 \) in the interval \([ \epsilon + \frac{3}{4}N_0, \epsilon + \frac{5}{4}N_0 ]\). It thus follows that \( y \) is symmetric about \( \epsilon \) in the interval \([ \epsilon - \frac{1}{4}N_0, \epsilon + \frac{1}{4}N_0 ]\) and about \( \epsilon + \frac{1}{2}N_0 \) in the interval \([ \epsilon + \frac{3}{4}N_0, \epsilon + \frac{5}{4}N_0 ]\).

Second, we consider the even polynomial nonlinearity \[ y = W(v) = v^2. \] Figure 2b illustrates the resulting signals \( u(k), v(k), \) and \( y(k) \) in harmonic steady state. The signals \( u \) and \( v \) are equal to the signals shown in Figure 2a. However, in addition to the two points of symmetry shown in Figure 2a, note that \( y \) has two additional points of symmetry, specifically, \( y \) is symmetric about \( \epsilon + \frac{1}{4}N_0 \) in the interval \([ \epsilon, \epsilon + \frac{1}{2}N_0 ]\) and about \( \epsilon + \frac{3}{4}N_0 \) in the interval \([ \epsilon + \frac{1}{2}N_0, \epsilon + N_0 ]\).

### 3.1 Symmetry Search Algorithm

We now present an algorithm to determine \( \epsilon \) from \( y \). We then use \( \epsilon \) to estimate the nonharmonic phase shift of \( y \) relative to \( u \). For convenience, we assume that the harmonic steady state begins at \( k = 0 \).

Consider the signal \( y \) shown in Figure 3, and let \( n \geq 6m \) denote the width of the data window so that it includes at least one and a half periods. To encompass a complete signal period, we construct a sliding window with \( N_0 + 1 \) data points. The window is divided into quarters as shown in Figure 3.

Next, for \( k = 0, \ldots, n - N_0 \), define

\[
\beta_1(k) \triangleq \sum_{i=1}^{2m-1} |y(k + i - 1) - y(k + 2m - i + 1)|, \tag{3.5}
\]

which is the sum of the absolute difference in magnitude for each pair of candidate symmetric
points in the first and second quarters about the point \( k + \frac{1}{4}N_0 \) for the sliding window starting at time step \( k \). Likewise, for \( k = 0, \ldots, n - N_0 \), define

\[
\beta_2(k) \triangleq \sum_{i=1}^{2m-1} |y(k + 2m + i - 1) - y(k + 4m - i + 1)|,
\]

(3.6)

for each pair of candidate symmetric points in the third and fourth quarters about the point \( k + \frac{3}{4}N_0 \). The values of \( \beta_1 \) and \( \beta_2 \) quantify the symmetry error about the points \( k + \frac{1}{4}N_0 \) and \( k + \frac{3}{4}N_0 \), respectively, for each allowable value of \( k \). Thus, using (3.5) and (3.6), we define the symmetry error index

\[
\beta(k) \triangleq \beta_1(k) + \beta_2(k),
\]

(3.7)

corresponding to the sliding window starting at point \( k \), for \( k = 0, \ldots, n - N_0 \).

For \( k = 0, \ldots, n - N_0 \), let \( k_0 < n - N_0 \) be the minimizer of \( \beta(k) \). We use knowledge of \( k_0 \) to determine the location of the points of symmetry \( \varepsilon \) and \( \varepsilon + \frac{1}{2}N_0 \) for the sliding window starting at point \( k_0 \). In particular, since \( k_0 \) is the starting point of the window that minimizes \( \beta \) and since \( \varepsilon \) and \( \varepsilon + \frac{1}{2}N_0 \) are, respectively, the quarter point and three quarter point of the same window, it follows that

\[
\varepsilon = k_0 + \frac{1}{4}N_0,
\]

(3.8)

\[
\varepsilon + \frac{1}{2}N_0 = k_0 + \frac{3}{4}N_0.
\]

(3.9)

Note that, in general, \( \beta(k_0) \neq 0 \). However if \( \frac{G(e^{i\Theta_0})}{\pi} \) is an integer, then \( \beta(k_0) = 0 \), which indicates exact symmetry about \( k_0 + \frac{1}{4}N_0 \) in the interval \([k_0, k_0 + \frac{1}{2}N_0]\) and about \( k_0 + \frac{3}{4}N_0 \) in the interval \([k_0 + \frac{1}{2}N_0, k_0 + N_0]\).

To illustrate the symmetry search algorithm, we reconsider the example considered in Figures 2a and 3, where \( y = W(v) = 0.6(v + 1)^3 - 1 \). Note that \( W \) is not even. Figure 4a shows the values of \( \beta \) calculated for \( y(k) \) on the interval \([k_0, k_0 + 2N_0]\). Since, in Figure 4a, the data window of \( y \) is selected to start at \( k_0 = \varepsilon - \frac{1}{4}N_0 \), the minimum values of \( \beta(k) \) occur at \( k_0 \) and \( k_0 + N_0 \), where \( k_0 + N_0 \) is the start of the next period and, thus, need not be considered. Thus, using the
unique minimizer $k_0$ of $\beta(k)$, it follows that the locations of the points of symmetry are given by (3.8) and (3.9).

