NOTICE CONCERNING COPYRIGHT RESTRICTIONS

The copyright law of the United States [Title 17, United States Code] governs the making of photocopies or other reproductions of copyrighted material.

Under certain conditions specified in the law, libraries and archives are authorized to furnish a photocopy or other reproduction. One of these specified conditions is that the reproduction is not to be used for any purpose other than private study, scholarship, or research. If a user makes a request for, or later uses, a photocopy or reproduction for purposes in excess of “fair use” that use may be liable for copyright infringement.

The institution reserves the right to refuse to accept a copying order if, in its judgment, fulfillment of the order would involve violation of copyright law. No further reproduction and distribution of this copy is permitted by transmission or any other means.
Stable Stabilization with $H_2$ and $H_\infty$
Performance Constraints*

Y. William Wang† Wassim M. Haddad†
Dennis S. Bernstein†

Abstract

In this paper, we derive sufficient conditions for stable compensators with closed-loop $H_2$ and $H_\infty$ performance criteria. We first generalize prior results in which $H_2$-suboptimal stable compensators were considered. Then using two different approaches, we derive conditions for constructing mixed-norm $H_2/H_\infty$ stable compensators.

Key words: mixed-norm $H_2/H_\infty$ control, strong stabilization, reduced-order control

1 Introduction

It was shown in [10] that a system is stabilizable by means of a stable compensator if and only if the real, unstable poles and zeros of the system satisfy the parity interlacing property. Consequently, certain plants can only be stabilized by unstable compensators. Issues related to strong stabilization were further studied by many authors, for example, [2] and [3]. In [7] and [8], the authors modified the closed-loop Lyapunov equation along with parameter optimization to guarantee controller stability. In this a priori approach controller stability is guaranteed prior to carrying out the optimization. An a posteriori approach, given in [5] and [9] is based upon the modified LQG Riccati equations to yield a stable compensator.

In this paper, we first generalize the results in [9] by using the a posteriori approach for $H_2$ strong stabilization. Specifically, we show that

*Received July 23, 1993, received in final form November 29, 1993.
†This research was supported in part by the Air Force Office of Scientific Research under Grant F49620-92-J-0127 and the National Science Foundation under Research Initiation Grant ECS-9109558.
there exist several modifications to the LQG result that yield a stable compensator. Then using both the \textit{a priori} and \textit{a posteriori} approaches, we derive sufficient conditions for constructing stable compensators with the closed-loop system satisfying $H_2$ and $H_\infty$ performance constraints.

It is important to note that the conditions given in this paper for stable stabilization are sufficient. The more difficult problem of determining conditions for stable stabilization that are both necessary and sufficient in the context of $H_2$ and $H_2/H_\infty$ control objectives remains an open problem for future research.

2 \ H_2-Suboptimal Strong Stabilization

In this section, we generalize the \textit{a posteriori} approach of [9]. Consider the $n$th-order plant

$$
\dot{x}(t) = Ax(t) + Bu(t) + D_1w(t),
$$

$$
y(t) = Cx(t) + D_2w(t),
$$

with performance variables

$$
z(t) = E_1x(t) + E_2u(t),
$$

where $w(t)$ is a standard white noise process. Using the $n_c$th-order dynamic compensator

$$
\dot{x}_c(t) = A_cx_c(t) + B_cy(t),
$$

$$
u(t) = C_cx_c(t),
$$

we obtain the closed-loop system

$$
\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{D}w(t),
$$

$$
z(t) = \tilde{E}\tilde{x}(t),
$$

where

$$
\tilde{x}(t) \triangleq \begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix}, \quad \tilde{A} \triangleq \begin{bmatrix} A & BC_c \\ B_cC & A_c \end{bmatrix}, \quad \tilde{D} \triangleq \begin{bmatrix} D_1 \\ B_cD_2 \end{bmatrix},
$$

$$
\tilde{E} \triangleq \begin{bmatrix} E_1 & E_2C_c \end{bmatrix}.
$$

The performance index is given by

$$
J(A_c, B_c, C_c) = \lim_{t \to \infty} \mathcal{E}[x^T(t)x(t) + u^T(t)r_2u(t)],
$$

where '\mathcal{E}' denotes expectation and $R_1 \triangleq E_1^TE_1$, $R_2 \triangleq E_2^TE_2 > 0$. For convenience, we define $V_1 \triangleq D_1D_1^T$, $V_2 \triangleq D_2D_2^T > 0$, and $\Sigma \triangleq BR_2^{-1}B^T$,
$\Sigma \triangleq C^T V_2^{-1} C$. For simplicity, we assume $R_{12} \triangleq E_1^T E_2 = 0$ and $V_{12} \triangleq D_1 D_2^T = 0$.