Next, for the even nonlinearity $y = W(v) = v^2$ considered in Figure 2b, Figure 4b shows the values of $\beta(k)$ calculated for $y(k)$ on the interval $[k_0, k_0 + 2N_0]$. In this case, the minimum values of $\beta(k)$ occur at $k_0$, $k_0 + \frac{1}{2}N_0$, and $k_0 + N_0$, where $k_0 + N_0$ is the start of the next period and, thus, need not be considered. Thus, using $k_0$, it follows that the locations of the points of symmetry are given by (3.8) and (3.9). Also, using $k_0 + \frac{1}{2}N_0$, we obtain two additional points of symmetry given by

$$\varepsilon + \frac{1}{4}N_0 = k_0 + \frac{1}{2}N_0,$$

$$\varepsilon + \frac{3}{4}N_0 = k_0 + N_0.$$  \hfill (3.10)

$$\varepsilon + \frac{1}{4}N_0 = k_0 + \frac{1}{2}N_0,$$

$$\varepsilon + \frac{3}{4}N_0 = k_0 + N_0.$$  \hfill (3.11)

This ambiguity is due to the fact that $\varepsilon$ and $\varepsilon + \frac{1}{2}N_0$ are the midpoints of two identical symmetric portions of $y$. Thus, the start of the data window within which the function has the symmetry properties illustrated in Figure 3 can be taken as either $k_0$ or $k_0 + \frac{1}{2}N_0$. Note that the second minimizer $k_0 + \frac{1}{2}N_0$ appears only for even nonlinearities.

### 3.2 Nonparametric Approximation of the Static Nonlinearity

Using $\delta$, which is assumed to be known from the harmonic input $u$, and the estimate of $\varepsilon$ obtained from $y$ in Section 3.1, we now determine an estimate $\hat{\phi}$ of the nonharmonic phase shift of $y$ relative to $u$ by

$$\hat{\phi} \triangleq \Omega_0(\varepsilon - \delta),$$  \hfill (3.12)

which is an estimate of $\angle G(e^{i\Omega_0})$. Moreover, define the virtual signal

$$\tilde{v}(k) \triangleq A_0 \sin(\Omega_0 k + \hat{\phi}),$$  \hfill (3.13)

which is an approximation of the intermediate signal $v$ given by (3.2) divided by the constant $|G(e^{i\Omega_0})|$. Note that, if $\hat{\phi} = \angle G(e^{i\Omega_0})$, then $|G(e^{i\Omega_0})|\tilde{v} = v$. Also, note that the amplitude of $\tilde{v}(k)$
is irrelevant due to the scaling factor $\lambda$ shown in Figure 1b.

Using $\tilde{v}$ and $y$, the nonparametric estimate of $\mathcal{W}$ is given by

$$\hat{\mathcal{W}} \triangleq \{(\tilde{v}(k_0), y(k_0)), (\tilde{v}(k_0 + 1), y(k_0 + 1)), \ldots, (\tilde{v}(n), y(n))\},$$

(3.14)

where each pair $(\tilde{v}(k), y(k))$, for $k = k_0, \ldots, n$, determines a value of the nonparametric estimate $\hat{\mathcal{W}}$ of $\mathcal{W}$.

Figure 4 shows that, depending on the type of nonlinearity, $\beta(k)$ has either one or two minima within each period. For a non-even polynomial nonlinearity, $\beta(k)$ has one minimum within each period. Therefore, the estimate of the nonharmonic phase shift has two candidate values, namely, $\hat{\phi}$ and $\hat{\phi} + \pi$. For an even nonlinearity, $\beta(k)$ has two minima within each period. Therefore, the estimate of the nonharmonic phase shift has four candidate values, namely, $\hat{\phi}$, $\hat{\phi} + \frac{\pi}{2}$, $\hat{\phi} + \pi$, and $\hat{\phi} + \frac{3\pi}{2}$. However, for the even case, $\hat{\phi}$ and $\hat{\phi} + \pi$ yield the same nonparametric model $\hat{\mathcal{W}}$, while $\hat{\phi} + \frac{\pi}{2}$ and $\hat{\phi} + \frac{3\pi}{2}$ yield the same $\hat{\mathcal{W}}$.

Therefore, for both non-even and even cases, there are two candidate nonparametric estimates of $\mathcal{W}$, both of which are constructed using (3.13) and (3.14). The correct nonparametric model will become apparent when identifying the dynamic block of the Wiener system.

4 Parametric Identification of the Linear Time-Invariant Dynamics

Using the nonparametric model $\hat{\mathcal{W}}$ of $\mathcal{W}$, we now identify a model of $\mathcal{L}$ given by $\hat{\mathcal{L}}$ using retrospective cost optimization (RCO) [20]. The RCO algorithm is presented in [7, 19, 20] together with guidelines for choosing its tuning parameters, namely, $n_c$, $p$, and $\alpha$.

Consider the adaptive feedback architecture for $\hat{\mathcal{L}}$ shown in Figure 5, where $\hat{\mathcal{L}}_m$ denotes the initial model with input $w \in \mathbb{R}$ and output $\hat{v} \in \mathbb{R}$, and where $\hat{\mathcal{L}}_{\Delta}$ denotes the feedback delta model
with inputs $u, \hat{v} \in \mathbb{R}$ and output $w$.