The $H_2$ optimal control problem can be stated as follows: minimize the $H_2$ performance criterion given by (2.8), or, equivalently,

$$J(A_c, B_c, C_c) = \| \tilde{E} (sI - \tilde{A})^{-1} \tilde{D} \|_2^2 = \text{tr} \tilde{Q} \tilde{R},$$  \hspace{1cm} (2.9)$$

subject to

$$0 = \tilde{A} \tilde{Q} + \tilde{Q} \tilde{A}^T + \tilde{V},$$  \hspace{1cm} (2.10)$$

where

$$\tilde{R} \triangleq \tilde{E}^T \tilde{E} = \begin{bmatrix} R_1 & 0 \\ 0 & C_c^T R_2 C_c \end{bmatrix}, \hspace{1cm} \tilde{V} \triangleq \tilde{D} \tilde{D}^T = \begin{bmatrix} V_1 & 0 \\ 0 & B_c V_2 B_c^T \end{bmatrix}.$$  

Throughout the paper we make the standard assumptions that both $(A, B)$ and $(A, D_1)$ are stabilizable and both $(C, A)$ and $(E_1, A)$ are detectable. In this case, it is well known that there exist nonnegative-definite matrices $Q, P$ satisfying

$$0 = A^T P + PA + R_1 - P \Sigma P,$$  \hspace{1cm} (2.11)$$

$$0 = AQ + QA^T + V_1 - Q \Sigma Q,$$  \hspace{1cm} (2.12)$$

such that the optimal full-order controller is given by

$$A_c = A + BC_c - B_c C_c,$$  \hspace{1cm} (2.13)$$

$$B_c = QC^T V_2^{-1},$$  \hspace{1cm} (2.14)$$

$$C_c = -R_2^{-1} B^T P.$$  \hspace{1cm} (2.15)$$

Although LQG controllers minimize the $H_2$ performance criterion (2.8) and guarantee closed-loop asymptotic stability, the resulting compensator is not necessarily either stable in the sense of Lyapunov or asymptotically stable. Hence, in this section we derive sufficient conditions that guarantee that the compensator is asymptotically stable along with suboptimal $H_2$ closed-loop performance.

For the full-order case $n_c = n$, Theorem 2.1 gives sufficient conditions for constructing stable controllers.

**Theorem 2.1** Let $R_1 > 0$ and let $\Omega(P, \hat{P}) \geq 0$ satisfy

$$(A + \Sigma \hat{P})^T P + P (A + \Sigma \hat{P}) + \Omega(P, \hat{P}) \geq 0,$$  \hspace{1cm} (2.16)$$

for all $n \times n$ nonnegative-definite matrices $P$ and $\hat{P}$. Furthermore, suppose there exist $n \times n$ nonnegative-definite matrices $Q, P,$ and $\tilde{P}$ satisfying

$$0 = AQ + QA^T + V_1 - Q \tilde{\Sigma} Q,$$  \hspace{1cm} (2.17)$$

183
\[ 0 = A^T P + P A + R_1 + \Omega(P, \hat{P}) - P \Sigma P, \quad (2.18) \]
\[ 0 = (A - Q \hat{\Sigma})^T \hat{P} + \hat{P} (A - Q \hat{\Sigma}) + P \Sigma P, \quad (2.19) \]
and let \((A_c, B_c, C_c)\) be given by (2.13)-(2.15). Then \(A_c\) and \(\hat{A}\) are asymptotically stable.

**Proof:** Adding (2.18) to (2.19) and using (2.16) and \(R_1 > 0\) yields \(A_c^T \hat{P} + \hat{P} A_c < 0\). Hence \(A_c\) is asymptotically stable. Now defining \(\bar{R}_1 = R_1 + \Omega(P, \hat{P}) > 0\), it can be seen that (2.17) and (2.18) are in the form of the standard LQG Riccati equations. Thus \(\hat{A}\) is asymptotically stable. \(\square\)

Note that by letting \(\Omega(P, \hat{P}) = 0\), we recover the standard LQG result where \(A_c\) is not necessarily stable. Since the ordering induced by the cone of nonnegative-definite matrices is only a partial ordering, there does not exist a unique function \(\Omega(\cdot, \cdot)\) satisfying (2.16). The next result gives nine such functions.

**Proposition 2.1** For arbitrary \(\alpha, \beta > 0\), the following matrix functions \(\Omega(P, \hat{P})\) satisfy (2.16) for all nonnegative-definite matrices \(P, \hat{P}\):

(i) \[ \Omega(P, \hat{P}) = \alpha^2 (A + \Sigma \hat{P})^T (A + \Sigma \hat{P}) + \alpha^{-2} P^2, \]

(ii) \[ \Omega(P, \hat{P}) = [\alpha (A + \Sigma \hat{P}) - \alpha^{-1} P]^T [\alpha (A + \Sigma \hat{P}) - \alpha^{-1} P], \]

(iii) \[ \Omega(P, \hat{P}) = [\alpha (A + \Sigma \hat{P}) - \alpha^{-1} I]^T P [\alpha (A + \Sigma \hat{P}) - \alpha^{-1} I], \]

(iv) \[ \Omega(P, \hat{P}) = \alpha^2 A^T A + \alpha^{-2} P^2 + \beta^2 P \Sigma^2 P + \beta^{-2} \hat{P}^2, \]

(v) \[ \Omega(P, \hat{P}) = \alpha^2 A^T A + (\alpha^{-2} - \beta^{-2}) P^2 + \beta^2 \hat{P} \Sigma^2 \hat{P}, \]

(vi) \[ \Omega(P, \hat{P}) = \alpha^2 A^T A + \alpha^{-2} P^2 + (\beta P - \beta^{-1} \hat{P}) \Sigma (\beta P - \beta^{-1} \hat{P}), \]

(vii) \[ \Omega(P, \hat{P}) = (\alpha A - \alpha^{-1} I)^T P (\alpha A - \alpha^{-1} I) + \beta^2 P \Sigma^2 P + \beta^{-2} P^2, \]

(viii) \[ \Omega(P, \hat{P}) = (\alpha A - \alpha^{-1} I)^T P (\alpha A - \alpha^{-1} I) + \beta^2 \hat{P} \Sigma^2 \hat{P} + \beta^{-2} P^2, \]

(ix) \[ \Omega(P, \hat{P}) = (\alpha A - \alpha^{-1} I)^T P (\alpha A - \alpha^{-1} I) + (\beta P - \beta^{-1} \hat{P}) \Sigma (\beta P - \beta^{-1} \hat{P}). \]

**Proof:** The proof involves straightforward matrix manipulations and hence is omitted. \(\square\)

For reduced-order dynamic compensation \(n_c < n\), we recall from [6] the necessary conditions for \(H_2\) optimality.

**Theorem 2.2** Let \(n_c \leq n\) and suppose \((A_c, B_c, C_c)\) minimizes \(J(A_c, B_c, C_c)\). Then there exist \(n \times n\) nonnegative-definite matrices \(Q, \hat{Q}, P, \hat{P}\) such that \(A_c, B_c, C_c\) are given by

\[ A_c = \Gamma (A - Q \hat{\Sigma} - \Sigma P) G^T, \quad (2.20) \]
\[ B_c = \Gamma Q C^T V_2^{-1}, \quad (2.21) \]
\[ C_c = -R_2^{-1} B^T P G^T, \quad (2.22) \]
where \( Q, \hat{Q}, P, \hat{P}, \Gamma \) and \( G \) satisfy
\[
0 = AQ + QA^T + V_1 - Q\hat{\Sigma}Q + \tau_\perp Q\hat{\Sigma}Q\tau_\perp^T, \tag{2.23}
\]
\[
0 = (A - \Sigma P)\hat{Q} + \hat{Q}(A - \Sigma P)^T + Q\hat{\Sigma}Q - \tau_\perp Q\hat{\Sigma}Q\tau_\perp^T, \tag{2.24}
\]
\[
0 = A^T P + PA + R_1 - P\Sigma P + \tau_\perp^T P\Sigma P\tau_\perp, \tag{2.25}
\]
\[
0 = (A - Q\hat{\Sigma})^T \hat{P} + \hat{P}(A - Q\hat{\Sigma}) + P\Sigma P - \tau_\perp^T P\Sigma P\tau_\perp, \tag{2.26}
\]
\[
\text{rank } \hat{Q} = \text{rank } \hat{P} = n_c, \tag{2.27}
\]
\[
\hat{Q}\hat{P} = G^T M\Gamma, \quad \Gamma G^T = I_{n_c}, \quad M \in \mathbb{R}^{n_c \times n_c}, \tag{2.28}
\]
\[
\tau \triangleq G^T \Gamma, \quad \tau_\perp \triangleq I_n - \tau, \tag{2.29}
\]
\[
\hat{Q} = \tau \hat{Q}, \quad \hat{P} = \hat{P}\tau. \tag{2.30}
\]

Next, we modify Theorem 2.2 to construct reduced-order stable controllers.

**Theorem 2.3** Let \( R_1 > 0 \) and let \( \Omega(P, \hat{P}) \geq 0 \) satisfy (2.16) for all \( n \times n \) nonnegative-definite matrices \( P, \hat{P} \). Suppose there exist \( n \times n \) nonnegative-definite matrices \( Q, \hat{Q}, P, \hat{P} \) satisfying (2.27)-(2.30) and
\[
0 = AQ + QA^T + V_1 - Q\hat{\Sigma}Q + \tau_\perp Q\hat{\Sigma}Q\tau_\perp^T, \tag{2.31}
\]
\[
0 = (A - \Sigma P)\hat{Q} + \hat{Q}(A - \Sigma P)^T + Q\hat{\Sigma}Q - \tau_\perp Q\hat{\Sigma}Q\tau_\perp^T, \tag{2.32}
\]
\[
0 = A^T P + PA + R_1 + \Omega(P, \hat{P}) - P\Sigma P + \tau_\perp^T P\Sigma P\tau_\perp, \tag{2.33}
\]
\[
0 = (A - Q\hat{\Sigma})^T \hat{P} + \hat{P}(A - Q\hat{\Sigma}) + P\Sigma P - \tau_\perp^T P\Sigma P\tau_\perp. \tag{2.34}
\]