The goal is to adaptively tune $\hat{L}_\Delta$ so that the performance variable

$$z(k) \triangleq y(k) - \hat{y}(k)$$

is minimized in the presence of the identification signal $u$. For simplicity, we choose $\hat{L}_m$ to be the one-step delay $1/z$. Together, $\hat{L}$ and $\hat{W}$ comprise a semiparametric model of the Wiener system.

From Section 3.2, recall that there are two candidates for the nonparametric estimate of $\hat{W}$. Thus, we run RCO for each nonparametric estimate of $\hat{W}$ and obtain a corresponding parametric model of $L$. Note that the performance variable $z$ is calculated for both semiparametric models. We choose the semiparametric model whose performance variable has a smaller norm.

### 4.1 Retrospective Cost Optimization

We now review the RCO adaptive control algorithm and show how it is used to identify linear time-invariant dynamic systems using $\hat{W}$. A detailed discussion of RCO and as well as the theoretical foundations of the algorithm are found in [13, 14, 19].

RCO depends on several parameters that are selected \textit{a priori}. Specifically, $n_c$ is the estimated plant order, $p \geq 1$ is the data window size used to estimate $\hat{L}_\Delta$, and $\mu$ is the number of Markov parameters of $\hat{L}_\Delta$. The methodology for choosing these parameters is as follows. $n_c$ is overestimated, that is, chosen to be greater than the expected order of $\hat{L}$. From Section 4, recall that we assume that the controller $\hat{L}_\Delta$ is placed in feedback with a unit delay. Therefore, there is only one nonzero Markov parameter, so $\mu = 1$ in all example cases. The adaptive update law is based on a quadratic cost function, which involves a time-varying weighting parameter $\alpha(k) > 0$, referred to as the \textit{learning rate} since it affects the convergence speed of the adaptive control algorithm. In [19], RCO is presented for MIMO systems, where $l_u, l_v, l_w,$ and $l_y$ denote the sizes of $u, v, w,$ and $y$, respectively. However, in this paper, we consider only the SISO system (2.1)-(2.3). For convenience, we keep the notation of [19] and set $l_u = l_v = l_w = l_y = 1$. 

9
Let $\hat{L}_m \sim \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ as given by (2.1), (2.2), where $x(k) \in \mathbb{R}^l_x$, $A \in \mathbb{R}^{l_x \times l_x}$, $B \in \mathbb{R}^{l_x \times 1}$, $C \in \mathbb{R}^{1 \times l_x}$. Since $\hat{L}_m$ is set as unit delay, it follows that $A = 0_{l_x \times l_x}$, $B = 1_{l_x \times 1}$, and $C = 1_{1 \times l_x}$, yielding

$$\hat{v}(k) = w(k - 1),$$

where $w(k - 1)$ is the output of $\hat{L}_\Delta$, which was obtained using RCO in the previous iteration. Note that, to compute (4.1), $\hat{y}(k)$ is assumed to be known. To accomplish that, we use the estimated intermediate signal $\hat{v}(k)$ with $\hat{W}$ as follows. Note that, in general, $\hat{v}(k)$ is not in the set defined by (3.14). We thus suggest two methods by which this issue may be overcome. For simplicity, the first case is to use the closest value of $\hat{v}(k)$ in the set (3.14) to $\hat{v}(k)$. Second, interpolation between the closest bounding values may be used. For convenience, henceforth, we use the first method.

Next, to compute $w(k)$ we use an exactly proper time-series controller of order $n_c$ such that the control $w(k)$ is given by

$$w(k) = \sum_{i=1}^{n_c} M_i(k)w(k - i) + \sum_{i=0}^{n_c} N_i(k) \begin{bmatrix} \hat{v}(k - i) \\ u(k - i) \end{bmatrix}, \quad (4.2)$$

where $M_i \in \mathbb{R}^{l_x \times l_w}, i = 1, \ldots, n_c$, and $N_i \in \mathbb{R}^{l_w \times (l_w + l_u)}, i = 0, \ldots, n_c$, are given by an adaptive update law. Note that the ARX model given in (4.2) is a model of $\hat{L}_\Delta$. The control can be expressed as

$$w(k) = \Theta(k)\psi(k), \quad (4.3)$$

where

$$\Theta(k) \equiv \begin{bmatrix} N_0(k) & \cdots & N_{n_c}(k) & M_1(k) & \cdots & M_{n_c}(k) \end{bmatrix}$$
is the controller parameter block matrix and the regressor vector $\psi(k)$ is given by

$$\psi(k) \triangleq \begin{bmatrix} \hat{v}(k) \\ \vdots \\ \hat{v}(k - n_c) \\ u(k) \\ \vdots \\ u(k - n_c) \\ w(k - 1) \\ \vdots \\ w(k - n_c) \end{bmatrix} \in \mathbb{R}^{n_{l_w} + (n_c + 1)(l_c + l_w)}.$$  

For positive integers $p$ and $\mu$, we define the extended performance vector $Z(k)$ and the extended control vector $W(k)$ by

$$Z(k) \triangleq \begin{bmatrix} z(k) \\ \vdots \\ z(k - p + 1) \end{bmatrix}, \quad W(k) \triangleq \begin{bmatrix} w(k) \\ \vdots \\ w(k - p_c + 1) \end{bmatrix},$$

where $p_c \triangleq \mu + p$.