Furthermore, let \((A_c, B_c, C_c)\) be given by (2.20)-(2.22) and assume that \((\hat{A}, \hat{V})\) is stabilizable and \((C_c, A_c)\) is observable. Then \( A_c \) and \( \hat{A} \) are asymptotically stable.

**Proof:** Adding (2.33) to (2.34) yields
\[
0 = (A - Q\hat{\Sigma})^T \hat{P} + \hat{P}(A - Q\hat{\Sigma}) + A^T P + PA + R_1 + \Omega(P, \hat{P}),
\]
which can be written as
\[
0 = (A - Q\hat{\Sigma} - \Sigma P)^T \hat{P} + \hat{P}(A - Q\hat{\Sigma} - \Sigma P) + R_1 + (A + \Sigma \hat{P})^T P + P(A + \Sigma \hat{P}) + \Omega(P, \hat{P}).
\]

Since \((C_c, A_c)\) is observable, it follows that \( P_2 \triangleq G\hat{P}G^T > 0 \) [6]. Using (2.16) and \( P_2 > 0 \) with the fact that \( R_1 > 0 \) and \( \hat{P}\tau = \hat{P} \), it then follows that
\[
A_c^T P_2 + P_2 A_c = -G[R_1 + (A + \Sigma \hat{P})^T P + P(A + \Sigma \hat{P}) + \Omega(P, \hat{P})]G^T < 0,
\]

185
which implies that $A_c$ is asymptotically stable. Now since $(\tilde{A}, \tilde{V})$ is stabilizable, it follows that $\tilde{A}$ is asymptotically stable. 

\textbf{Remark 2.1} Setting $G = \Gamma = \tau = I$ in Theorem 2.3, which corresponds to $n_c = n$, we recover Theorem 2.1 for constructing full-order stable compensators.

\textbf{Remark 2.2} Note that the modification $\Omega(P, \hat{P})$ for constructing stable compensators involves equations (2.18) and (2.33) for $P$ and $\hat{P}$ as shown in both Theorem 2.1 and Theorem 2.3. By duality, we can similarly modify the $Q$ and $\hat{Q}$ equations to obtain stable compensators.

\section{Mixed-Norm $H_2/H_{\infty}$-Suboptimal Strong Stabilization}

In this section, we generalize the results in the previous section to the case of $H_2/H_{\infty}$-suboptimal stable stabilization. The goal is to obtain an asymptotically stable compensator dynamics matrix $A_c$ such that

(i) the closed-loop system (2.6), (2.7) is asymptotically stable;

(ii) the closed-loop transfer function $\tilde{G}_\infty(s) \sim \begin{bmatrix} \tilde{A} & \tilde{D} \\ \tilde{E}_\infty \end{bmatrix}$ from the disturbance $w(t)$ to performance variables $z(t) = E_{1\infty}x(t) + E_{2\infty}u(t)$, satisfies the constraint

$$\|\tilde{G}_\infty(s)\|_\infty \leq \gamma,$$

where $\gamma > 0$ is a given constant and $\tilde{E}_\infty \triangleq [E_{1\infty} \quad E_{2\infty}C_c]$ ; and

(iii) the $H_2$ performance measure (2.8) is minimized.

Following the \textit{a posteriori} approach discussed in the previous section, we state the main result for reduced-order mixed-norm $H_2/H_{\infty}$ stable stabilization. For notational convenience, we define $R_{1\infty} \triangleq E_{1\infty}^T E_{1\infty}$, $R_{2\infty} \triangleq E_{2\infty}^T E_{2\infty} = \kappa R_2$ and $S \triangleq (I + \kappa \gamma^{-2} \hat{Q} \hat{P})^{-1}$, where $\kappa \geq 0$ is a given constant.

\textbf{Theorem 3.1} Let $R_1 > 0$ and let $\Phi(Q, \hat{Q}, P, \hat{P}) \geq 0$ satisfy

$$[A + \gamma^{-2}(Q + \hat{Q}) R_{1\infty}]^T P + P [A + \gamma^{-2}(Q + \hat{Q}) R_{1\infty}] + S^T P \Sigma \hat{P} + \hat{P} \Sigma PS + \Phi(Q, \hat{Q}, P, \hat{P}) \geq 0,$$

for all $n \times n$ nonnegative-definite matrices $Q, \hat{Q}, P, \hat{P}$. Suppose there exist $n \times n$ nonnegative-definite matrices $Q, \hat{Q}, P, \hat{P}$ satisfying (2.27)-(2.30)
and

\[
0 = A^T Q + QA^T + V_1 + \gamma^{-2} QR_{1\infty} Q - Q \tilde{\Sigma} Q + \tau_\perp Q \tilde{\Sigma} Q \tau_\perp^T, \tag{3.3}
\]

\[
0 = (A - \Sigma PS + \gamma^{-2} QR_{1\infty}) \hat{Q} + \hat{Q} (A - \Sigma PS + \gamma^{-2} QR_{1\infty})^T
+ Q \tilde{\Sigma} Q - \tau_\perp Q \tilde{\Sigma} Q \tau_\perp^T + \gamma^{-2} \hat{Q} (R_{1\infty} + \kappa S^T P \Sigma PS) \hat{Q}, \tag{3.4}
\]

\[
0 = [A + \gamma^{-2} (Q + \hat{Q}) R_{1\infty}]^T P + P [A + \gamma^{-2} (Q + \hat{Q}) R_{1\infty}]
+ R_1 + \Phi(Q, \hat{Q}, P, \hat{P}) - S^T P \Sigma PS + \tau_\perp^T S^T P \Sigma PS \tau_\perp, \tag{3.5}
\]

\[
0 = (A - Q \tilde{\Sigma} + \gamma^{-2} QR_{1\infty})^T \hat{P} + \hat{P} (A - Q \tilde{\Sigma} + \gamma^{-2} QR_{1\infty})
+ S^T P \Sigma PS - \tau_\perp^T S^T P \Sigma PS \tau_\perp. \tag{3.6}
\]

Furthermore, let

\[
A_c = \Gamma (A - Q \tilde{\Sigma} - \Sigma PS + \gamma^{-2} QR_{1\infty}) G^T, \tag{3.7}
\]

\[
B_c = \Gamma QC^T V_2^{-1}, \tag{3.8}
\]

\[
C_c = -R_2^{-1} B^T P S G^T, \tag{3.9}
\]

and assume that \((A, \hat{V})\) is stabilizable and \((C_c, A_c)\) is observable. Then \(\hat{A}\) and \(A_c\) are asymptotically stable, and \(\|\hat{G}_\infty(s)\|_\infty \leq \gamma\).

**Proof:** Defining

\[
\hat{R} \triangleq R_1 + \Phi(Q, \hat{Q}, P, \hat{P}) > 0,
\]

it can be seen that (3.3)-(3.6) correspond to the standard reduced-order \(H_2/H_\infty\) controller [1] with \(R_1\) replaced by \(\hat{R}\) in (3.5) and hence \(\|\hat{G}_\infty(s)\|_\infty \leq \gamma\) is immediately satisfied. Using the fact that \((\hat{A}, \hat{V})\) is stabilizable, it follows that \(\hat{A}\) is asymptotically stable. To show that \(A_c\) is asymptotically stable, add (3.5) to (3.6) to obtain

\[
0 = (A - Q \tilde{\Sigma} - \Sigma PS + \gamma^{-2} QR_{1\infty})^T \hat{P} + \hat{P} (A - Q \tilde{\Sigma} - \Sigma PS + \gamma^{-2} QR_{1\infty})
+ R_1 + [A + \gamma^{-2} (Q + \hat{Q}) R_{1\infty}]^T P + P [A + \gamma^{-2} (Q + \hat{Q}) R_{1\infty}]
+ S^T P \Sigma \hat{P} + \hat{P} \Sigma PS + \Phi(Q, \hat{Q}, P, \hat{P}).
\]

Since \((C_c, A_c)\) is observable, it can be shown that \(P_2 = G \hat{P} G^T > 0\) [1]. Using (3.2), \(\hat{P} r = \hat{P}\), and \(P_2 > 0\), it follows that

\[
A_c^T P_2 + P_2 A_c = -G \{ R_1 + [A + \gamma^{-2} (Q + \hat{Q}) R_{1\infty}]^T P + P [A + \gamma^{-2} (Q + \hat{Q}) R_{1\infty}] + S^T P \Sigma \hat{P} + \hat{P} \Sigma PS + \Phi(Q, \hat{Q}, P, \hat{P}) \} G^T < 0,
\]

which implies that \(A_c\) is asymptotically stable. \(\square\)

**Proposition 3.1** For arbitrary \(\alpha, \beta > 0\), the following matrix functions \(\Phi(Q, \hat{Q}, P, \hat{P})\) satisfy (3.2) for all nonnegative-definite matrices \(Q, \hat{Q}, P, \hat{P}\):
(i) \[ \Phi(Q, \dot{Q}, P, \dot{P}) = \alpha^2 [A + \gamma^{-2}(Q + \dot{Q})R_{1\infty}]^T[A + \gamma^{-2}(Q + \dot{Q})R_{1\infty}] + \alpha^{-2}P^2 + \beta^2S^TP\Sigma^2PS + \beta^{-2}\hat{P}^2, \]