From (4.3), it follows that the extended control vector $W(k)$ can be written as

$$W(k) = \sum_{i=1}^{p_c} L_i \theta(k - i + 1) \psi(k - i + 1),$$

where

$$L_i \triangleq \begin{bmatrix} 0_{(i-1)l_w \times l_w} \\ I_{l_w} \\ 0_{(p_c-i)l_w \times l_w} \end{bmatrix} \in \mathbb{R}^{p_c l_w \times l_w}.$$  

We define the surrogate performance vector $\hat{Z}(\hat{\theta}(k), k)$ by

$$\hat{Z}(\hat{\theta}(k), k) \triangleq Z(k) - \underbrace{\mathbf{E}}_{z_w} \left( W(k) - W(k) \right),$$

where $z_w$.
where
\[
\hat{W}(k) \triangleq \sum_{i=1}^{p_k} L_i \hat{\theta}(k) \psi(k-i+1),
\]
and \(\hat{\theta}(k) \in \mathbb{R}^{l_w \times [n_c l_w + (n_c+1)(l_o + l_u)]}\) is the surrogate controller parameter block matrix. The block-Toeplitz surrogate control matrix \(\overline{B}_{zw}\) is given by
\[
\overline{B}_{zw} \triangleq \begin{bmatrix}
0_{l_z \times l_w} & \cdots & 0_{l_z \times l_w} & H_d & \cdots & 0_{l_z \times l_w} & \cdots & 0_{l_z \times l_w} \\
0_{l_z \times l_w} & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0_{l_z \times l_w} & \cdots & 0_{l_z \times l_w} & 0_{l_z \times l_w} & \cdots & 0_{l_z \times l_w} & H_d & \cdots & H_d
\end{bmatrix},
\]
where the relative degree \(d\) is the smallest positive integer \(i\) such that the \(i\)th Markov parameter \(H_i = CA^{-1}B\) of \(\hat{L}_m\) is nonzero. The leading zeros in \(\overline{B}_{zw}\) account for the nonzero relative degree \(d\). The algorithm places no constraints on either the value of \(d\) or the rank of \(H_d\) or \(\overline{B}_{zw}\). For the SISO case when \(\hat{L}_m\) is a unit delay,
\[
\overline{B}_{zw} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}.
\] (4.9)

Furthermore, we define
\[
D(k) \triangleq \sum_{i=1}^{n_c+\mu-1} \psi^T(k-i+1) \otimes L_i,
\] (4.10)
\[
f(k) \triangleq Z(k) - \overline{B}_{zw} W(k).
\] (4.11)

We now consider the cost function
\[
J(\hat{\theta}, k) \triangleq \hat{Z}^T(\hat{\theta}, k) R_1(k) \hat{Z}(\hat{\theta}, k) + \text{tr} \left[ R_2(k) \left( \hat{\theta} - \theta \right) \right]^T R_3(k) \left( \hat{\theta} - \theta \right),
\] (4.12)
where \(R_1(k) \triangleq I_{l_w}, \ R_2(k) \triangleq \alpha(k) I_{n_c (l_w + (l_t + l_u))}, \) and \(R_3(k) \triangleq I_{l_w}\). Note that the cost function is quadratic in the retrospective term \(\hat{Z}\), while the second term penalizes the difference \(\theta(k+1) - \theta(k)\); therefore, \(R_2\) and \(R_3\) can be used to control how much the controller parameters will change in a given step.
Substituting (4.7) and (4.8) into (4.12), $J$ is written as the quadratic form

$$J(\hat{\theta}, k) = c(k) + b^T \text{vec} \hat{\theta} + \left(\text{vec} \hat{\theta}\right)^T A(k) \text{vec} \hat{\theta},$$

(4.13)

where

$$A(k) = D^T(k) D(k) + \alpha(k) I,$$

(4.14)

$$b(k) = 2D^T(k) f(k) - 2\alpha(k) \text{vec} \theta(k),$$

(4.15)

$$c(k) = f(k)^T R_1(k) f(k) + \text{tr} \left[ R_2(k) \theta^T(k) R_3(k) \theta(k) \right].$$

(4.16)

Since $A(k)$ is positive definite, $J(\hat{\theta}, k)$ has the strict global minimizer

$$\hat{\theta}(k) = \frac{1}{2} \text{vec}^{-1} (A(k)^{-1} b(k)).$$

(4.17)

The controller gain update law is

$$\theta(k + 1) = \hat{\theta}(k),$$

(4.18)

such that $w(k)$ is computed using (4.3). The key feature of the adaptive control algorithm (4.3) is the surrogate performance variable $Z(k)$ based on the difference between the actual past control inputs $W(k)$ and the recomputed past control inputs based on the current control law $\hat{W}(k)$. The parameter $\alpha$ is chosen to be as small as possible while guaranteeing that $A(k)$ is positive definite.