(ii) \[ \Phi(Q, \dot{Q}, P, \dot{P}) = \alpha^2 [A + \gamma^{-2}(Q + \dot{Q})R_{1\infty}]^T[A + \gamma^{-2}(Q + \dot{Q})R_{1\infty}] + \alpha^{-2}P^2 + (\beta PS - \beta^{-1}\hat{P})^T\Sigma(\beta PS - \beta^{-1}\hat{P}), \]

(iii) \[ \Phi(Q, \dot{Q}, P, \dot{P}) = [\alpha(A + \gamma^{-2}(Q + \dot{Q})R_{1\infty}) - \alpha^{-1}I]^T\Sigma[\alpha(A + \gamma^{-2}(Q + \dot{Q})R_{1\infty}) - \alpha^{-1}I] + \beta^2S^TP\Sigma^2PS + \beta^{-2}\hat{P}^2, \]

(iv) \[ \Phi(Q, \dot{Q}, P, \dot{P}) = [\alpha(A + \gamma^{-2}(Q + \dot{Q})R_{1\infty}) - \alpha^{-1}I]^T\Sigma[\alpha(A + \gamma^{-2}(Q + \dot{Q})R_{1\infty}) - \alpha^{-1}I] + (\beta PS - \beta^{-1}\hat{P})^T\Sigma(\beta PS - \beta^{-1}\hat{P}). \]

Proof: The proof involves straightforward matrix manipulations and hence is omitted. \qed

Next, we specialize Theorem 3.1 to full-order dynamic compensation.

**Corollary 3.1 (Full-Order H₂/H_∞ Strong Stabilization)** Let \( R_1 > 0 \) and let \( \Phi(Q, \dot{Q}, P, \dot{P}) \geq 0 \) satisfy (3.2). Suppose there exist \( n \times n \) nonnegative-definite matrices \( Q, \dot{Q}, P, \dot{P} \) satisfying

\[
0 = AQ + QA^T + V_1 + \gamma^{-2}QR_{1\infty}Q - Q\dot{\Sigma}Q, \tag{3.10}
\]

\[
0 = (A - \Sigma PS + \gamma^{-2}QR_{1\infty})\dot{Q} + \dot{Q}(A - \Sigma PS + \gamma^{-2}QR_{1\infty})^T + Q\dot{\Sigma}Q + \gamma^{-2}R_{1\infty}\dot{\Sigma}, \tag{3.11}
\]

\[
0 = [A + \gamma^{-2}(Q + \dot{Q})R_{1\infty}]^TP + P[A + \gamma^{-2}(Q + \dot{Q})R_{1\infty}] + R_1 + \Phi(Q, \dot{Q}, P, \dot{P}) - S^TP\Sigma PS, \tag{3.12}
\]

\[
0 = (A - Q\dot{\Sigma} + \gamma^{-2}QR_{1\infty})^T\dot{P} + \dot{P}(A - Q\dot{\Sigma} + \gamma^{-2}QR_{1\infty}) + S^TP\Sigma PS. \tag{3.13}
\]

Furthermore, let

\[
A_c = A - Q\dot{\Sigma} - \Sigma PS + \gamma^{-2}QR_{1\infty}, \tag{3.14}
\]

\[
B_c = QC^TV_2^{-1}, \tag{3.15}
\]

\[
C_c = -R_2^{-1}B^TPS. \tag{3.16}
\]

and assume that \( (\tilde{A}, \tilde{V}) \) is stabilizable and \( (C_c, A_c) \) is observable. Then \( \tilde{A} \) and \( A_c \) are asymptotically stable and \( \|\tilde{G}_{\infty}(s)\|_{\infty} \leq \gamma \).

Proof: The result is a direct consequence of Theorem 3.1 by setting \( n_c = n \) so that \( \tau = \Gamma = G = I_n \). \qed

**Remark 3.1** Note that by letting \( \gamma \to \infty \) in Theorem 3.1 and Corollary 3.1, we recover the H₂ results obtained in Section 2 for both full- and reduced-order dynamic compensation. It is interesting to note that in contrast to the full-order H₂ case described in Section 2 and [9] which involves three matrix equations for constructing stable controllers, the full-order H₂/H_∞ problem involves four matrix equations for obtaining stable controllers.
Next, we use the cost modification approach proposed in [9] to design $H_2/H_\infty$ stable controllers. This approach seeks to minimize

$$\mathcal{J}(A_c, B_c, C_c) = \text{tr } Q \tilde{R}$$

subject to

$$0 = \tilde{A}Q + Q \tilde{A}^T + \gamma^{-2} Q \tilde{R}_\infty Q + \tilde{V} + \Omega(Q),$$

where

$$\tilde{R} \triangleq \begin{bmatrix} R_1 & 0 \\ 0 & C_c R_2 C_c \end{bmatrix}, \quad \tilde{R}_\infty \triangleq \begin{bmatrix} R_{1\infty} & 0 \\ 0 & C_c^T R_{2\infty} C_c \end{bmatrix},$$

$$\tilde{V} \triangleq \begin{bmatrix} V_1 & 0 \\ 0 & B_c V_2 B_c^T \end{bmatrix}.$$

As shown in [1], minimizing (3.17) subject to (3.18) with $\Omega(Q) = 0$ guarantees closed-loop stability, $H_\infty$ disturbance attenuation, and a worst-case bound on $H_2$ performance. To also guarantee stability of the compensator dynamics $A_c$, we set $\Omega(Q)$ to

$$\Omega(Q) = \begin{bmatrix} 0 & 0 \\ 0 & Q_{12}^T \Sigma Q_{12} \end{bmatrix},$$

as in [9], where $Q = \begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{bmatrix}$. Now using the fixed-structure optimization approach outlined in [1] to minimize (3.17) subject to (3.18) with $\Omega(Q)$ given by (3.19) yields the following theorem.

**Theorem 3.2** Suppose there exist $n \times n$ nonnegative-definite $Q$, $\hat{Q}$, $P$, $\hat{P}$ satisfying (2.27)-(2.30) and

$$0 = \begin{bmatrix} AQ + QA^T + V_1 + \gamma^{-2} Q R_{1\infty} Q - Q \tilde{\Sigma} Q + \tau_\perp Q \tilde{\Sigma} Q \tau_\perp^T + \hat{Q} \tilde{\Sigma} \hat{Q}, \\ \end{bmatrix}$$

$$0 = \begin{bmatrix} (A - \Sigma PS + \gamma^{-2} Q R_{1\infty}) \hat{Q} + \hat{Q} (A - \Sigma PS + \gamma^{-2} Q R_{1\infty})^T + Q \tilde{\Sigma} Q \\ - \tau_\perp Q \tilde{\Sigma} Q \tau_\perp^T - \hat{Q} \tilde{\Sigma} \hat{Q} + \gamma^{-2} \hat{Q} (R_{1\infty} + \kappa S^T P \Sigma PS) \hat{Q}, \\ \end{bmatrix}$$

$$0 = \begin{bmatrix} A + \gamma^{-2} (Q + \hat{Q}) R_{1\infty}^T P + P [A + \gamma^{-2} (Q + \hat{Q}) R_{1\infty}^T + R_1 \\ + \hat{P} \tilde{\Sigma} + \tilde{\Sigma} \hat{P} - \gamma S^T P \Sigma PS + \tau_\perp^T \gamma S^T P \Sigma PS \tau_\perp, \\ \end{bmatrix}$$

$$0 = \begin{bmatrix} A - (Q + \hat{Q}) \tilde{\Sigma} + \gamma^{-2} Q R_{1\infty}^T \hat{P} + \hat{P} A - (Q + \hat{Q}) \tilde{\Sigma} + \gamma^{-2} Q R_{1\infty} \\ + \gamma S^T P \Sigma PS - \tau_\perp^T \gamma S^T P \Sigma PS \tau_\perp, \\ \end{bmatrix}$$

and let

$$A_c = \Gamma [A - (Q + \hat{Q}) \tilde{\Sigma} - \Sigma PS + \gamma^{-2} Q R_{1\infty}] G^T,$$

$$B_c = \Gamma Q C_c^T V_2^{-1},$$

$$C_c = -R_2^{-1} B_c^T P S G^T.$$
If \((A_c, B_c)\) is stabilizable, \((C_c, A_c)\) is observable and \(R_{1\infty} > 0\), then \(\hat{A}\) and \(A_c\) are asymptotically stable, \(\| \hat{G}_\infty(s) \|_\infty \leq \gamma\), and the \(H_2\) performance satisfies the bound

\[ J(A_c, B_c, C_c) \leq \text{tr} \left[ (Q + \hat{Q})R_1 + \hat{Q}S^TP\Sigma PS \right]. \]  
(3.27)

**Proof:** Since \(\Omega(Q) \geq 0\), it can be seen from (3.18) that if there exist nonnegative-definite \(Q\) and \(\hat{Q}\) satisfying (3.18), then \(\hat{A}\) is asymptotically stable and \(\| \hat{G}_\infty(s) \|_\infty \leq \gamma\). Now applying the Lagrange multiplier method to minimize (3.17) subject to (3.18), yields

\[ A_cQ_2 + Q_2A_c^T = -\Gamma[(Q + \hat{Q})\Sigma(Q + \hat{Q}) + \gamma^{-2}(R_{1\infty} + \kappa S^TP\Sigma PS)\hat{Q}]\Gamma^T < 0, \]

which implies that \(A_c\) is asymptotically stable. Equations (3.20)-(3.26) follow from algebraic manipulations. See [1] for details of a similar proof. \(\Box\)

**Remark 3.2** In the full-order case, set \(n_c = n\) so that \(\Gamma = G = \tau = I_n\) to obtain sufficient conditions for constructing stable \(H_2/H_\infty\) compensators using the cost modification approach. Finally, relaxing the \(H_\infty\) constraint, i.e., \(\gamma \to \infty\), Theorem 3.2 specializes to Theorem 4.2 of [9].