5 Numerical Examples: Nominal Case

To demonstrate semiparametric model identification, we consider various static nonlinearities. For each example, we choose $G$ to have poles $0.34 \pm 0.87 j, -0.3141 \pm 0.9 j, 0.05 \pm 0.3122 j, -0.6875$ and zeros $0.14 \pm 0.97 j, -0.12 \pm 0.62 j, -0.89$ with monic numerator and denominator. Also, $u(k)$ is chosen to be a realization of zero-mean Gaussian white noise with standard deviation $\sigma_u = 3.5$.

Note that $A_0$ should, in practice, be chosen to be greater than the expected operating range of the Wiener system. This guarantees that the inputs to the model can be interpolated from the
nonparametric map. For the following examples we choose $m$ to be much larger than required. Although we show in Section 7.2 that very little performance gain is attained from choosing $m$ large, it is visually easier to compare the identified nonparametric map to the true nonlinearity when using more data points. Finally, the parameter $\alpha(k)$ discussed in the previous section is chosen as a constant value for all examples. We choose varying values for $\alpha(k)$ to demonstrate that the final estimate of the Wiener system is not sensitive to this parameter.

**Example 5.1.** (Non-even Polynomial) Consider $W$ defined by

$$y = W(v) = v^3 + 4v + 7.$$  \hfill (5.1)

The parameters for nonparametric identification of $W$ are $m = 500$ and $A_0 = 5$. Figure 6 compares the true and identified nonlinearities. The RCO parameters used to identify the linear dynamic system are set as $n_c = 9$, $p = 1$, and $\alpha = 1$. Figure 7 shows the frequency response of the true dynamic model $G$ and the identified model using RCO with the identified nonlinearity shown in Figure 6.

**Example 5.2.** (Even Polynomial) Consider $W$ defined by

$$y(k) = W(v) = 7v^4 + v^2.$$  \hfill (5.2)

The parameters for nonparametric identification of $W$ are $m = 500$ and $A_0 = 5$. Figure 8 compares the true and identified nonlinearities. The RCO parameters used to identify the linear dynamic system are set as $n_c = 9$, $p = 1$, and $\alpha = 50$. Figure 9 shows the frequency response of $G$ and the identified model using RCO with the identified nonlinearity shown in Figure 8.

Next, to illustrate the ambiguity discussed in Section 3.2, we select the incorrect nonharmonic phase shift, specifically, $\hat{\phi} + \frac{\pi}{2}$. Figure 10 shows a comparison of the true and identified nonlinearities. Note that the incorrect nonharmonic phase shift produces an erroneous nonparametric model of the nonlinearity. Figure 11 shows a frequency response comparison of $G$ and the model identified using RCO with the identified nonlinearity shown in Figure 10.
To determine the appropriate phase shift $\hat{\phi}$ or $\hat{\phi} + \frac{\pi}{2}$, we examine the performance variable $z$ given by (4.1), which provides insight into which candidate value yields the correct semiparametric model. The upper plot of Figure 12 shows the RCO performance variable $z$ for the incorrect nonparametric model of $W$, while the lower plot shows the performance variable for the correct nonparametric model of $W$. The correct semiparametric model clearly outperforms the incorrect model.

6 Numerical Examples: Off-Nominal Cases

We now reconsider the Wiener system (2.1)-(2.3) with noise, as shown in Figure 13. The input $u(k)$ is a realization of zero-mean Gaussian white noise with standard deviation $\sigma_u = 3.5$, while $d_1(k) \in \mathbb{R}$ and $d_2(k) \in \mathbb{R}$ are unknown zero-mean Gaussian white disturbances with standard deviations $\sigma_{d_1}$ and $\sigma_{d_2}$, respectively. The output

$$y(k) = W(v(k)) + d_3(k),$$

has standard deviation $\sigma_y$ about its mean, and $d_3(k) \in \mathbb{R}$ is an unknown zero-mean Gaussian white disturbance with standard deviation $\sigma_{d_3}$. The disturbance signals $d_1(k)$, $d_2(k)$, and $d_3(k)$ are process, input, and output noise, respectively.

We now consider additional static nonlinearities, where, for each example, we choose $G$ as in Section 5.

**Example 6.1.** (Deadzone) Consider $W$ defined by

$$y = W(v) = \begin{cases} 
0, & \text{if } |v| \leq 0.17; \\
v, & \text{if } |v| > 0.17.
\end{cases} \quad (6.2)$$

Furthermore, we consider process and output noise $\sigma_{d_1} = \frac{1}{15} \sigma_u$, $\sigma_{d_3} = \frac{1}{10} \sigma_y$ and $d_2 = 0$. For this problem, the parameters for nonparametric identification are $m = 250$ and $A_0 = 5$. In this example we also parameterize the estimated nonlinearity for comparison with the nonparametric estimate.
The parametric model is a 25\textsuperscript{th} order polynomial. Figure 14 compares the true, nonparametric identified and parametric identified nonlinearities. The RCO parameters used to identify the linear dynamic system are set as \( n_c = 9, p = 1, \) and \( \alpha = 10. \) Figure 15 shows the frequency response of \( G \) and the identified model using RCO with the identified nonparametric model of the nonlinearity. Figure 16 shows the frequency response of \( G \) and the identified model using RCO with the identified parametric model of the nonlinearity. Figure 17 compares the output of the Wiener system \( y(k) \) and the output of the estimated semiparametric Wiener model \( \hat{y}(k) \), in response to a random input. Figure 18 compares the output of the Wiener system \( y(k) \) and the output of the estimated parametric Wiener model \( \hat{y}(k) \), in response to a random input.

Figure 19 is the difference between the error in the semiparametric Wiener model and the parametric Wiener model. Where the graph is negative, the semiparametric model has superior performance, and where the graph is positive the parametric model is superior.

**Example 6.2.** (Saturation) Consider \( W \) defined by

\[
y = W(v) = \begin{cases} 
8.64(v + 0.23) - 3.98, & \text{if } 0.1 < v < 0.4; \\
1.5, & \text{if } v \geq 0.4; \\
-1.2, & \text{if } v \leq 0.1.
\end{cases}
\]  

(6.3)

Furthermore, we consider input noise \( \sigma_{d_1} = \frac{1}{8} \sigma_u \) and \( d_2 = d_3 = 0. \) The parameters for non-parametric identification are \( m = 150 \) and \( A_0 = 5. \) Figure 20 compares the true and identified nonlinearities. The RCO parameters used to identify the linear dynamic system are set as \( n_c = 9, p = 1, \) and \( \alpha = 1. \) Figure 21 shows the frequency response of \( G \) and the identified model using RCO with the identified nonlinearity shown in Figure 20.

**Example 6.3.** (Switch function) Consider \( W \) defined by

\[
y = W(v) = \begin{cases} 
0, & \text{if } |v| = 0; \\
8.64v + \text{sgn}(v)4.5, & \text{if } 0 < |v| \leq 1.5.
\end{cases}
\]  

(6.4)

Furthermore, we consider process, input, and output noise \( \sigma_{d_1} = \frac{1}{15} \sigma_u, \sigma_{d_2} = \frac{1}{15} \sigma_w, \) and \( \sigma_{d_3} = \frac{1}{15} \sigma_y. \)
The parameters for nonparametric identification are \( m = 100 \) and \( A_0 = 5 \). Figure 22 compares the true and identified nonlinearities. The RCO parameters used to identify the linear dynamic system are set as \( n_c = 9, \ p = 1, \) and \( \alpha = 1 \). Figure 23 shows the frequency response of \( G \) and the identified model using RCO with the identified nonlinearity shown in Figure 22.

**Example 6.4.** (Stairs Function) Consider \( \mathcal{W} \) defined by

\[
y = \mathcal{W}(v) = \begin{cases} 
0, & \text{if } |v| = 0; \\
\text{sgn}(v)1, & \text{if } 0 < |v| \leq 0.17; \\
\text{sgn}(v)3, & \text{if } 0.17 < |v| \leq 0.35; \\
\text{sgn}(v)4.5, & \text{if } 0.35 < |v| \leq 0.52; \\
\text{sgn}(v)6, & \text{if } 0.52 < |v|;
\end{cases}
\]

Furthermore, we consider process, input, and output noise \( \sigma_{d1} = \frac{1}{8} \sigma_u, \ \sigma_{d2} = \frac{1}{8} \sigma_w, \) and \( \sigma_{d3} = \frac{1}{8} \sigma_y \).

The parameters for nonparametric identification are \( m = 75 \) and \( A_0 = 5 \). Figure 24 compares the true and identified nonlinearities. The RCO parameters used to identify the linear dynamic system are set as \( n_c = 9, \ p = 1, \) and \( \alpha = 1 \). Figure 25 is a frequency response comparison of \( G \) and the system identified using RCO with the identified nonlinearity shown in Figure 24. Figure 26 compares the output of the Wiener system \( y(k) \) and the output of the estimated semiparametric Wiener model \( \hat{y}(k) \), in response to a random input.

### 7 Numerical Examples: Error Metrics

We now investigate the effect of systematically decreasing the amount of available output data that is used to identify the linear block of the Wiener system. Moreover, we investigate the effect of decreasing \( m \), which determines the number of points in the nonparametric model, and therefore affects the fidelity of \( \hat{\mathcal{W}} \).

To quantify the accuracy of the identified semiparametric model, we compute the root-mean-square error (RMSE) for the first 15 Markov parameters of the true linear system and the
identified linear system. The linear model is the same as in Sections 5 and 6, while \( W \) is given by (5.1).

7.1 Effect of Disturbances

To evaluate the effect of \( \sigma_{d_1}, \sigma_{d_2}, \text{ and } \sigma_{d_3} \), we decrease the number of available data points from 4000 to 10. For each case, we perform a 100-run Monte Carlo simulation with a signal-to-noise ratio of 10. We consider the effect of \( d_1, d_2, \text{ and } d_3 \) individually, as well as the effect of all three noise signals, which may be uncorrelated or correlated. Furthermore we consider when \( d_1 \) and \( d_3 \) are correlated, and \( d_2 \) and \( d_3 \) are correlated.

Figure 27 demonstrates the increase in error for decreasing amounts of available data. Furthermore, we see that the cases with correlated disturbances yield similar results compared to the case with uncorrelated disturbances.

7.2 Nonparametric Model Accuracy

We now perform a Monte Carlo simulation to evaluate how \( m \) affects the accuracy of the identified linear system. Specifically, we vary \( m \) from 1 to 100. For each value of \( m \) we average the result over 100 simulations. We consider the nominal case, that is, without noise.