### 4 Illustrative Numerical Examples

We now use the numerical procedure proposed in [9] to solve the coupled matrix Riccati equations developed in the previous sections. This procedure involves first-order parameter variations of \(Q, \hat{Q}, P\) and \(\hat{P}\) and is based upon Newton's method.

**Example 1** Consider the two-mass system shown in Figure 1, where

\[ A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, \]

with disturbance weighting matrices given by

\[ D_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \]

and performance weighting matrices given by

\[ E_1 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 \\ 0.01 \end{bmatrix}. \]
$H_2/H_\infty$ STRONG STABILIZATION

Figure 1: Two Mass System

Note that the results in [9] are based upon (vi) of Proposition 2.1 in which $\Omega(P, \hat{P})$ involves two parameters $\alpha$ and $\beta$. In this paper, we alternatively choose (i) of Proposition 2.1 to guarantee the asymptotic stability of $A_c$ while implementing Theorem 2.1. Since there is only one parameter $\alpha$ present in (i) of Proposition 2.1, the numerical procedure for obtaining an asymptotically stable compensator is somewhat simpler. The closed-loop poles are

$$\{-97.6052, -1.0277 \pm j2.8161, -2.4941 \pm j1.1604, -0.9728, -0.0033 \pm j1.0\}$$

while the eigenvalues of $A_c$ are

$$\{-97.2507, -8.3768, -0.0003 \pm j1.0364\}.$$ 

The modified cost is 264.4246 while the LQG optimal cost is 261.6534. Hence the cost increment is 1.06%. Comparing to the cost increment 26.94% of the modified design given in [9], we have a lower cost increment using (i) of Corollary 2.1 while implementing Theorem 2.1. In general, the choice of $\Omega(P, \hat{P})$ in Corollary 2.1 which leads to the minimal cost increment is problem dependent.

Our next example involves the design of a stable $H_2/H_\infty$ controller.

**Example 2** Consider again the system given in Example 1 with plant dynamics $A, B, C$, disturbance matrix $D_1$ and sensor noise matrix $D_2$ with the performance weightings

$$E_1 = \begin{bmatrix} 10 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix},$$

and $H_\infty$ weighting matrices $R_{1\infty} = 0.05R_1$ and $R_{2\infty} = 0$. Applying Corollary 3.1 with $\Phi$ given by (ii) in Proposition 3.1 and using the numerical
algorithm developed in [9], we obtain controllers for $H_2/H_\infty$ strong stabilization. In the full-order case, the LQG cost of the $H_2$-optimal solution is $2.6165 \times 10^4$. The LQG design places the closed-loop poles at $\{-99.985, -2.4941 \pm j1.1604, -1.0277 \pm j2.8161, -1.0001, -0.0035 \pm j1.0\}$ while the poles of $A_c$ are $\{-99.637, -8.399, .0003 \pm j1.0396\}$ which yields an unstable LQG controller. Now choosing $\alpha = 0.5$ and $\beta = 2$ in Proposition 3.1, the resulting stable $H_2/H_\infty$ controller has closed-loop poles at $\{-96.3944, -5.575, -1.2085 \pm j2.5093, -2.6172, -1.0003, -0.0022 \pm j1.0\}$ with $A_c$ poles $\{-95.7916, -12.2165, -0.0005 \pm j1.0282\}$. The total cost is $3.5453 \times 10^4$. Furthermore, the peak magnitude of the maximum singular value $\tilde{G}(s) = \tilde{E}(sI - \tilde{A})^{-1}\tilde{D}$ with the LQG gains is $49.368$dB whereas the peak magnitude of $\|\tilde{G}(s)\|_\infty$ with the stable $H_2/H_\infty$ gains is $47.62$dB as shown in Figure 2.

5 Conclusion

Several alternative modifications of the full- and reduced-order $H_2$ optimality conditions have been shown to enforce the stability of the compensator. The modification terms, once selected, involve free parameters whose values can be determined by an iterative search.
This paper considered both \textit{a priori} and \textit{a posteriori} approaches, discussed in [9], to derive results for strong stabilization with $H_2$ and $H_\infty$ performance measures. It is noted that in the full-order case with the \textit{a posteriori} approach, the stable controller is characterized by three modified Riccati equations in the $H_2$ setting whereas four modified Riccati equations are needed to characterize the full-order stable mixed-norm $H_2/H_\infty$ controller.

\textbf{References}