Figure 28 shows that RMSE generally decreases as \( m \) increases. Note that, for this example, only a slight decrease in RMSE is observed for \( m \geq 20 \).

8 Conclusions

In this paper we develop a two-step method to identify semiparametric models for SISO discrete-time Wiener systems. We make only two assumptions about the system, namely, the linear dynamic block is assumed to be asymptotically stable, and the static nonlinearity is assumed
to be piecewise continuous. Furthermore, this method requires identification signals with specific properties for each of the two steps, as discussed as follows.

First, we choose a single harmonic input and measure the system output when the state trajectory is in harmonic steady state. By exploiting symmetry properties of these signals, we approximate the nonharmonic phase shift and, therefore, estimate the intermediate signal. Using the estimate of the intermediate signal, a nonparametric model of the static nonlinearity is obtained.

Second, using the identified nonparametric model, we use retrospective cost optimization to identify a parametric model of the dynamic system. As commonly assumed in the system identification literature, the identification signal for this step is assumed to be sufficiently persistent such that the dynamic linear system can be identified.

It is important to point out that the method investigated in this work does not require invertibility of the nonlinearity, which is a common assumption in Wiener identification. However, the cost of removing this assumption is the need for two steps, and the requirement that the signal for the first step be a single harmonic. Furthermore, the user must wait until the system has reached a steady state before useful data can be obtained. On the other hand, from Section 1, recall that there are methods based on multiple harmonic inputs in the literature. Finally, it should be noted that, although a nonparametric model of the nonlinearity was used in this discussion, the data which represents the nonparametric map could be parameterized.

The two-step method presented in this paper is effectively demonstrated on several examples of increasing complexity, including nonlinearities in the form of both even and non-even polynomials, deadzone, saturation, and discontinuity, and disturbances on the form of process, input, and output noise.
References


Figure 1: (a) Block-structured Wiener model, where $u$ is the input, $v$ is the intermediate signal, $y$ is the output, $\mathcal{L}$ is a discrete-time linear time-invariant dynamic system, and $\mathcal{W}$ is a static nonlinearity. (b) An equivalent scaled model, where $\lambda$ is a scaling factor and $\mathcal{W}_\lambda$ is a scaled-domain modification of $\mathcal{W}$ satisfying $\mathcal{W}_\lambda(\lambda v) = \mathcal{W}(v)$. The scaling factor $\lambda$ is not identifiable.
Figure 2: Illustration of the symmetry properties of the signals $u$, $v$, and $y$ given by (3.1)-(3.3), respectively, for (a) the non-even polynomial nonlinearity $y = \mathcal{W}(v) = 0.6(v + 1)^3 - 1$ and (b) the even polynomial nonlinearity $y = \mathcal{W}(v) = v^2$. The signals $u$ and $v$ are harmonic, whereas $y$ is the output of the nonlinear block $\mathcal{W}$ and thus is not harmonic. Note that, for both cases, $u$ is symmetric about $\delta$ in the interval $[\delta - \frac{1}{4}N_0, \delta + \frac{1}{4}N_0]$ and about $\delta + \frac{1}{2}N_0$ in the interval $[\delta + \frac{1}{4}N_0, \delta + \frac{3}{4}N_0]$, while $v$ and $y$ are symmetric about $\varepsilon$ in the interval $[\varepsilon - \frac{1}{4}N_0, \varepsilon + \frac{1}{4}N_0]$ and about $\varepsilon + \frac{1}{2}N_0$ in the interval $[\varepsilon + \frac{1}{4}N_0, \varepsilon + \frac{3}{4}N_0]$. In addition, for the case of an even polynomial nonlinearity shown in (b), $y$ is also symmetric about $\varepsilon + \frac{1}{4}N_0$ in the interval $[\varepsilon, \varepsilon + \frac{1}{4}N_0]$ and about $\varepsilon + \frac{3}{4}N_0$ in the interval $[\varepsilon + \frac{1}{2}N_0, \varepsilon + N_0]$. 
Figure 3: Illustration of the symmetry search algorithm. The solid line box comprises the sliding window of length $N_0 + 1$ starting at time $k$, while the dashed lines indicate the windowed points of symmetry.
Figure 4: Illustration of the symmetry error index $\beta(k)$ given by (3.5). The values of $\beta(k)$ are shown for two static nonlinearities, namely, (a) a non-even polynomial and (b) an even polynomial.
Figure 5: Identification architecture for Wiener models using retrospective cost optimization.
Figure 6: Identified nonlinearity versus true nonlinearity (5.1), where $m = 500$ and $A_0 = 5$ (Example 5.1). The argument of the identified nonlinearity is scaled by $\frac{1}{|G(e^{j\Omega})|}$ to facilitate comparison with the true nonlinearity.
Figure 7: Frequency response comparison of the true $G$ and the identified LTI system obtained using $\hat{W}$ as an estimate of (5.1), where $k$ is the number of data points used to determine the identified dynamic model. The RCO controller order is $n_c = 9$ with $p = 1$ and $\alpha = 1$ (Example 5.1).
Figure 8: Identified nonlinearity versus true nonlinearity (5.2), where $m = 500$ and $A_0 = 5$ (Example 5.2).
Figure 9: Frequency response comparison of the true $G$ and the identified LTI system obtained using $\hat{W}$ as an estimate of (5.2), where $k$ is the number of data points used to determine the identified dynamic model. The RCO controller order is $n_c = 9$ with $p = 1$, and $\alpha = 50$ (Example 5.2).
Figure 10: Identified nonlinearity versus true nonlinearity (5.2), where $m = 500$ and $A_0 = 5$ (Example 5.2). Both candidate values for the nonharmonic phase shift, namely, $\hat{\phi}$ and $\hat{\phi} + \frac{\pi}{2}$, are used to build the two candidate identified nonlinearities.
Figure 11: Frequency response comparison of the true $G$ and the identified LTI system obtained using $\hat{W}$ corresponding to the incorrect phase shift as an estimate of (5.2), where $k$ is the number of data points used to determine the identified dynamic model. The RCO controller order is $n_c = 9$ with $p = 1$, and $\alpha = 50$ (Example 5.2).
Figure 12: Retrospective optimization performance comparison for Example 5.2. The upper plot shows the performance variable $z$ for the case in which the nonparametric model is generated using the incorrect candidate for the nonharmonic phase shift $\hat{\phi} + \frac{\pi}{2}$. The lower plot shows $z$ for the case in which the correct candidate $\hat{\phi}$ is used.
Figure 13: Block-structured Wiener model with process, input, and output noise, where $d_1$, $d_2$, and $d_3$ are unknown zero-mean Gaussian disturbances.
Figure 14: Identified nonlinearity versus true nonlinearity (6.2), where $m = 250$ and $A_0 = 5$. In this example, we also parameterize the estimated nonlinearity using a $25^{\text{th}}$ order polynomial. (Example 6.1).
Figure 15: Frequency response comparison of the true $G$ and the identified LTI system obtained using the nonparametric $\hat{W}$ as an estimate of (6.2), where $k$ is the number of data points used to determine the identified dynamic model. The RCO controller order is $n_c = 9$ with $p = 1$ and $\alpha = 10$. (Example 6.1).
Figure 16: Frequency response comparison of the true $G$ and the identified LTI system obtained using the parametric $\hat{W}$ as an estimate of (6.2), where $k$ is the number of data points used to determine the identified dynamic model. The RCO controller order is $n_c = 9$ with $p = 1$ and $\alpha = 10$ (Example 6.1).
Figure 17: Performance comparison for Example 6.1 using the nonparametric estimate of the nonlinearity. The top plot is the output of the Wiener system $y(k)$, and the output of the estimated system $\hat{y}(k)$. The bottom plot is the performance $z(k)$. 
Figure 18: Performance comparison for Example 6.1 using the parametric estimate of the nonlinearity. The top plot is the output of the Wiener system $y(k)$, and the output of the estimated system $\hat{y}(k)$. The bottom plot is the performance $z(k)$. 
Figure 19: This plot is the difference between the error in the semiparametric Wiener model and the parametric Wiener model. Where the graph is negative, the semiparametric model has superior performance, and where the graph is positive the parametric model is superior.
Figure 20: Identified nonlinearity versus true nonlinearity (6.3), where $m = 150$ and $A_0 = 5$ (Example 6.2).
Figure 21: Frequency response comparison of the true $G$ and the identified LTI system obtained using $\hat{W}$ as an estimate of (6.3), where $k$ is the number of data points used to determine the identified dynamic model. The RCO controller order is $n_c = 9$ with $p = 1$ and $\alpha = 1$ (Example 6.2).
Figure 22: Identified nonlinearity versus true nonlinearity (6.4), where $m = 100$ and $A_0 = 5$ (Example 6.3).
Figure 23: Frequency response comparison of the true $G$ and the identified LTI system obtained using $\hat{W}$ as an estimate of (6.4), where $k$ is the number of data points used to determine the identified dynamic model. The RCO controller order is $n_c = 9$ with $p = 1$ and $\alpha = 1$ (Example 6.3).
Figure 24: Identified nonlinearity versus true nonlinearity (6.5), where $m = 75$ and $A_0 = 5$ (Example 6.4).
Figure 25: Frequency response comparison of the true $G$ and the identified LTI system obtained using $\hat{\mu}$ as an estimate of (6.5), where $k$ is the number of data points used to determine the identified dynamic model. The RCO controller order is $n_c = 9$ with $p = 1$ and $\alpha = 1$ (Example 6.4).
Figure 26: Performance comparison for Example 6.4 of the output of the Wiener system $y(k)$, and the output of the estimated system $\hat{y}(k)$. The bottom plot is the performance $z(k)$. 

Figure contains two subplots. The top subplot shows the difference between the true model and the identified model over time, with data points indicating the performance at various data steps. The bottom subplot shows the performance of the estimated system $z(k)$ against the true output $y(k)$.
Figure 27: RMSE Markov parameter error versus number of data points. For each number of data points we perform a 100-run Monte Carlo simulation.
Figure 28: RMSE Markov parameter error for an increasing number of points in the nonparametric model. For each value of $m$, a 100-run Monte Carlo simulation is performed.